Discontinuous Galerkin methods for solid mechanics: Application to fracture, shells & strain gradient elasticity

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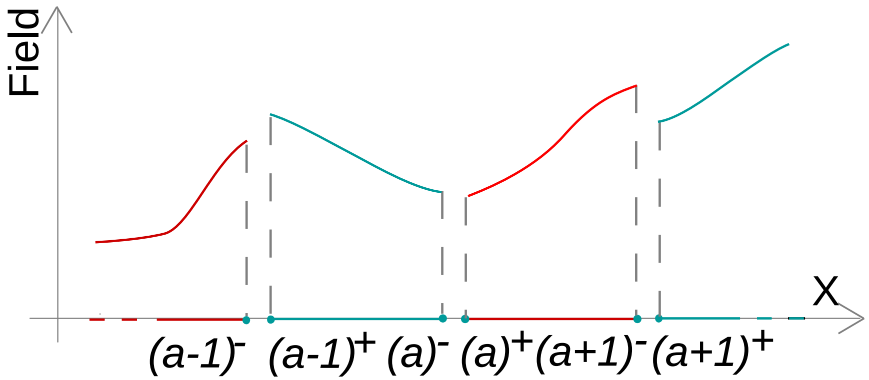
Discontinuous Galerkin Methods

- Main idea
  - Finite-element discretization
  - Same **discontinuous** polynomial approximations for the
    - **Test** functions φ_h and
    - **Trial** functions $\delta\varphi$



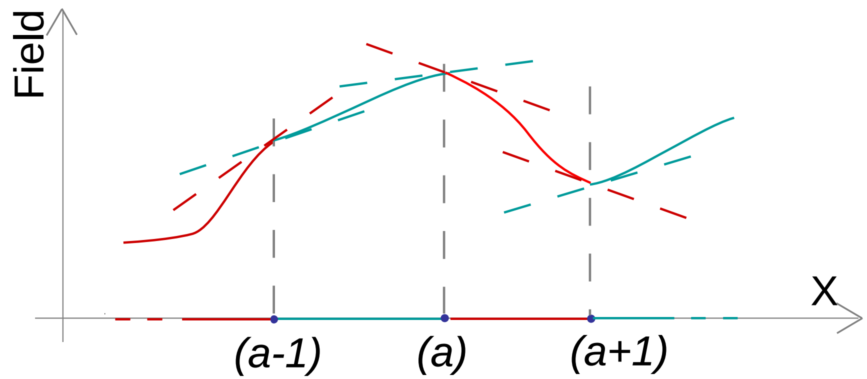
  - Definition of operators on the interface trace:
    - **Jump** operator: $[\![\bullet]\!] = \bullet^+ - \bullet^-$
    - **Mean** operator: $\langle\bullet\rangle = \dfrac{\bullet^+ + \bullet^-}{2}$
  - Continuity is weakly enforced, such that the method
    - Is consistent
    - Is stable
    - Has the optimal convergence rate

Discontinuous Galerkin Methods

- Discontinuous Galerkin methods vs Continuous
  - More expensive (more degrees of freedom)
  - More difficult to implement
  - …
- So why discontinuous Galerkin methods?
  - Weak enforcement of C^1 continuity for high-order equations
    - Strain-gradient effect
    - Shells with complex material behaviors
    - Toward high-order computational homogenization
  - Exploitation of the discontinuous mesh to simulate dynamic fracture [Seagraves, Jérusalem, Noels, Radovitzky, col. ULg-MIT]:
    - Correct wave propagation before fracture
    - Easy to parallelize & scalable

Discontinuous Galerkin Methods

- Continuous field / discontinuous derivative
  - No new nodes
  - Weak enforcement of C^1 continuity
  - Displacement formulations of high-order differential equations
  - Usual shape functions in 3D (no new requirement)
  - Applications to
    - Beams, plates [Engel et al., CMAME 2002; Hansbo & Larson, CALCOLO 2002; Wells & Dung, CMAME 2007]
    - Linear & non-linear shells [Noels & Radovitzky, CMAME 2008; Noels IJNME 2009]
    - Damage & Strain Gradient [Wells et al., CMAME 2004; Molari, CMAME 2006; Bala-Chandran et al. 2008]



Topics

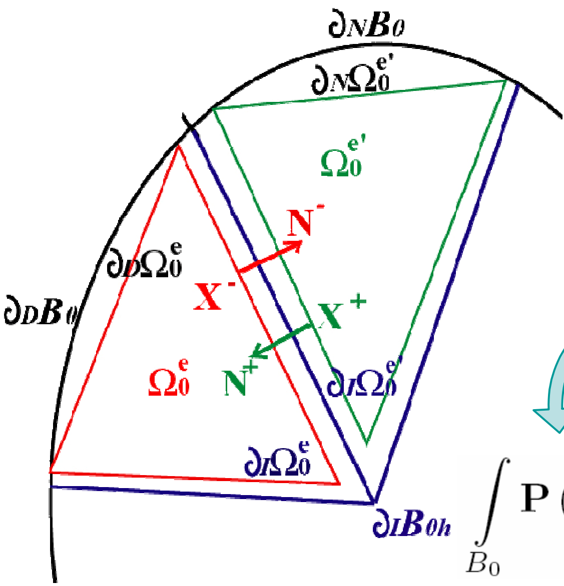
- Key principles of DG methods
  - Illustration on volume FE
- Discontinuous Mesh & Dynamic Fracture
- Kirchhoff-Love shells
  - Kinematics
  - Non-Linear shells
  - Numerical examples
- Strain gradient elasticity
- Conclusions & Perspectives

Key principles of DG methods

- Application to non-linear mechanics
  - Formulation in terms of the first Piola stress tensor $\mathbf{P}$

$$\nabla_0 \cdot \mathbf{P}^T = 0 \text{ in } \Omega \quad \& \quad \begin{cases} \mathbf{P} \cdot \boldsymbol{N} = \bar{\boldsymbol{T}} \text{ on } \partial_N \Omega \\ \varphi_h = \bar{\varphi}_h \text{ on } \partial_D B \end{cases}$$

  - New weak formulation obtained by integration by parts **on each element** Ω^e



$$\sum_e \int_{\Omega_0^e} \nabla_0 \cdot \mathbf{P}^T(\varphi_h) \cdot \delta\varphi \, dB = 0$$

$$\sum_e \int_{\Omega_0^e} -\mathbf{P}(\varphi_h) : \nabla_0 \delta\varphi \, dB + \sum_e \int_{\partial \Omega_0^e} \delta\varphi \cdot \mathbf{P}(\varphi_h) \cdot \boldsymbol{N} \, d\partial B = 0$$

$$\int_{B_0} \mathbf{P}(\varphi_h) : \nabla_0 \delta\varphi \, dB + \int_{\partial_I B_0} [\![\delta\varphi \cdot \mathbf{P}(\varphi_h)]\!] \cdot \boldsymbol{N}^- \, d\partial B = \int_{\partial_N B_0} \bar{\boldsymbol{T}} \cdot \delta\varphi \, d\partial B$$

?

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Key principles of DG methods

- Interface term rewritten as the sum of 3 terms
  - Introduction of the numerical flux ***h***

$$\int_{\partial_I B_0} [\![\delta\boldsymbol{\varphi}\cdot\mathbf{P}(\boldsymbol{\varphi}_h)]\!]\cdot\boldsymbol{N}^-\,d\partial B \rightarrow \int_{\partial_I B_0} [\![\delta\boldsymbol{\varphi}]\!]\cdot\boldsymbol{h}\left(\mathbf{P}^+,\,\mathbf{P}^-,\,\boldsymbol{N}^-\right)\,d\partial B$$

    - Has to be consistent: $\begin{cases}\boldsymbol{h}\left(\mathbf{P}^+,\,\mathbf{P}^-,\,\boldsymbol{N}^-\right) = -\boldsymbol{h}\left(\mathbf{P}^-,\,\mathbf{P}^+,\,\boldsymbol{N}^+\right)\\ \boldsymbol{h}\left(\mathbf{P}_{\text{exact}},\,\mathbf{P}_{\text{exact}},\,\boldsymbol{N}^-\right) = \mathbf{P}_{\text{exact}}\cdot\boldsymbol{N}^-\end{cases}$
    - One possible choice: $\boldsymbol{h}\left(\mathbf{P}^+,\,\mathbf{P}^-,\,\boldsymbol{N}^-\right) = \langle\mathbf{P}\rangle\cdot\boldsymbol{N}^-$
  - Weak enforcement of the compatibility

$$\int_{\partial_I B_0} [\![\boldsymbol{\varphi}_h]\!]\cdot\left\langle\frac{\partial\mathbf{P}}{\partial\mathbf{F}}:\boldsymbol{\nabla}_0\delta\boldsymbol{\varphi}\right\rangle\cdot\boldsymbol{N}^-\,d\partial B$$

  - Stabilization controlled by parameter β, for all mesh sizes h^s

$$\int_{\partial_I B_0} [\![\boldsymbol{\varphi}_h]\!]\otimes\boldsymbol{N}^- : \left\langle\frac{\beta}{h^s}\frac{\partial\mathbf{P}}{\partial\mathbf{F}}\right\rangle : [\![\delta\boldsymbol{\varphi}]\!]\otimes\boldsymbol{N}^-\,d\partial B$$

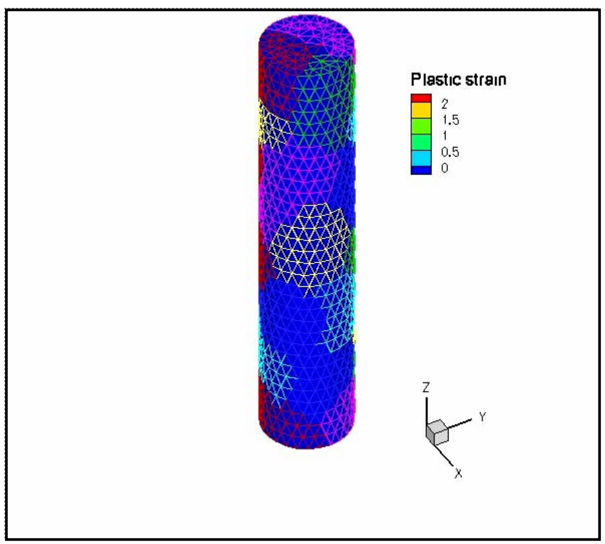
Noels & Radovitzky, IJNME 2006 & JAM 2006

  - Those terms can also be explicitly derived from a variational formulation (Hu-Washizu-de Veubeke functional)

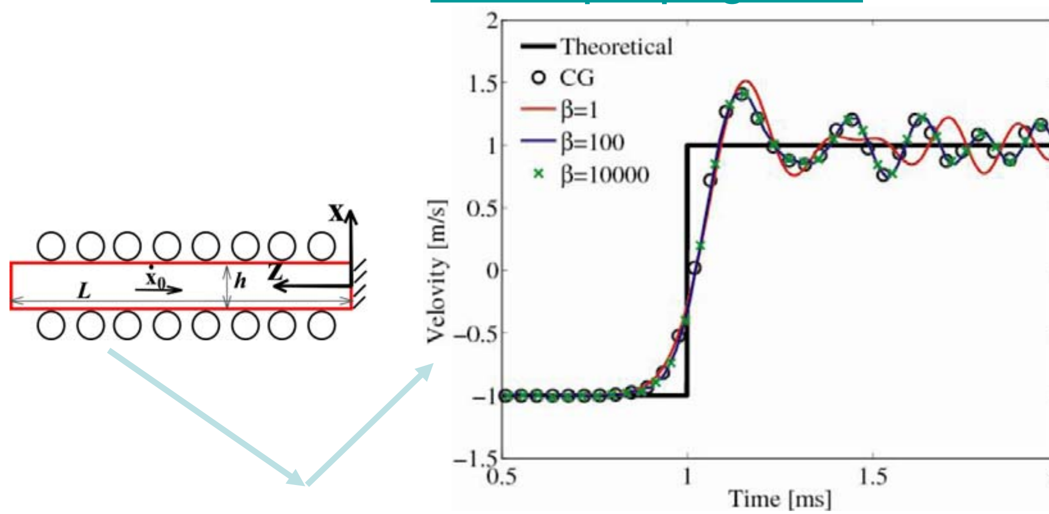
Key principles of DG methods

- Numerical applications
  - Properties for a polynomial approximation of order k
    - Consistent, stable for $\beta > C^k$, convergence in the e-norm in k
    - Explicit time integration with conditional stability $\Delta t_{\text{crit}} = \frac{h^s}{\sqrt{\beta}}\sqrt{\frac{\rho_0}{E}}$
    - High scalability
  - Examples

Taylor's impact



Wave propagation



Time evolution of the free face velocity

Key principles of DG methods

- Numerical applications
  - Application to oligo-crystal plasticity

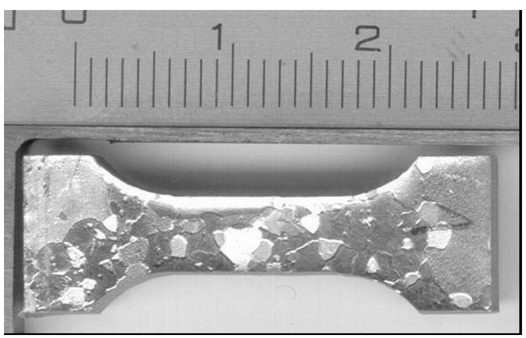
Oligo crystal sample preparation: MIT + Alcoa.

Sample cutting, polishing & Electron Backscatter Diffraction (EBSD): Caltech + MIT (Z. Zhao).

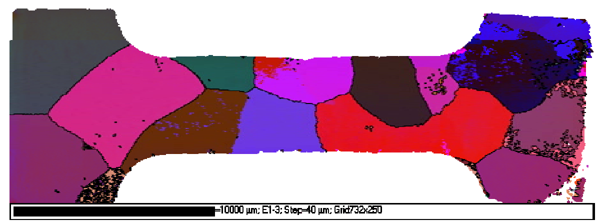
Tensile test & Digital image correlation (DIC): Rutgers (S. Kuchnicki, A. Cuitino) + MIT.

Theoretical polycrystal model: Rutgers + MIT.

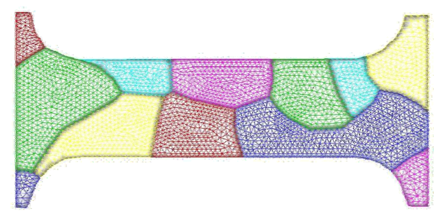
Aluminum oligo crystal sample



Dogbone tensile test sample



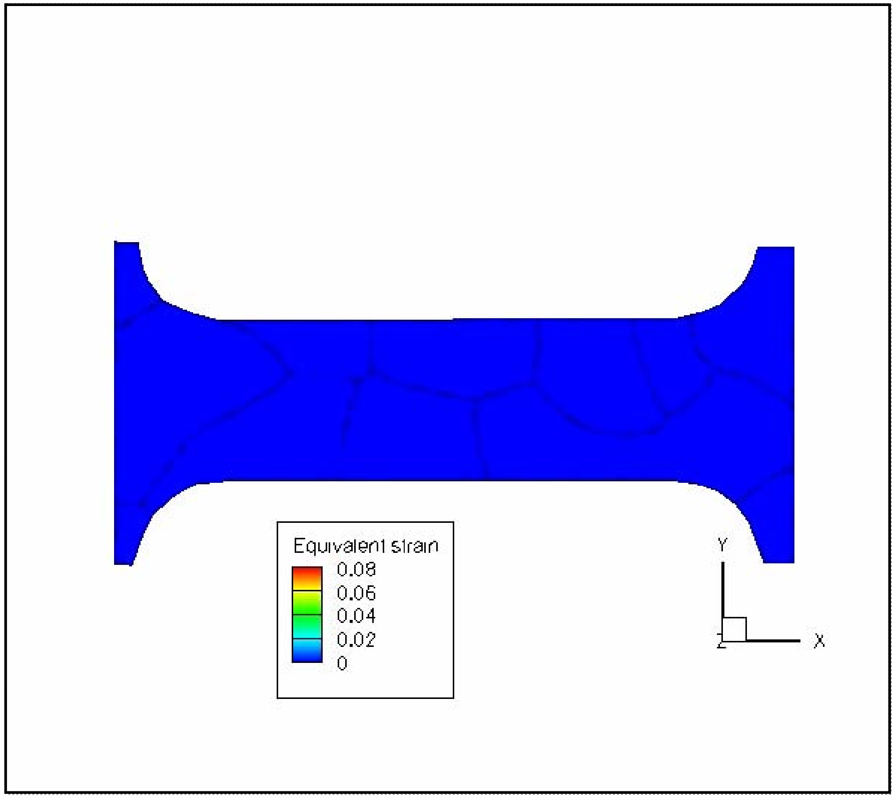
Grain profile from EBSD measurement

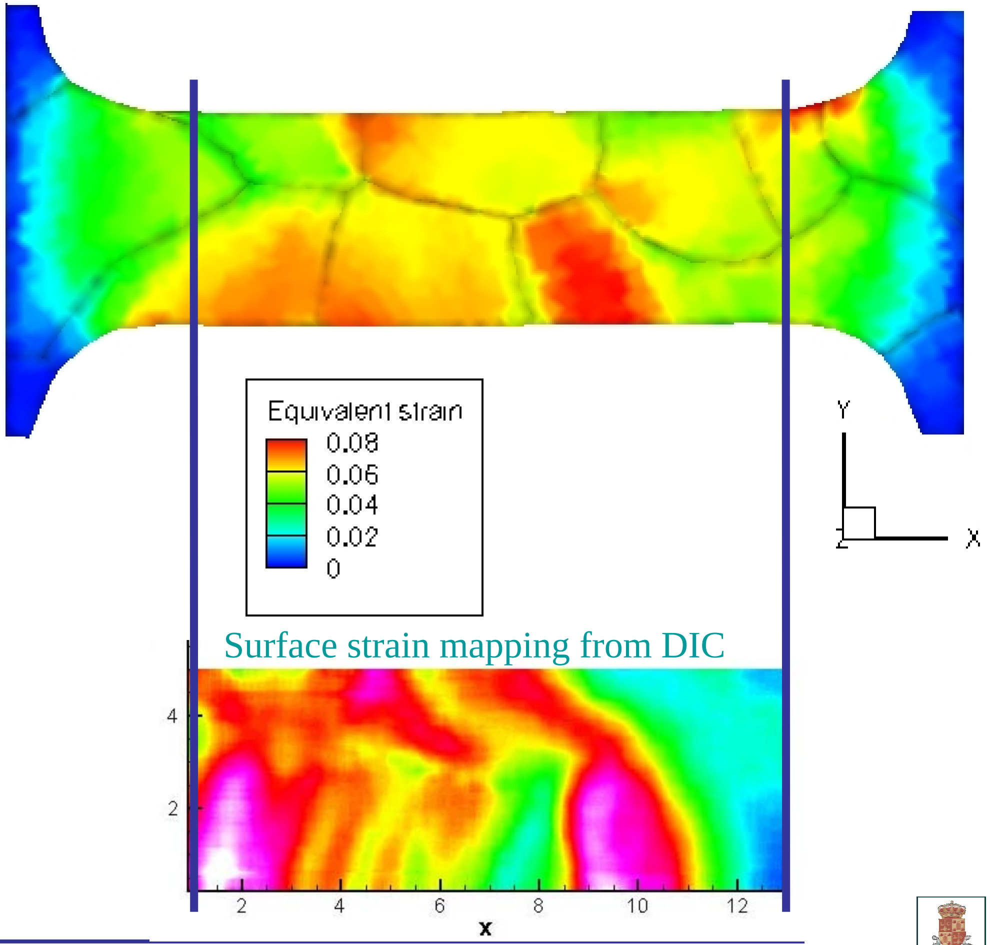


Mesh setup according to orientation mapping

Key principles of DG methods

- Numerical applications
  - Application to oligo-crystal plasticity





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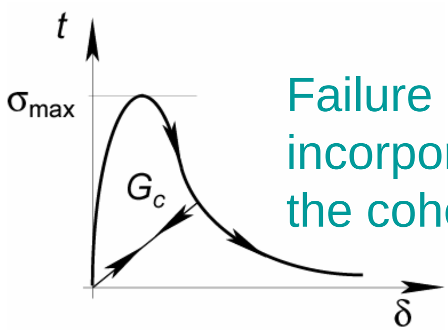
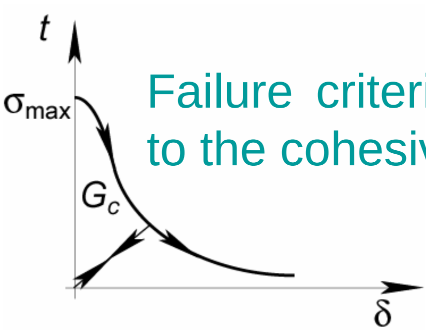
Discontinuous Mesh & Dynamic Fracture

- Dynamic fracture
  - Fracture: a gradual process of separation which occurs in small regions of material adjacent to the tip of a forming crack: the cohesive zone [Dugdale 1960, Barrenblatt 1962, …]
  - Separation is resisted to by a cohesive traction
  - 2-parameter cohesive law
    - Peak cohesive traction σ_{max} (spall strength)
    - Fracture energy G_c
    - Automatically accounts for time scale [Camacho & Ortiz, 1996]
    - Intrinsic law vs Extrinsic law



Failure criterion incorporated within the cohesive law



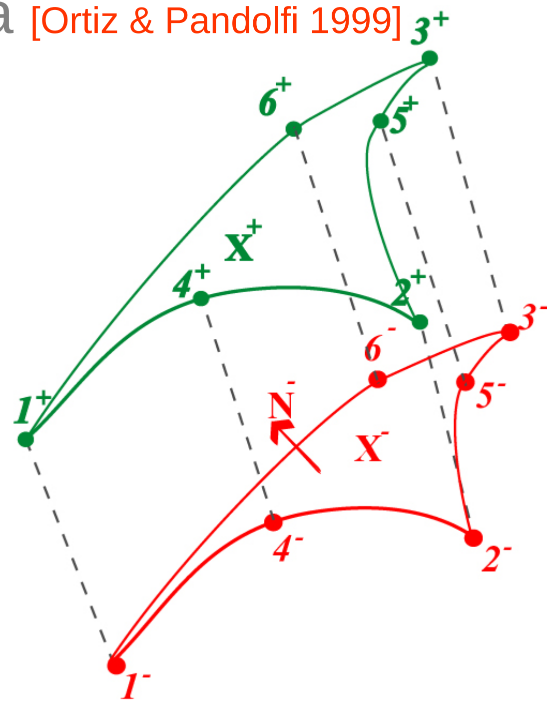
Failure criterion external to the cohesive law

Discontinuous Mesh & Dynamic Fracture

- Finite element discretization & interface elements
  - The cohesive law is integrated on an interface element inserted between two adjacent tetrahedra [Ortiz & Pandolfi 1999]
  - Potential structure of the cohesive law: [Ortiz & Pandolfi 1999]
    - Effective opening in terms of β_c the ratio between the shear and normal critical tractions:

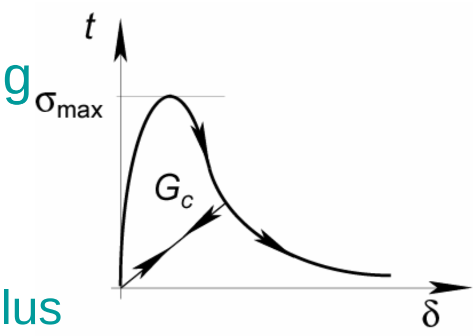
$$\delta = \sqrt{\underbrace{\left|[\![\varphi]\!]\cdot \boldsymbol{N}^-\right|^2}_{\delta_n^2 = \|\boldsymbol{\delta}_n\|^2} + \beta_c^2 \underbrace{\left\|[\![\varphi]\!] - [\![\varphi]\!]\cdot \boldsymbol{N}^- \boldsymbol{N}^-\right\|^2}_{\delta_s^2 = \|\boldsymbol{\delta}_s\|^2}}$$

    - Definition of a potential: $\phi = \phi(\delta)$
    - Interface traction: $\boldsymbol{t} = \frac{\partial \phi}{\partial \boldsymbol{\delta}} = \frac{\partial \phi}{\partial \delta_n}\boldsymbol{N}^- + \frac{\partial \phi}{\partial \delta_s}\frac{\boldsymbol{\delta_s}}{\delta_s}$

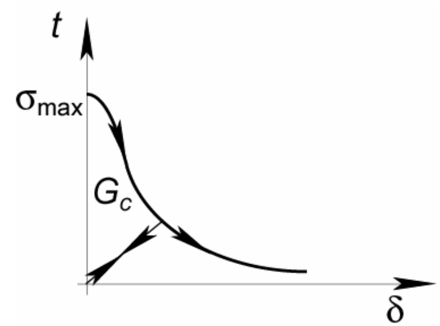


Discontinuous Mesh & Dynamic Fracture

- Two methods
  - Intrinsic Law
    - Cohesive elements inserted from the beginning
    - Drawbacks:
      - Efficient if a priori knowledge of the crack path
      - Mesh dependency [Xu & Needelman, 1994]
      - Initial slope modifies the effective elastic modulus
      - This slope should tend to infinity [Klein et al. 2001]:
        - » Alteration of a wave propagation
        - » Critical time step is reduced



  - Extrinsic Law
    - Cohesive elements inserted on the fly when failure criterion is verified [Ortiz & Pandolfi 1999]
    - Drawback
      - Complex implementation in 3D (parallelization)



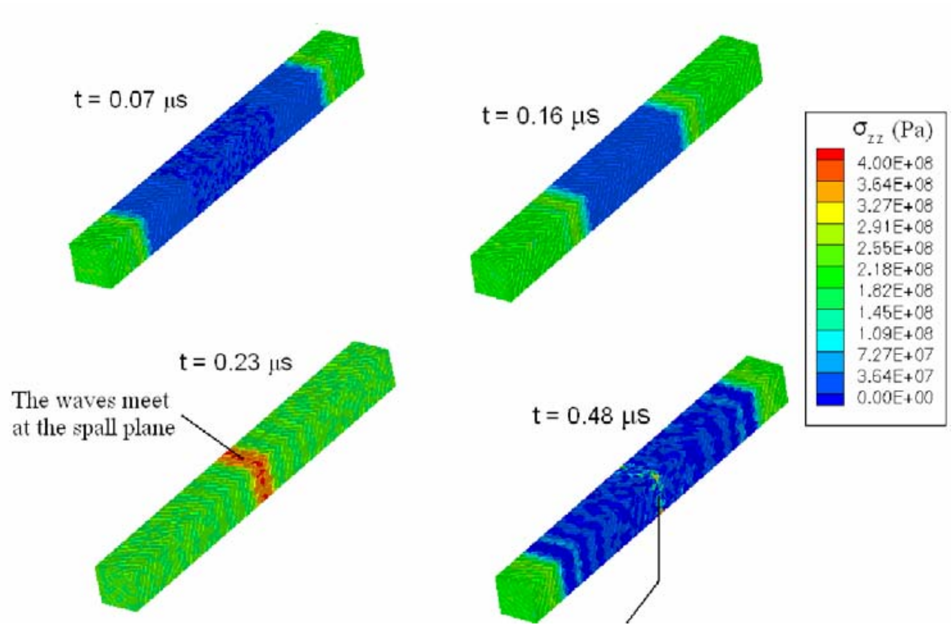
- New DG/extrinsic method [Seagraves, Jerusalem, Radovitzky, Noels]
  - Interface elements inserted from the beginning
  - Interface law corresponds initially to the DG interface forces

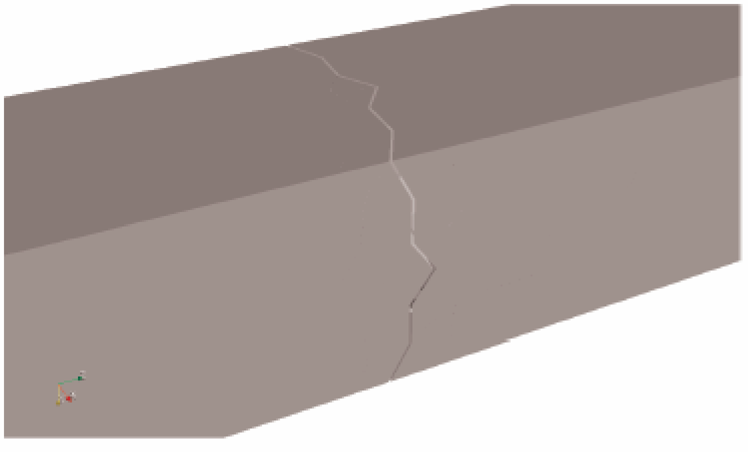
Discontinuous Mesh & Dynamic Fracture

- New DG/extrinsic method:

[Seagraves, Jerusalem, Radovitzky, Noels, col. MIT & ULg]

  - Numerical application: the spall test
    - Two opposite waves interact at the center of the specimen
    - The interaction leads to stresses higher than the spall stress
    - The specimen breaks exactly at its middle





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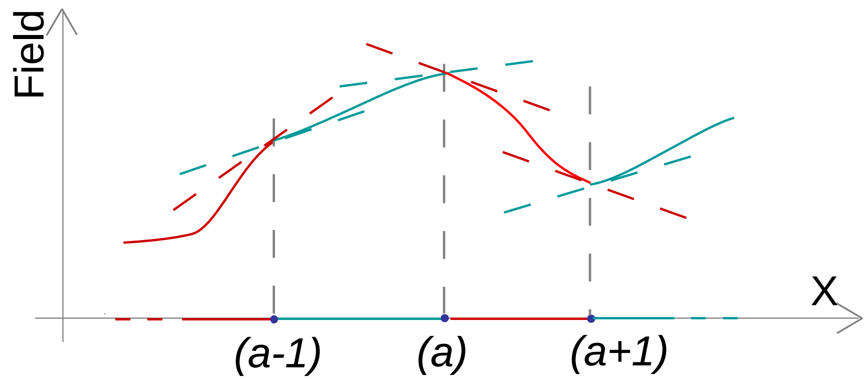


Kirchhoff-Love Shells

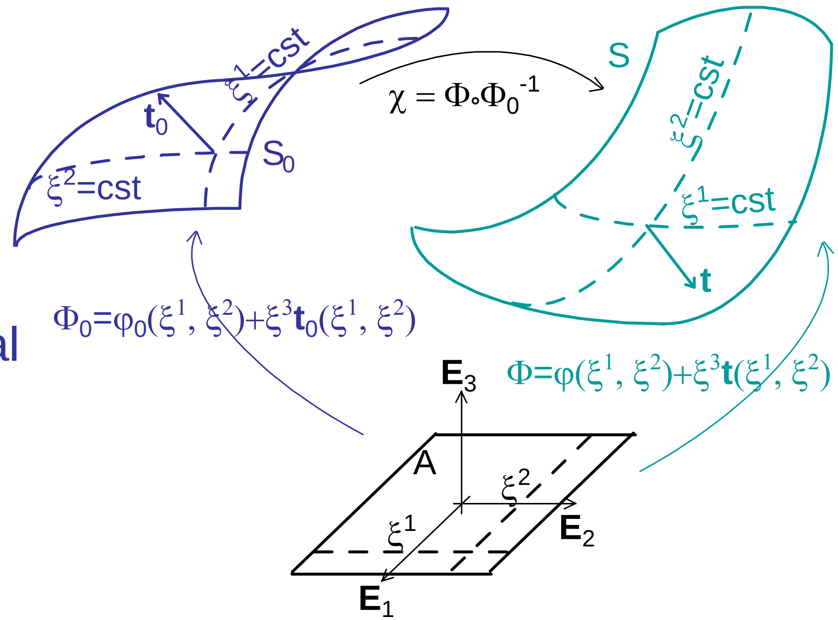
- Description of the thin body

$$\boldsymbol{x} = \boldsymbol{\Phi}\left(\xi^I\right) = \boldsymbol{\varphi}\left(\xi^\alpha\right) + \xi^3 \lambda_h \boldsymbol{t}\left(\xi^\alpha\right)$$

Mapping of the mid-surface

Thickness stretch

Mapping of the normal to the mid-surface



- Deformation mapping

$\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\Phi} \circ \left[\boldsymbol{\nabla}\boldsymbol{\Phi}_0\right]^{-1}$ with

$\boldsymbol{\nabla}\boldsymbol{\Phi} = \boldsymbol{g}_i \otimes \boldsymbol{E}^i$ & $\boldsymbol{g}_i = \boldsymbol{\nabla}\boldsymbol{\Phi}\boldsymbol{E}_i = \dfrac{\partial \boldsymbol{\Phi}}{\partial \xi^i}$

$\Longrightarrow$ $\boldsymbol{g}_\alpha = \dfrac{\partial \boldsymbol{\Phi}}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{,\alpha} + \xi^3 \lambda_h \boldsymbol{t}_{,\alpha} + \xi^3 \boldsymbol{t} \lambda_{h,\alpha}$ & $\boldsymbol{g}_3 = \dfrac{\partial \boldsymbol{\Phi}}{\partial \xi^3} = \lambda_h \boldsymbol{t}$

- Shearing is neglected

$\Longrightarrow$ $\boldsymbol{t} = \dfrac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\left\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\right\|}$ & the gradient of thickness stretch $\lambda_{h,\alpha}$ neglected

Higher order equation

Kirchhoff-Love Shells

- Resultant equilibrium equations:
  - Linear momentum $\frac{1}{\bar{j}}\left(\bar{j}\boldsymbol{n}^{\alpha}\right)_{,\alpha}+\boldsymbol{n}^{\mathcal{A}}=0$
  - Angular momentum $\frac{1}{\bar{j}}\left(\bar{j}\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha}-\boldsymbol{l}+\lambda\boldsymbol{t}+\tilde{\boldsymbol{m}}^{\mathcal{A}}=0$
  - In terms of resultant stresses:

$$\left\{\begin{aligned}\boldsymbol{n}^{\alpha} &= \frac{1}{\bar{j}}\int_{h_{\min 0}}^{h_{\max 0}}\boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\det\left(\boldsymbol{\nabla}\boldsymbol{\Phi}\right)d\xi^{3}\\ \tilde{\boldsymbol{m}}^{\alpha} &= \frac{1}{\bar{j}}\int_{h_{\min 0}}^{h_{\max 0}}\xi^{3}\boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\det\left(\boldsymbol{\nabla}\boldsymbol{\Phi}\right)d\xi^{3}\\ \boldsymbol{l} &= \frac{1}{\bar{j}}\int_{h_{\min 0}}^{h_{\max 0}}\boldsymbol{\sigma}\boldsymbol{g}^{3}\det\left(\boldsymbol{\nabla}\boldsymbol{\Phi}\right)d\xi^{3}\end{aligned}\right.$$

    of resultant applied tension $\boldsymbol{n}^{\mathcal{A}}$ and torque $\tilde{\boldsymbol{m}}^{\mathcal{A}}$

    and of the mid-surface Jacobian $\bar{j}=\left\|\boldsymbol{\varphi}_{,1}\wedge\boldsymbol{\varphi}_{,2}\right\|$

Kirchhoff-Love Shells

- Non-linear material behavior
  - Through the thickness integration by Simpson's rule
  - At each Simpson point
    - Internal energy $W(\mathbf{C}=\mathbf{F}^T\mathbf{F})$ with

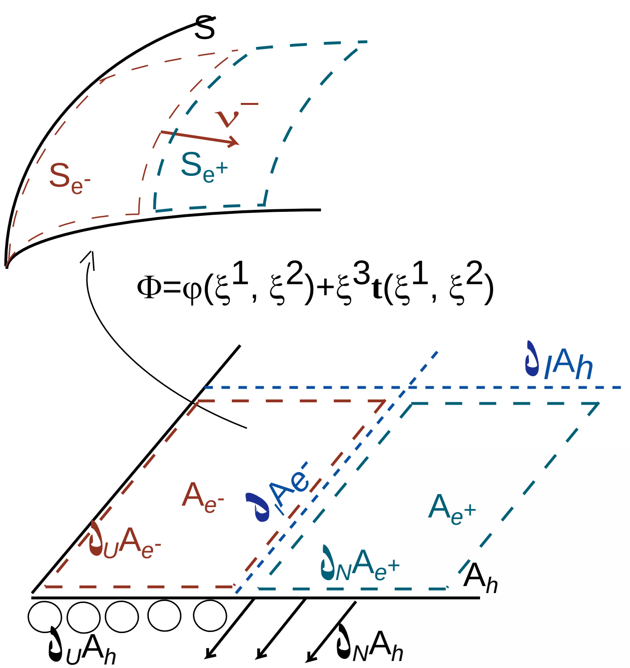
$$\begin{cases} \mathbf{C} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \, \boldsymbol{g}_0^i \otimes \boldsymbol{g}_0^j = \mathrm{g}_{ij} \, \boldsymbol{g}_0^i \otimes \boldsymbol{g}_0^j \\ \boldsymbol{\sigma} = \sigma^{ij} \, \boldsymbol{g}_i \otimes \boldsymbol{g}_j = 2 \frac{\det(\boldsymbol{\nabla \Phi}_0)}{\det(\boldsymbol{\nabla \Phi})} \frac{\partial W}{\partial \mathrm{g}_{ij}} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \end{cases}$$

    - Iteration on the thickness ratio $\lambda_h = \frac{h_{\max} - h_{\min}}{h_{\max 0} - h_{\min 0}}$ in order to reach the plane stress assumption $\sigma^{33}=0$
  - Simpson's rule leads to the resultant stresses:

$$\begin{cases} \boldsymbol{n}^\alpha = \frac{1}{\bar{j}} \int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma} \boldsymbol{g}^\alpha \det(\boldsymbol{\nabla \Phi}) \, d\xi^3 \\ \tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}} \int_{h_{\min 0}}^{h_{\max 0}} \xi^3 \boldsymbol{\sigma} \boldsymbol{g}^\alpha \det(\boldsymbol{\nabla \Phi}) \, d\xi^3 \\ \boldsymbol{l} = \frac{1}{\bar{j}} \int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma} \boldsymbol{g}^3 \det(\boldsymbol{\nabla \Phi}) \, d\xi^3 \end{cases}$$

Kirchhoff-Love Shells

- Non-linear discontinuous Galerkin formulation
  - New weak form obtained from the momentum equations
  - Integration by parts on each element $\mathcal{A}^e$
  - Across 2 elements $\delta \boldsymbol{t}$ is discontinuous



$$0 = \int_{\mathcal{A}_e} \left(\bar{j}\boldsymbol{n}^\alpha(\boldsymbol{\varphi}_h)\right)_{,\alpha}\cdot\delta\boldsymbol{\varphi} d\mathcal{A} + \int_{\mathcal{A}_e} \boldsymbol{n}^{\mathcal{A}}\cdot\delta\boldsymbol{\varphi}\bar{j}d\mathcal{A} + \int_{\mathcal{A}_e}\left[\left(\bar{j}\tilde{\boldsymbol{m}}^\alpha(\boldsymbol{\varphi}_h)\right)_{,\alpha} - \bar{j}\boldsymbol{l}\right]\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_e}\tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\delta\boldsymbol{t}\lambda_h\bar{j}d\mathcal{A}$$

$$-\sum_e\int_{\bar{\mathcal{A}}_e}\bar{j}\tilde{\boldsymbol{m}}^\alpha(\boldsymbol{\varphi}_h)\cdot(\delta\boldsymbol{t}\lambda_h)_{,\alpha}d\mathcal{A} + \sum_e\int_{\partial\mathcal{A}_e}\bar{j}\tilde{\boldsymbol{m}}^\alpha(\boldsymbol{\varphi}_h)\cdot\delta\boldsymbol{t}\lambda_h\nu_\alpha d\mathcal{A}$$

$$\int_{\mathcal{A}_h}\bar{j}\boldsymbol{n}^\alpha(\boldsymbol{\varphi}_h)\cdot\delta\boldsymbol{\varphi}_{,\alpha}d\mathcal{A} + \int_{\mathcal{A}_h}\bar{j}\boldsymbol{l}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h}\bar{j}\tilde{\boldsymbol{m}}^\alpha(\boldsymbol{\varphi}_h)\cdot(\delta\boldsymbol{t}\lambda_h)_{,\alpha}d\mathcal{A} + \int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h}[\![\delta\boldsymbol{t}\cdot\bar{j}\lambda_h\tilde{\boldsymbol{m}}^\alpha]\!]\nu_\alpha^- d\partial\mathcal{A} = \int_{\partial_N\mathcal{A}_h}\bar{j}\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{\varphi}d\mathcal{A} + \int_{\partial_M\mathcal{A}_h}\bar{j}\bar{\tilde{\boldsymbol{m}}}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h}\boldsymbol{n}^{\mathcal{A}}\cdot\delta\boldsymbol{\varphi}\bar{j}d\mathcal{A} + \int_{\mathcal{A}_h}\tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\delta\boldsymbol{t}\lambda_h\bar{j}d\mathcal{A}$$

Kirchhoff-Love Shells

- Interface terms rewritten as the sum of 3 terms
  - Introduction of the numerical flux $\boldsymbol{h}$

$$\int_{\partial_I \mathcal{A}_h} [\![\bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h)\cdot\delta\boldsymbol{t}\lambda_h]\!]\,\nu_{\alpha}^{-}\,d\mathcal{A} \rightarrow \int_{\partial_I \mathcal{A}_h} [\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{h}\left((\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha})^{+},\,(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha})^{-},\,\nu_{\alpha}^{-}\right)d\mathcal{A}$$

    - Has to be consistent: $\boldsymbol{h}\left(\lambda_h\bar{j}\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}},\,\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}},\,\nu_{\alpha}\right)=\lambda_h\bar{j}\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}}\nu_{\alpha}^{-}$
    - One possible choice: $\boldsymbol{h}\left((\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha})^{+},\,(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha})^{-},\,\nu_{\alpha}^{-}\right)=\nu_{\alpha}^{-}\left\langle\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right\rangle$
  - Weak enforcement of the compatibility

$$\int_{\partial_I \mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\left\langle\delta\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)\right\rangle\nu_{\alpha}^{-}\,d\partial\mathcal{A}$$

Linearization leads to the material tangent modulii $\mathcal{H}_m$

$$\int_{\partial_I \mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\left\langle\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\delta\boldsymbol{\varphi}_{,\gamma}\cdot\boldsymbol{t}_{,\delta}+\boldsymbol{\varphi}_{,\gamma}\cdot\delta\boldsymbol{t}_{,\delta}\right)\boldsymbol{\varphi}_{,\beta}+\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\cdot\boldsymbol{\varphi}_{,\beta}\;\;\delta\boldsymbol{\varphi}_{,\beta}\right\rangle\nu_{\alpha}^{-}\,d\partial\mathcal{A}$$

  - Stabilization controlled by parameter β, for all mesh sizes h^s

$$\int_{\partial_I \mathcal{A}_h\cup\partial_T\mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\boldsymbol{\varphi}_{,\beta}\left\langle\frac{\beta\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}}{h^s}\right\rangle[\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{\varphi}_{,\gamma}\nu_{\alpha}^{-}\nu_{\delta}^{-}\,d\partial\mathcal{A}$$

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Kirchhoff-Love Shells

- New weak formulation

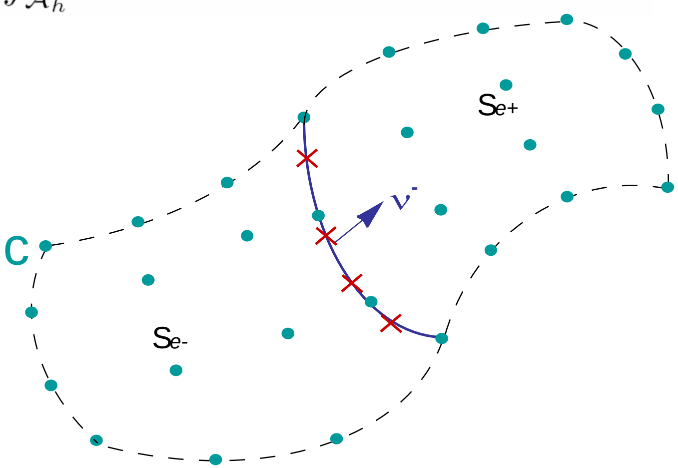
$$\int_{\mathcal{A}_h} \bar{j}\boldsymbol{n}^{\alpha}(\boldsymbol{\varphi}_h)\cdot\delta\boldsymbol{\varphi}_{,\alpha}d\mathcal{A} + \int_{\mathcal{A}_h} \bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h)\cdot(\delta\boldsymbol{t}\lambda_h)_{,\alpha}d\mathcal{A} + \int_{\mathcal{A}_h}\bar{j}\boldsymbol{l}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} +$$

$$\int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\left\langle \bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}(\delta\boldsymbol{\varphi}_{,\gamma}\cdot\boldsymbol{t}_{,\delta}+\boldsymbol{\varphi}_{,\gamma}\cdot\delta\boldsymbol{t}_{,\delta})\boldsymbol{\varphi}_{,\beta}+\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\cdot\boldsymbol{\varphi}_{,\beta}\ \delta\boldsymbol{\varphi}_{,\beta}\right\rangle \nu_{\alpha}^{-}d\partial\mathcal{A}$$

$$\int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h} [\![\delta\boldsymbol{t}]\!]\cdot\langle\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\rangle\nu_{\alpha}^{-}d\partial\mathcal{A} + \int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h}[\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\boldsymbol{\varphi}_{,\beta}\left\langle\frac{\beta\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}}{h^s}\right\rangle[\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{\varphi}_{,\gamma}\nu_{\alpha}^{-}\nu_{\delta}^{-}d\partial\mathcal{A} =$$

$$\int_{\partial_N\mathcal{A}_h}\bar{j}\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{\varphi}d\mathcal{A} + \int_{\partial_M\mathcal{A}_h}\bar{j}\bar{\tilde{\boldsymbol{m}}}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h}\boldsymbol{n}^{\mathcal{A}}\cdot\delta\boldsymbol{\varphi}\bar{j}d\mathcal{A} + \int_{\mathcal{A}_h}\tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\delta\boldsymbol{t}\lambda_h\bar{j}d\mathcal{A}$$

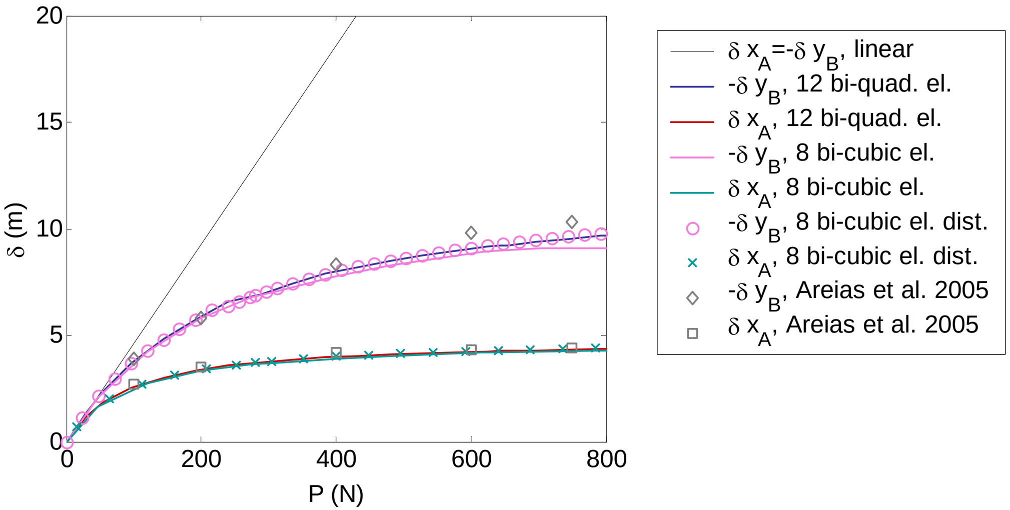
- Implementation
  - Shell elements
    - Membrane and bending responses
    - 2x2 (4x4) Gauss points for bi-quadratic (bi-cubic) quadrangles
  - Interface elements
    - 3 contributions
    - 2 (4) Gauss points for quadratic (cubic) meshes
    - Contributions of neighboring shells evaluated at these points

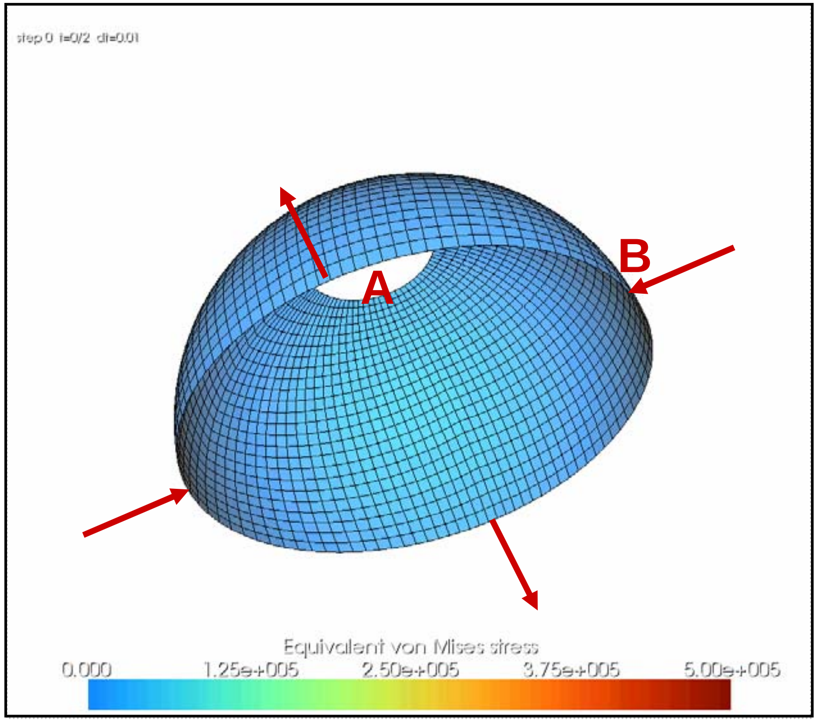


Kirchhoff-Love Shells

- Pinched open hemisphere
  - Properties:
    - 18-degree hole
    - Thickness 0.04 m; Radius 10 m
    - Young 68.25 MPa; Poisson 0.3
  - Comparison of the DG methods
    - Quadratic, cubic & distorted el.

    with literature





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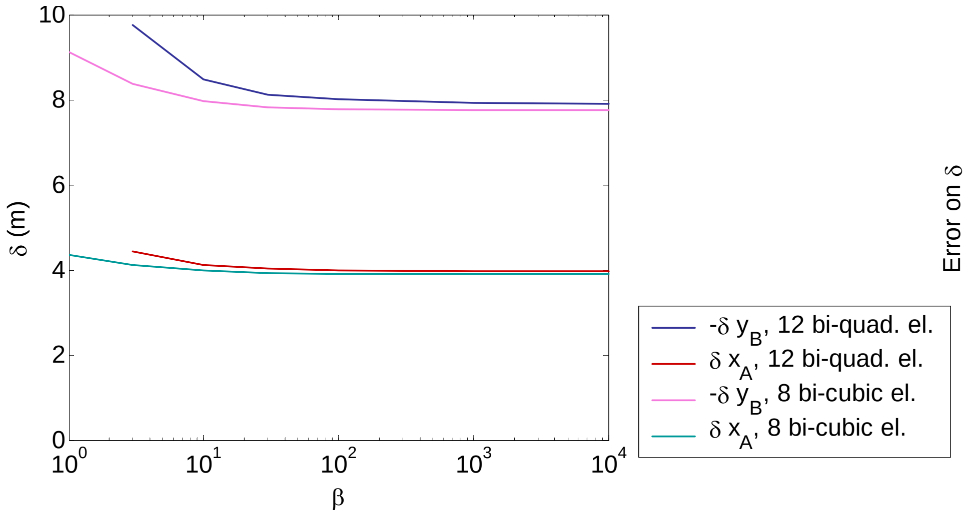
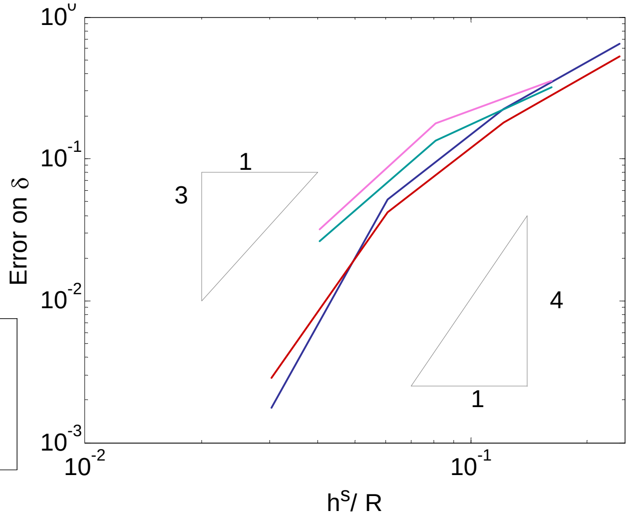
Kirchhoff-Love Shells

- Pinched open hemisphere

Influence of the stabilization parameter



Influence of the mesh size

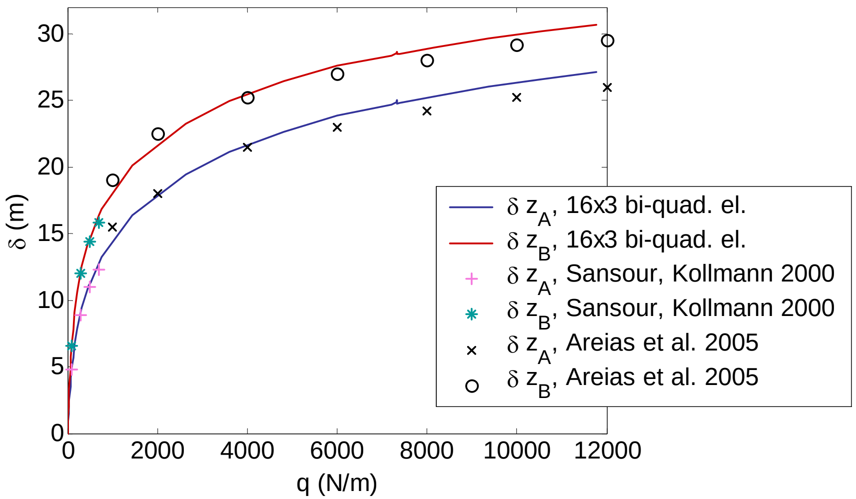


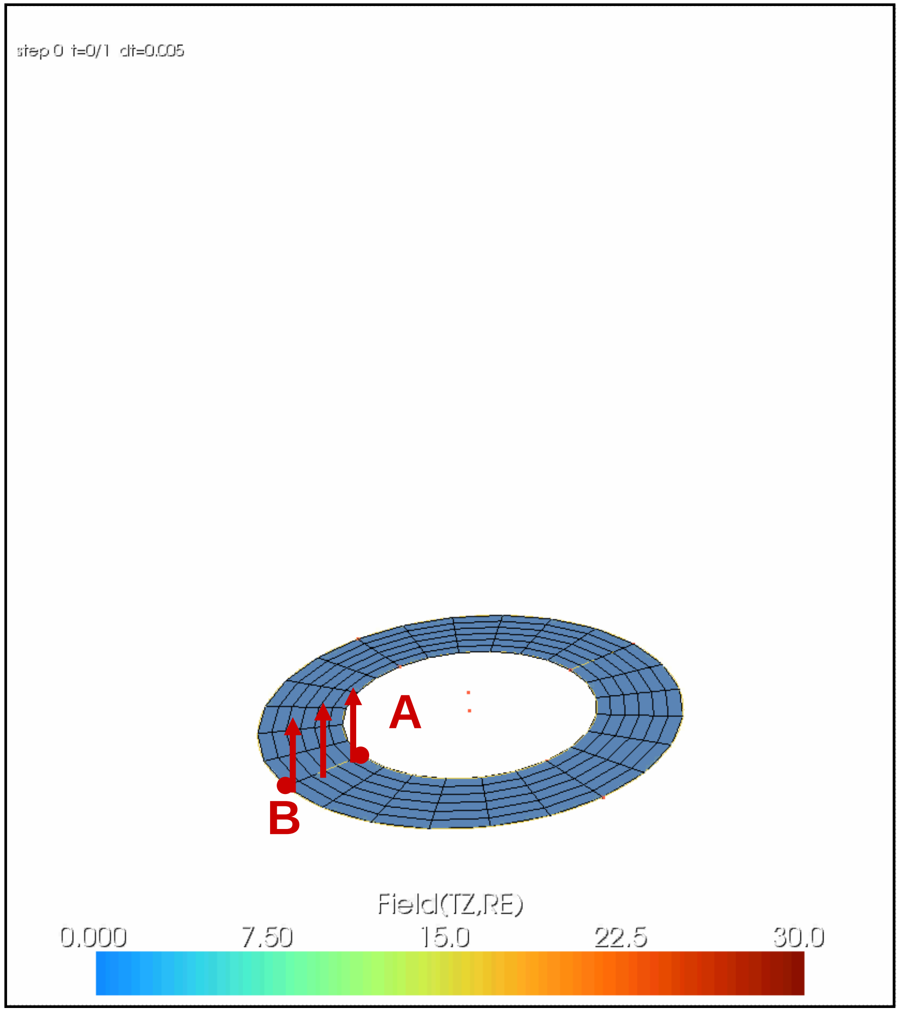
  - Stability if $\beta > 10$
  - Order of convergence in the L^2-norm in k+1

Kirchhoff-Love Shells

- Plate ring
  - Properties:
    - Radii 6 -10 m
    - Thickness 0.03 m
    - Young 12 GPa; Poisson 0
  - Comparison of DG methods
    - Quadratic elements

  with literature

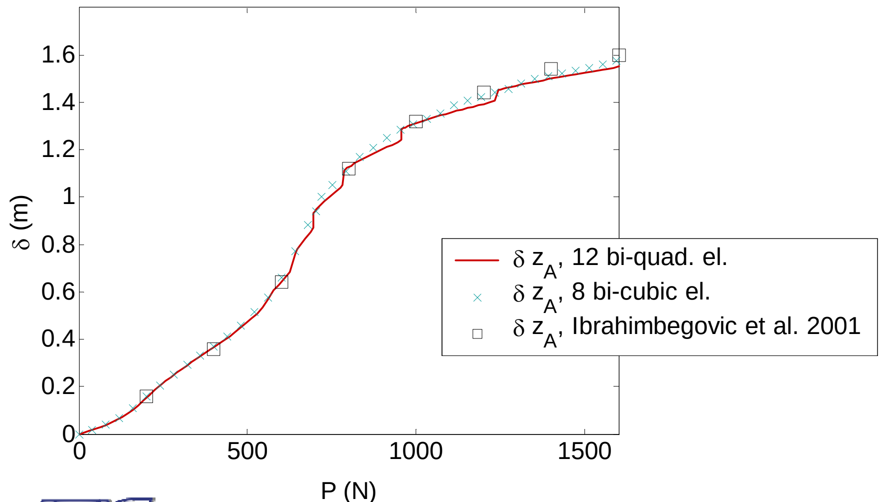


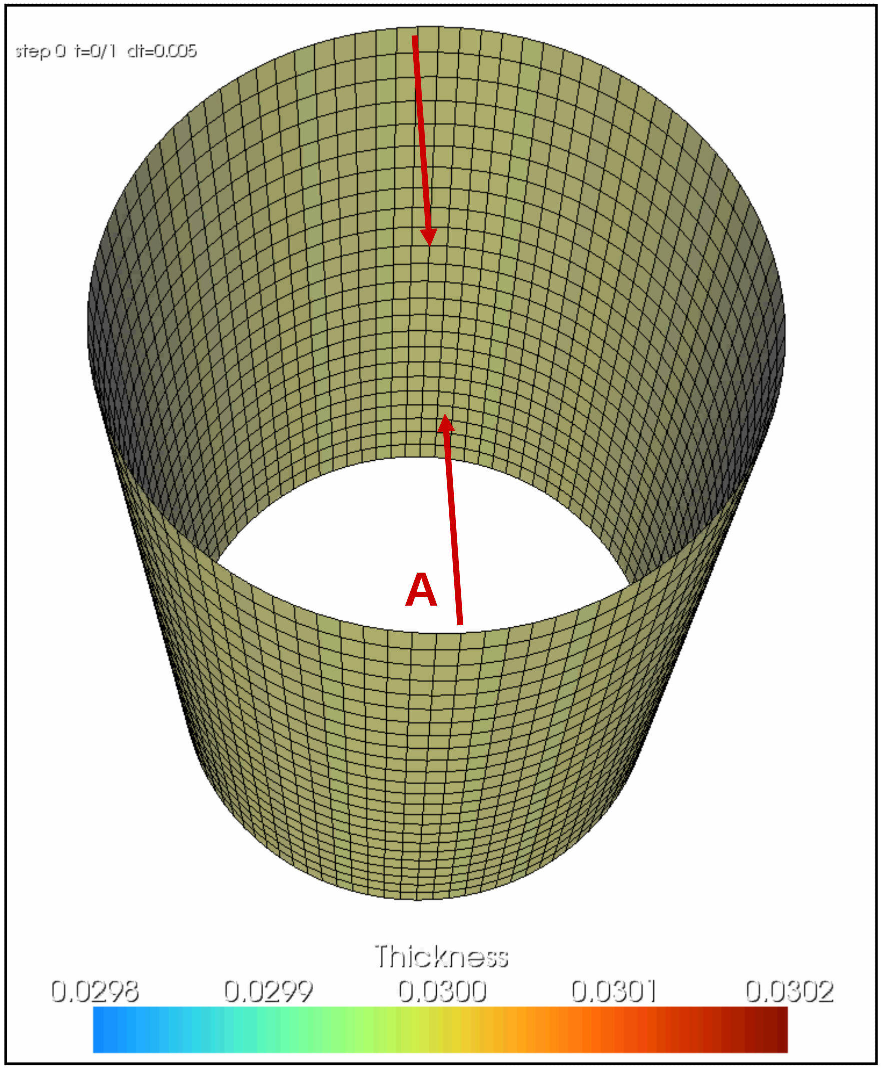


Kirchhoff-Love Shells

- Clamped cylinder
  - Properties:
    - Radius 1.016 m; Length 3.048 m; Thickness 0.03 m
    - Young 20.685 MPa; Poisson 0.3
  - Comparison of DG methods
    - Quadratic & cubic elements

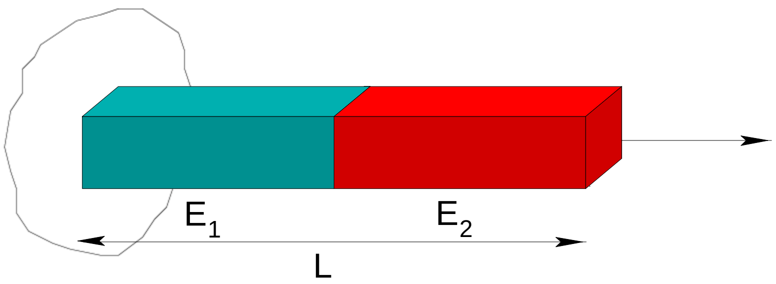
    with literature

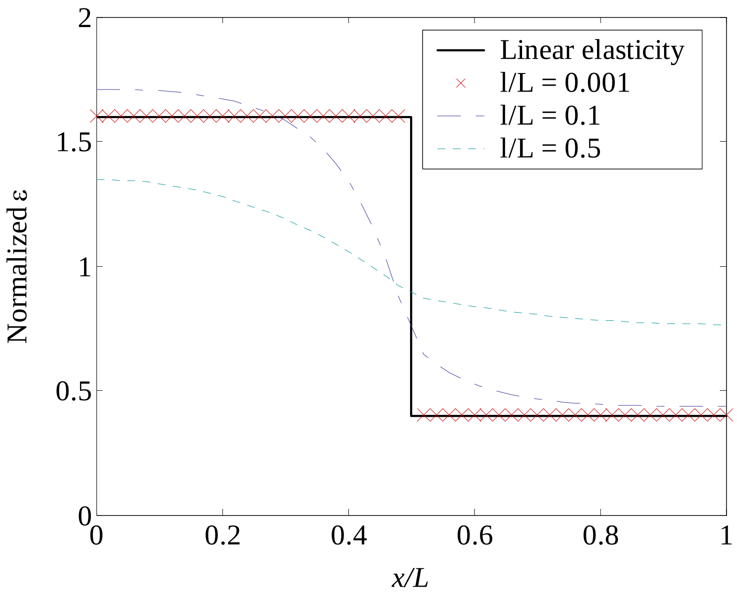




Strain-gradient elasticity

- Strain-gradient effect
  - Length scales in modern technology are now of the order of the micrometer or nanometer
  - At these scales, material laws depend on the strain but also on the strain-gradient
  - Example:
    - Bi-material tensile test:





    - $E_1/E_2 = 4$
    - Characteristic length l
    - Differential equation: $E\left(u_{,xx} - l^2 u_{,xxxx}\right) = 0$

Strain-gradient elasticity

- Strain-gradient theory of linear elasticity
  - At a material point stress is a function of the strain and of the gradient of strain [Toupin 1962, Mindlin 1964]:
    - Strain energy $W = W(\boldsymbol{\varepsilon}, \mathbb{E})$ depends on the strain and its gradient
    - Low and high order stresses introduced as the work conjugate of low and high order strains: $\boldsymbol{\sigma} = \dfrac{\partial W}{\partial \boldsymbol{\varepsilon}} = \mathcal{H} : \boldsymbol{\varepsilon}$ and $\mathbb{K} = \dfrac{\partial W}{\partial \mathbb{E}} = \mathcal{J} \,\vdots\, \mathbb{E}$
  - Governing PDE obtained from the virtual work statement:

$$\int_B \left(\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} + \mathbb{K} \,\vdots\, \delta\mathbb{E} \right) dB = \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta\boldsymbol{u} \; d\partial B + \int_{\partial_M B} \boldsymbol{r} \cdot \underbrace{\boldsymbol{\nabla}\delta\boldsymbol{u} \cdot \boldsymbol{n}}_{D\delta\boldsymbol{u}} \; d\partial B$$

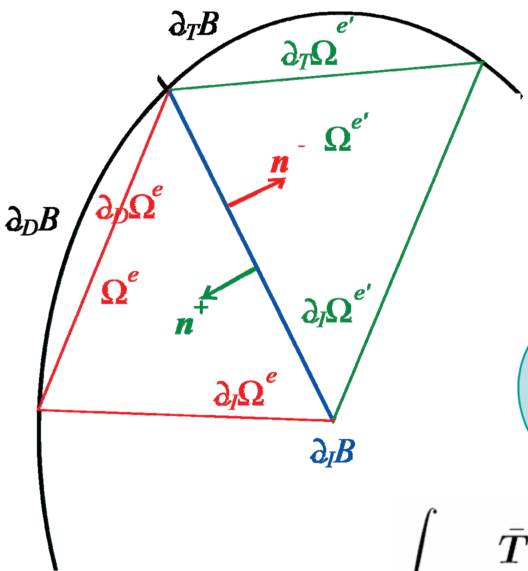
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} - \boldsymbol{\nabla} \cdot \mathbb{K}) = 0 \;\text{ in }\; B$$

&

$$\begin{cases} (\boldsymbol{n} \otimes \boldsymbol{n}) : \mathbb{K} = \boldsymbol{r} \;\text{ in }\; \partial_M B \\ D\boldsymbol{u} = \overline{D\boldsymbol{u}} \;\text{ in }\; \partial_T B \end{cases} \quad \& \quad \begin{cases} \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\nabla} \cdot \mathbb{K}) + \left(\boldsymbol{n} \overset{s}{\boldsymbol{\nabla}} \cdot \boldsymbol{n} - \overset{s}{\boldsymbol{\nabla}} \right) \cdot (\boldsymbol{n} \cdot \mathbb{K}) = \bar{\boldsymbol{T}} \;\text{ in }\; \partial_N B \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \;\text{ in }\; \partial_D B \end{cases}$$

Strain-gradient elasticity

- Discontinuous Galerkin formulation for strain-gradient theory
  - Test functions $\boldsymbol{u}_h$ and trial functions $\delta\boldsymbol{u}$ are $\mathcal{C}^0$
  - New weak formulation obtained by repeated integrations by parts on each element Ω^e :



$$\sum_e \int_{\Omega^e} \boldsymbol{\nabla} \cdot \left[\boldsymbol{\sigma}\left(\boldsymbol{u}_h\right) - \boldsymbol{\nabla} \cdot \mathbb{K}\left(\boldsymbol{u}_h\right)\right] \cdot \delta\boldsymbol{u}\; dB = 0$$

$$\sum_e \int_{\partial\mathcal{A}_e} \boldsymbol{n} \cdot \left[\boldsymbol{\sigma}\left(\boldsymbol{u}_h\right) - \boldsymbol{\nabla} \cdot \mathbb{K}\left(\boldsymbol{u}_h\right)\right] \cdot \delta\boldsymbol{u}\; d\partial B + \sum_e \int_{\partial\mathcal{A}_e} \boldsymbol{n} \cdot \mathbb{K}\left(\boldsymbol{u}_h\right) : \boldsymbol{\nabla}\delta\boldsymbol{u}\; d\partial B -$$

$$\sum_e \int_{\Omega^e} \boldsymbol{\sigma}\left(\boldsymbol{u}_h\right) : \boldsymbol{\varepsilon}\left(\delta\boldsymbol{u}\right)\; dB - \sum_e \int_{\Omega^e} \mathbb{K}\left(\boldsymbol{u}_h\right) \vdots \mathbb{E}\left(\delta\boldsymbol{u}\right)\; dB = 0$$

$$\int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta\boldsymbol{u} d\partial B + \int_{\partial_T B} \left(\boldsymbol{n} \otimes \boldsymbol{n}\right) : \mathbb{K}\left(\boldsymbol{u}_h\right) \cdot D\delta\boldsymbol{u}\; d\partial B - \int_{\partial_I B} \boldsymbol{n}^- \cdot [\![\mathbb{K} : \boldsymbol{\nabla}\delta\boldsymbol{u}]\!]\; d\partial B +$$

$$\int_{\partial_M B} \boldsymbol{r} \cdot D\delta\boldsymbol{u}\; d\partial B - \int_B \boldsymbol{\sigma}\left(\boldsymbol{u}_h\right) : \boldsymbol{\varepsilon}\left(\delta\boldsymbol{u}\right)\; dB - \int_B \mathbb{K}\left(\boldsymbol{u}_h\right) \vdots \mathbb{E}\left(\delta\boldsymbol{u}\right)\; dB = 0$$

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Strain-gradient elasticity

- Interface term rewritten as the sum of 3 terms
  - Introduction of the numerical flux $\boldsymbol{h}$

$$\int_{\partial_I B} \boldsymbol{n}^- \cdot [\![\mathbb{K}(\boldsymbol{u}_h) : \boldsymbol{\nabla}\delta\boldsymbol{u}]\!] \, d\partial B \rightarrow \int_{\partial_I B} [\![\boldsymbol{\nabla}\delta\boldsymbol{u}]\!] : \mathbf{h}\left(\mathbb{K}^+, \mathbb{K}^-, \boldsymbol{n}^-\right) \, d\partial B$$

    - Has to be consistent: $\begin{cases} \mathbf{h}\left(\mathbb{K}^+, \mathbb{K}^-, \boldsymbol{n}^-\right) = -\mathbf{h}\left(\mathbb{K}^-, \mathbb{K}^+, \boldsymbol{n}^+\right) \\ \mathbf{h}\left(\mathbb{K}_{\text{exact}}, \mathbb{K}_{\text{exact}}, \boldsymbol{n}\right) = \boldsymbol{n} \cdot \mathbb{K}_{\text{exact}} \end{cases}$
    - One possible choice: $\mathbf{h}\left(\mathbb{K}^+, \mathbb{K}^-, \boldsymbol{n}^-\right) = \boldsymbol{n}^- \cdot \langle \mathbb{K} \rangle$
  - Weak enforcement of the compatibility

$$\int_{\partial_I B} \left(\boldsymbol{n}^- \otimes [\![\boldsymbol{\nabla}\boldsymbol{u}_h]\!]\right) \vdots \left\langle \mathcal{J} \vdots \left(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \otimes \delta\boldsymbol{u}\right) \right\rangle \, d\partial B$$

  - Stabilization controlled by parameter β, for all mesh sizes h^s

$$\int_{\partial_I B} \left(\boldsymbol{n}^- \otimes [\![\boldsymbol{\nabla}\boldsymbol{u}_h]\!]\right) \vdots \left\langle \frac{\beta \mathcal{J}}{h^s} \right\rangle \vdots \left(\boldsymbol{n}^- \otimes [\![\boldsymbol{\nabla}\delta\boldsymbol{u}]\!]\right) \, d\partial B$$

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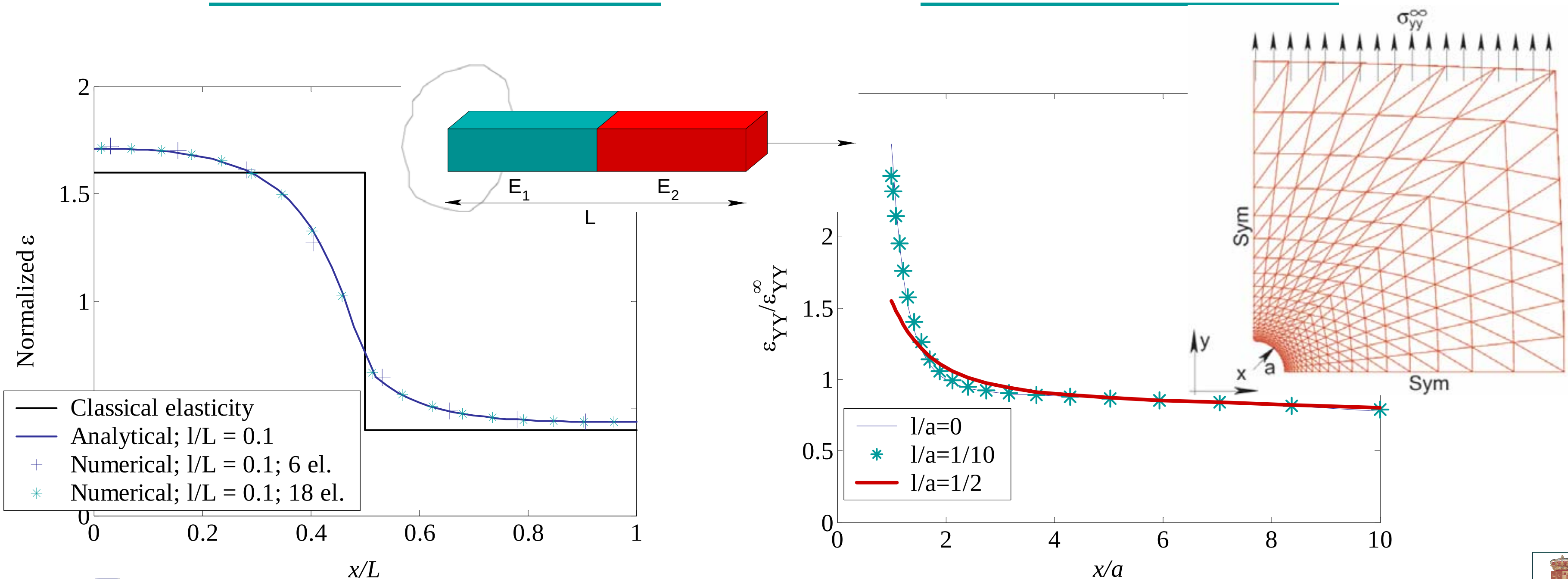
Strain-gradient elasticity

- Numerical applications
  - Properties for polynomial approximation of order *k*
    - Consistent, stable for $\beta > C^k$
    - Convergence in the e-norm in k-1, but in k+1 in the L^2-norm
  - Examples

Bi-material tensile test

Stress concentration



Conclusions & Perspectives

- Development of a discontinuous Galerkin formulation
  - A formulation has been proposed for non-linear dynamics
  - Application to high-order differential equations
    - Strain gradient elasticity
    - Shells
  - Application to dynamic fracture
- Works in progress
  - Fracture of thin structures
  - Fracture of composite structures (RVE)