Higher symmetries of the conformal Laplacian

J.-P. Michel, J. Silhan, R.

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Introduction

- On $(\mathbb{R}^2, \mathrm{g}_0)$, we consider the Schrödinger equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

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- Coordinates (u, v) separate this equation $\Longleftrightarrow$ $\exists$ solution of the form $f(u)g(v)$
- Coordinates (u, v) orthogonal $\Longleftrightarrow$ $\mathrm{g}_0(\partial_u, \partial_v) = 0$

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  4. Elliptic coordinates (α, β) :

$$\begin{cases} x &= \sqrt{d}\cos(\alpha)\cosh(\beta) \\ y &= \sqrt{d}\sin(\alpha)\sinh(\beta) \end{cases}$$

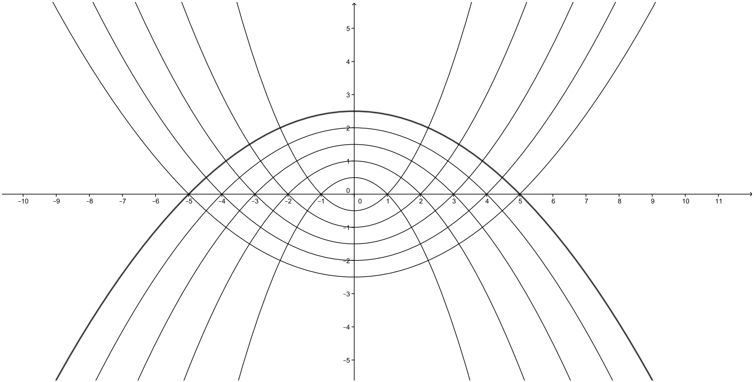

Figure: Coordinates lines corresponding to the parabolic coordinates system

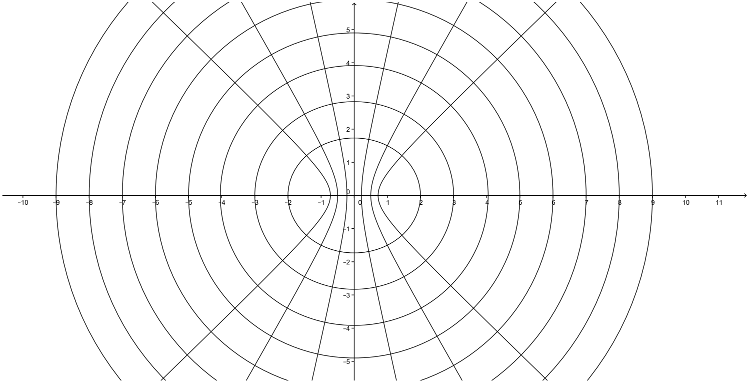

Figure: Coordinates lines corresponding to the elliptic coordinates system

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$$\begin{cases} \partial_x^2 f - E_1 f = 0 \\ \partial_y^2 g - (E - E_1)g = 0 \end{cases}$$

- Bijective correspondence

$$\{\text{Separating coordinates systems}\}$$

$$\longleftrightarrow$$

$$\{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\}$$

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Coordinates system	Symmetry
(x, y)	∂_x^2
(r, θ)	L_θ^2
(ξ, η)	$\frac{1}{2}(\partial_x L_\theta + L_\theta \partial_x)$
(α, β)	$L_\theta^2 + d\partial_x^2$

with $L_\theta = x\partial_y - y\partial_x$

- Link between the symmetry and the coordinates system : if the second-order part of D reads as

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eigenvectors of A in this case :

$$\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix}$$

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{Sc},$$

where Sc is the scalar curvature of g.

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- Conformal symmetry of Δ_Y : $D_1 \in \mathcal{D}(M)$ such that $\exists D_2 \in \mathcal{D}(M)$ such that $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$

- (M, g) conformally flat : for each $x \in M$, there exist a neighborhood U of x and a function f on U such that $e^{2f}g$ is flat on U

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- (M, g) Einstein : $\mathrm{Ric} = f\mathrm{g}$
  Existence of a second order symmetry (B. Carter)

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- $\sigma(D)$ can be viewed as a contravariant symmetric tensor of degree k :

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- Is this condition sufficient ?

Definition

A quantization on M is a linear bijection $\mathcal{Q}^M$ from the space of symbols $\mathrm{Pol}(T^*M)$ to the space of differential operators $\mathcal{D}(M)$ such that

$$\sigma(\mathcal{Q}^M(S)) = S, \quad \forall S \in \mathrm{Pol}(T^*M)$$

Definition

A natural and conformally invariant quantization is the data for every manifold M of a quantization $\mathcal{Q}^M$ depending on a pseudo-Riemannian metric defined on M such that

- *If Φ is a local diffeomorphism from M to a manifold N, then one has*

$$\mathcal{Q}^M(\Phi^*\mathrm{g})(\Phi^*S) = \Phi^*(\mathcal{Q}^N(\mathrm{g})(S)),$$

  for all pseudo-Riemannian metric g on N and all $S \in \mathrm{Pol}(T^*N)$

- *$\mathcal{Q}^M(\mathrm{g}) = \mathcal{Q}^M(\tilde{\mathrm{g}})$ whenever g and $\tilde{\mathrm{g}}$ are conformally equivalent, i.e. whenever there exists a function Υ such that $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$.*

- Proof of the existence of $\mathcal{Q}^M$:
  1. Work by A. Cap, J. Silhan
  2. Work by P. Mathonet, R.

- If K is a conformal Killing tensor of degree 2, there exists a conformal symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3A_{jl}{}^i \right)$$

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- C : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}(\mathrm{g}_{a[c}\mathrm{Ric}_{d]b} - \mathrm{g}_{b[c}\mathrm{Ric}_{d]a}) + \frac{2}{(n-1)(n-2)}\mathrm{Sc}\ \mathrm{g}_{a[c}\mathrm{g}_{d]b}$$

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- A : Cotton-York tensor :

$$A_{ijk} = \nabla_k \mathrm{Ric}_{ij} - \nabla_j \mathrm{Ric}_{ik} + \frac{1}{2(n-1)} \left(\nabla_j \mathrm{Sc}\ g_{ik} - \nabla_k \mathrm{Sc}\ g_{ij}\right)$$

- If $\mathrm{Obs}(K)^\flat = 2df$, the conformal symmetries of Δ_Y whose the principal symbol is given by K are of the form

$$\mathcal{Q}(K) - f + L_X + c,$$

where X is a conformal Killing vector field and where $c \in \mathbb{R}$

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  2. If $\mathrm{Ric} = \frac{1}{n}\mathrm{Sc}\,\mathrm{g}$ and if K is a Killing tensor of degree 2, then

$$\mathrm{Obs}(K)^\flat = d\left(\frac{2-n}{2(n+1)}(\nabla_i\nabla_j K^{ij}) + \frac{2-n}{2n(n-1)}\mathrm{Sc}\,\mathrm{g}_{ij}K^{ij}\right)$$

and $\nabla_i K^{ij}\nabla_j$ is a symmetry of Δ_Y

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  Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$:

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  Killing tensor K :

$$\frac{c(x_3) a(x_1, x_2) p_1^2 + c(x_3) b(x_1, x_2) p_2^2 - \gamma(x_1, x_2) p_3^2}{\gamma(x_1, x_2) + c(x_3)},$$

  $a, b, \gamma \in C^\infty(\mathbb{R}^2)$, $c \in C^\infty(\mathbb{R})$.

- If $\tilde{g} = \frac{1}{2(\gamma(x_1,x_2)+c(x_3))} g$, then

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- Symmetry of Δ_Y :

$$\nabla_i K^{ij} \nabla_j - \frac{1}{16}(\nabla_i \nabla_j K^{ij}) - \frac{1}{8}\mathrm{Ric}_{ij} K^{ij}$$

- Four-dimensional fiber bundle M over S^2 with coordinates (ψ, r, θ, ϕ)
- Taub-NUT metric g :

$$\left(1+\frac{2m}{r}\right)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right) + \frac{4m^2}{1+\frac{2m}{r}}\left(d\psi+\cos\theta d\phi\right)^2$$

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- g hyperkähler : there exist three complex structures J_i which are covariantly constant and which satisfy the quaternion relations

$$J_1^2=J_2^2=J_3^2=J_1J_2J_3=-\mathrm{Id}.$$

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$$K_i = p_\mu p_\nu \left(*Y_\lambda^{(\mu} J_i^{\nu)\lambda}\right)$$

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  conformal Killing tensors
- $\mathrm{Obs}(K_i)^\flat$ not exact, then there are no conformal symmetries whose principal symbols are the K_i

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- Schrödinger equation at zero energy : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential

- Schrödinger equation at zero energy R-separable in an orthogonal coordinates system (x^i) ($\mathrm{g}_{ij} = 0$ if $i \neq j$)

$$\Longleftrightarrow$$

$\exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R = \sum_{i=1}^{n} h_i L_i .$$

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$\forall\ E \in \mathbb{R}$, $\exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^{n} h_i L_i .$$

- $R\prod_{i=1}^{n}\phi_i(x^i)$ solution of one of the two previous equations

$$\Longleftrightarrow$$

$$L_i\phi_i = 0 \quad \forall i$$

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