Strategies for inversion of the additive relationship matrix among genotyped animals

P. Faux and N. Gengler

University of Liege, Gembloux Agro-Bio Tech, Animal Science Unit
Introduction: The case of $A_{22}$ vs. $A$

✓ $A_{22}$ = subpart of $A$ whose inversion is required, e.g. in ssGBLUP
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✓ The inverse of $A$ is computed as a sum of vector products (Henderson, 1976)

$$A^{-1}_{(i)} = \begin{bmatrix} A^{-1}_{(i-1)} & 0 \\ 0' & 0 \end{bmatrix} + \alpha_{(i)} \begin{bmatrix} -b_{(i)} \\ 1 \end{bmatrix} \begin{bmatrix} -b'_{(i)} & 1 \end{bmatrix}$$
Introduction: The case of $A_{22}$ vs. $A$

✓ The inverse of $A$ is computed as a sum of vector products.

$$
A^{-1}_{(i)} = \left[ \begin{array}{cc}
A^{-1}_{(i-1)} & 0 \\
0' & 0
\end{array} \right] + \alpha_{(i)} \left[ \begin{array}{c}
-b_{(i)} \\
1
\end{array} \right] \left[ \begin{array}{cc}
-b'_{(i)} & 1
\end{array} \right]
$$

$$
A^{-1}_{(i)} = \left( T^{-1}_{A(i)} \right)' D^{-1}_{A(i)} T^{-1}_{A(i)}
$$

$$
T^{-1}_{A(i)} = \left[ \begin{array}{cc}
T^{-1}_{A(i-1)} & 0 \\
-b'_{(i)} & 1
\end{array} \right]
$$

$$
D^{-1}_{A(i)} = \left[ \begin{array}{cc}
D^{-1}_{A(i-1)} & 0 \\
0' & \alpha_{(i)}
\end{array} \right]
$$
Introduction: The case of $A_{22}$ vs. $A$

$A_{22} = \text{subpart of } A \text{ whose inversion is required, e.g. in ssGBLUP}$
Sparsity in the inverse factor of $A_{22}$

✓ Example: An animal and its parents

A → C

B → C
Sparsity in the inverse factor of $A_{22}$

**Example:** An animal and its parents
Issues and Objective

✓ How sparse is the inverse of $A_{22}$?

... How sparse is the inverse factor $(T^{-1})$ of $A_{22}$?

✓ How a putative sparsity could be used in computation of the inverse?

→ Main objective: To avoid useless computations
Sparsity in the inverse factor of $A_{22}$

✓ How to deal with more complex cases?

✓ By a comprehensive search in the pedigree
  ✓ « SP Algorithm »
  ✓ Explores pedigree branches and apply simple rules
  ✓ Uses only pedigree and incidence vector
  ✓ Returns a symbolic inverse factorization
Sparsity in the inverse factor of $A_{22}$

✓ Some performances on different sizes of $A_{22}$:

![Graph showing performance times for different sizes of $A_{22}$]

Processor: Intel Xeon 64-bit
64Gb of RAM,
Clock speed: 3.16GHz
Strategies to take sparsity into account

1. Successive construction of the inverse

\[
A_{22(i)}^{-1} = \begin{bmatrix}
A_{22(i-1)}^{-1} & 0 \\
0' & 0
\end{bmatrix} + \alpha(i) \begin{bmatrix}
-b(i) \\
1
\end{bmatrix} \begin{bmatrix}
-b'(i) & 1
\end{bmatrix}
\]

How to get \(b\)?

1. \(b(i) = A_{22(i-1)}^{-1} A_{22(i-1)}(:,1:i-1)\)
2. \(A_{22(i-1)} b(i) = A_{22(i-1)}(:,1:i-1)\)
Strategies to take sparsity into account

1. Restricting the product only to elements of \( \mathbf{b} \) different from 0

\[
\mathbf{b}_{(i)} = \mathbf{A}_{22(i-1)}^{-1} \mathbf{A}_{22(i-1)} (:,1:i-1) \quad \rightarrow \quad \mathbf{x} = \mathbf{Zy}
\]
Strategies to take sparsity into account

2. Solving a linear system of lower size

\[ A_{22(i-1)} \mathbf{b}_i = A_{22(i-1)}(:,1:i-1) \rightarrow Z\mathbf{x} = \mathbf{y} \]

\[
\begin{array}{ccc}
\neq 0 & = 0 & \\
\neq 0 & 0 & = \\
= 0 & = 0 & \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{Z} & \mathbf{x} & \mathbf{y} \\
\mathbf{0} & = & \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{Z} & \mathbf{x} & \mathbf{y} \\
\end{array}
\]
Strategies to take sparsity into account

Processor: Intel Xeon 64-bit
8Gb of RAM,
Clock speed: 3 GHz
Strategies to take sparsity into account

✓ Order of $A_{22} = \text{Number of genotyped animals}$
✓ Depends on the pedigree (depth, lines, ...)

![Graph showing the percentage of zeros increasing with the number of genotyped animals.](image)
Strategies to take sparsity into account

3. Storing the inverse of $A_{22}$ from time to time and updating this inverse only for recent animals

$$A_{22}^{-1}(t) = \begin{bmatrix} A_{22}^{-1}(t-1) & 0 \\ 0' & 0 \end{bmatrix} + \alpha_i \begin{bmatrix} -\mathbf{b}_i \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{b}'_i \\ 1 \end{bmatrix}$$
Strategies to take sparsity into account

A

\[ A_{22(t)} \]

\[ A_{22(t+1)} \]
Strategies to take sparsity into account

A A

A_{22(t+1)}

A_{22(t)}
Take-home messages

1. Sparsity pattern of the inverse of $A_{22}$ can be set up without matrix computations, even for large matrices
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2. Using sparsity reduces time for inversion, if that inversion uses the inverse factor
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2. Using sparsity reduces time for inversion, if that inversion uses the inverse factor

3. As the order of $A_{22}$ increases, inversion shrinks to solve multiple small linear systems that are identified by SP algorithm
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