An energy momentum conserving algorithm using the incremental potential for visco-plasticity

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7th World Congress on Computational Mechanics
Los Angeles, California, USA, July 16-22, 2006

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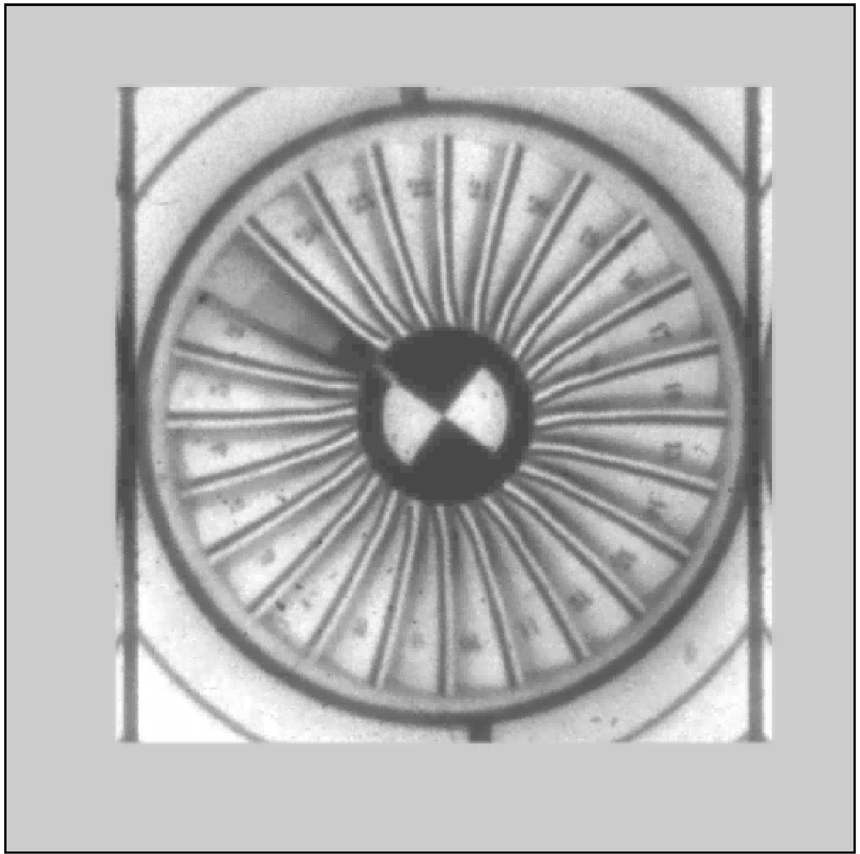
Introduction
Industrial problems

- Industrial context:
  - Structures must be able to resist to crash situations
  - Numerical simulations by finite elements is a key to design structures
  - Large deformations, plasticity,…
  - Efficient time integration in the non-linear range is needed
- Goal:
  - Numerical simulation of blade off and wind-milling in a turboengine
  - Example from SNECMA



Scope of the presentation

1. Scientific motivations
2. Conserving scheme in the non-linear range
3. New formulation using the variational update approach
4. Numerical examples
5. Conclusions

1. Scientific motivations
Dynamic simulations

- Spatial discretization into finite elements
- Temporal integration of the balance equations: $M\,\ddot{\vec{x}} + \vec{F}^{\text{int}} = \vec{F}^{\text{ext}}$
- 2 methods:
  - Explicit method

$$\left.\begin{matrix}\vec{x}_n\\ \dot{\vec{x}}_n\\ \ddot{\vec{x}}_n\end{matrix}\right\}\xrightarrow{\text{approximation}}\ddot{\vec{x}}_{n+1}=\mathbf{M}^{-1}\left(\vec{F}_n^{\text{ext}}-\vec{F}_n^{\text{int}}\right)\xrightarrow{\text{deduction}}\vec{x}_{n+1},\dot{\vec{x}}_{n+1}$$

    - Non iterative
    - Small memory requirement
    - Conditionally stable (small time step)

    } Very fast dynamics

  - Implicit method

$$\left.\begin{matrix}\vec{x}_n\\ \dot{\vec{x}}_n\\ \ddot{\vec{x}}_n\end{matrix}\right\}\xrightarrow{\text{extrapolation}}\left\{\begin{matrix}\vec{x}_{n+1}\\ \dot{\vec{x}}_{n+1}\\ \ddot{\vec{x}}_{n+1}\end{matrix}\right\}\xleftrightarrow{\text{iterations}}\left\{\begin{matrix}\mathbf{M}\ddot{\vec{x}}+\vec{F}^{\text{int}}=\vec{F}^{\text{ext}}\\ \vec{x}_{n+1}=f\left(\vec{x}_n,\dot{\vec{x}}_n,\dot{\vec{x}}_{n+1},\ddot{\vec{x}}_n,\ddot{\vec{x}}_{n+1}\right)\\ \dot{\vec{x}}_{n+1}=f\left(\dot{\vec{x}}_n,\vec{x}_n,\vec{x}_{n+1},\ddot{\vec{x}}_n,\ddot{\vec{x}}_{n+1}\right)\end{matrix}\right.$$

    - Iterative
    - Larger memory requirement
    - Unconditionally stable (large time step)

    } Slower dynamics

1. Scientific motivations
Implicit algorithm: our opinion

- If wave propagation effects are negligible

  → Implicit schemes are more suitable

  – Sheet metal forming (springback, superplastic forming, …)
  – Crashworthiness simulations (car crash, blade loss, shock absorber, …)
- Nowadays, people choose explicit scheme mainly because of difficulties linked to implicit scheme:
  – Lack of smoothness (contact, elasto-plasticity, …)

    → convergence can be difficult

  – Lack of available methods (commercial codes)
- Little room for improvement in explicit methods
- Complex problems can take advantage of combining explicit and implicit algorithms
- Necessity of developing robust and accurate implicit schemes

1. Scientific motivations
Conservation laws

- Conservation of linear momentum (Newton's law)
  - Continuous dynamics $\frac{\partial \mathbf{M}\dot{\vec{x}}}{\partial t} = \vec{F}^{\text{ext}}$
  - Time discretization $\sum_{nodes} \mathbf{M}\dot{\vec{x}}_{n+1} - \mathbf{M}\dot{\vec{x}}_n = \Delta t \sum_{nodes} \vec{F}^{\text{ext}}_{n+1/2}$ & $\sum_{nodes} \vec{F}^{\text{int}}_{n+1/2} = 0$
- Conservation of angular momentum
  - Continuous dynamics $\frac{\partial \vec{x} \wedge \mathbf{M}\dot{\vec{x}}}{\partial t} = \vec{x} \wedge \vec{F}^{\text{ext}}$
  - Time discretization $\sum_{nodes} \sum_{nodes} \vec{x}_{n+1} \wedge \mathbf{M}\dot{\vec{x}}_{n+1} - \vec{x}_n \wedge \mathbf{M}\dot{\vec{x}}_n = \Delta t \sum_{nodes} \vec{x}_{n+1/2} \wedge \vec{F}^{\text{ext}}_{n+1/2}$

    & $\sum_{nodes} \vec{x}_{n+1/2} \wedge \vec{F}^{\text{int}}_{n+1/2} = 0$
- Conservation of energy

  W^{int}: internal energy;
  W^{ext}: external energy;
  D^{int}: dissipation (plasticity …)

  - Continuous dynamics $\frac{\partial}{\partial t} K + \frac{\partial}{\partial t} W^{\text{int}} = \frac{\partial}{\partial t} W^{\text{ext}} - \dot{D}^{\text{int}}$
  - Time discretization $W^{\text{int}}_{n+1} - W^{\text{int}}_n + \Delta D^{\text{int}} = \sum_{nodes} \vec{F}^{\text{int}}_{n+1/2} \bullet [\vec{x}_{n+1} - \vec{x}_n]$ &

$$\sum_{nodes} \frac{1}{2}\mathbf{M}\dot{\vec{x}}_{n+1} \bullet \dot{\vec{x}}_{n+1} - \frac{1}{2}\mathbf{M}\dot{\vec{x}}_n \bullet \dot{\vec{x}}_n + W^{\text{int}}_{n+1} - W^{\text{int}}_n + \Delta D^{\text{int}} = \sum_{nodes} \vec{F}^{\text{ext}}_{n+1/2} \bullet [\vec{x}_{n+1} - \vec{x}_n]$$

1. Scientific motivations
Implicit algorithms

- α-generalized family (Chung & Hulbert [JAM, 1993])

  – Newmark relations:

$$\begin{cases} \ddot{\vec{x}}_{n+1} = \dfrac{1}{\beta\,\Delta t^2}\left[\vec{x}_{n+1} - \vec{x}_n - \Delta t\,\dot{\vec{x}}_n - \left[\dfrac{1}{2} - \beta\right]\Delta t^2\,\ddot{\vec{x}}_n\right] \\ \dot{\vec{x}}_{n+1} = \dfrac{\gamma}{\beta\,\Delta t}\left[\vec{x}_{n+1} - \vec{x}_n + \left[\dfrac{\beta}{\gamma} - 1\right]\Delta t\,\dot{\vec{x}}_n + \left[\dfrac{\beta}{\gamma} - \dfrac{1}{2}\right]\Delta t^2\,\ddot{\vec{x}}_n\right] \end{cases}$$

  – Balance equation:

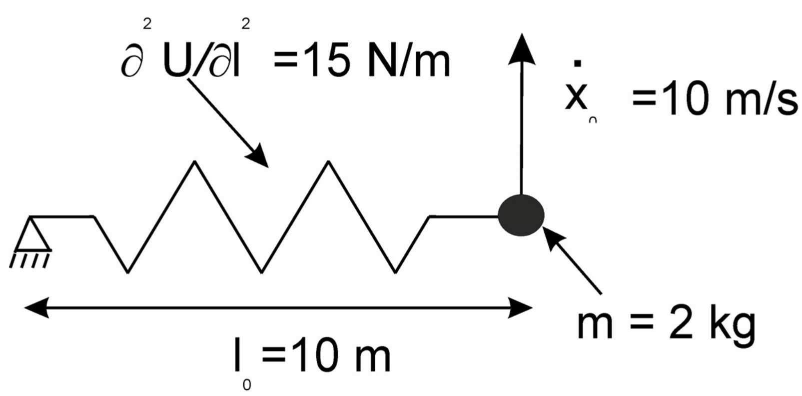
$$\frac{1-\alpha_M}{1-\alpha_F} M\,\ddot{\vec{x}}_{n+1} + \frac{\alpha_M}{1-\alpha_F} M\,\ddot{\vec{x}}_n + \left[\vec{F}^{\text{int}}_{n+1} - \vec{F}^{\text{ext}}_{n+1}\right] + \frac{\alpha_F}{1-\alpha_F}\left[\vec{F}^{\text{int}}_n - \vec{F}^{\text{ext}}_n\right] = 0$$

  – $\alpha_M = 0$ and $\alpha_F = 0$ (no numerical dissipation)
    - Linear range: consistency (i.e. physical results) demonstrated
    - Non-linear range with small time steps: consistency verified
    - Non-linear range with large time steps: total energy conserved but without consistency (e.g. plastic dissipation greater than the total energy, work of the normal contact forces > 0, …)

  – $\alpha_M \neq 0$ and/or $\alpha_F \neq 0$ (numerical dissipation)
    - Numerical dissipation is proved to be positive only in the linear range

1. Scientific motivations
Numerical example: mass-spring system

- Example: mass-spring system (2D) with an initial velocity perpendicular to the spring
  (Armero & Romero [CMAME, 1999])

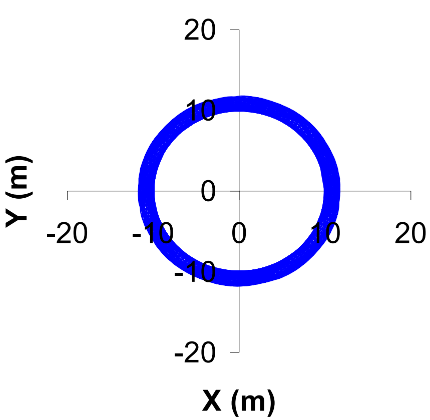


Explicit method: $\Delta t_{crit} \sim 0.72$s;

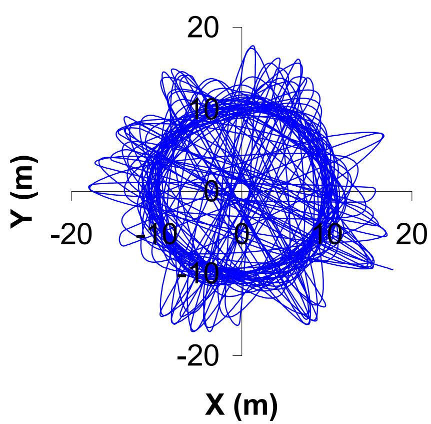
1 revolution ~ 4s

  – Newmark implicit scheme (no numerical damping)

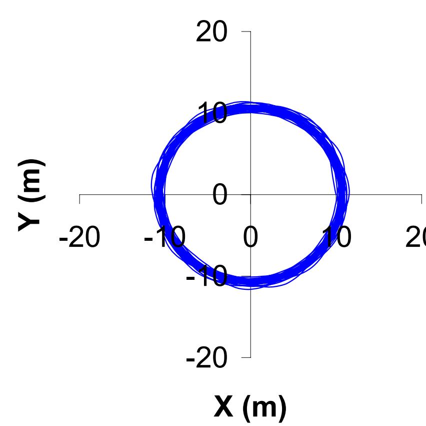
  – Chung-Hulbert implicit scheme (numerical damping)



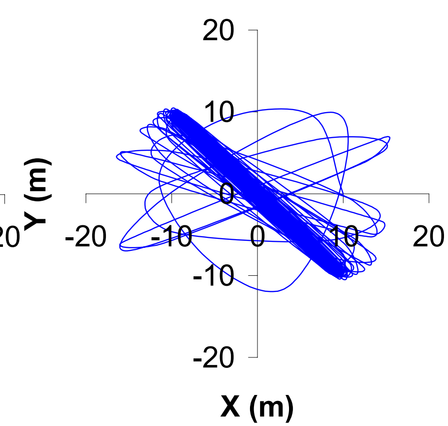
Δt=1s



Δt=1.5s



Δt=1s



Δt=1.5s

2. Conserving scheme in the non-linear range
Principle

- Consistent implicit algorithms in the non-linear range:
  - The Energy Momentum Conserving Algorithm or EMCA (Simo et al. [ZAMP 92], Gonzalez & Simo [CMAME 96]):
    - Conservation of the linear momentum
    - Conservation of the angular momentum
    - Conservation of the energy (no numerical dissipation)
  - The Energy Dissipative Momentum Conserving algorithm or EDMC (Armero & Romero [CMAME, 2001]):
    - Conservation of the linear momentum
    - Conservation of the angular momentum
    - Numerical dissipation of the energy is proved to be positive

2. Conserving scheme in the non-linear range
Principle

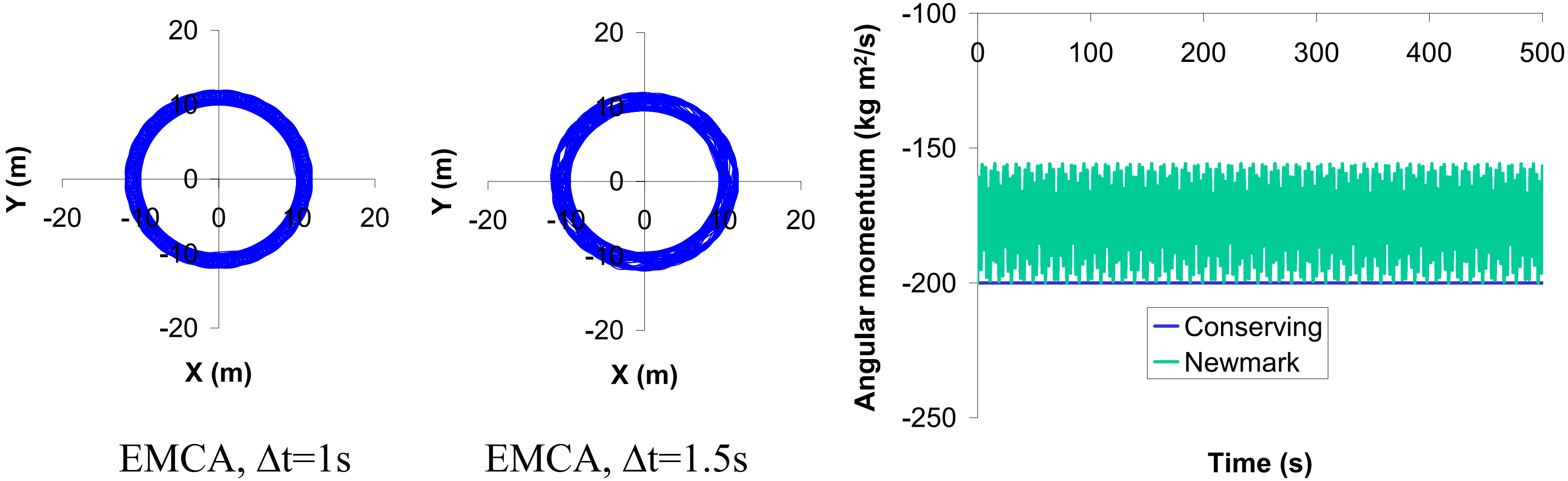
- Based on the mid-point scheme (Simo et al. [ZAMP, 1992]):
  - Relations between displacements, velocities, accelerations
    $$\begin{cases} \dfrac{\ddot{\vec{x}}_{n+1} + \ddot{\vec{x}}_n}{2} = \dfrac{\dot{\vec{x}}_{n+1} - \dot{\vec{x}}_n}{\Delta t} \\ \dfrac{\dot{\vec{x}}_{n+1} + \dot{\vec{x}}_n}{2} \left(+ \dot{\vec{x}}^{\text{diss}}_{n+1} \right) = \dfrac{\vec{x}_{n+1} - \vec{x}_n}{\Delta t} \end{cases}$$
  - Balance equation
    $$M \frac{\ddot{\vec{x}}_{n+1} + \ddot{\vec{x}}_n}{2} = \vec{F}^{\text{ext}}_{n+1/2} - \vec{F}^{\text{int}}_{n+1/2} \left(- \vec{F}^{\text{diss}}_{n+1/2} \right)$$
  - Energy Momentum Conserving Algorithm (EMCA):
    - With $\vec{F}^{\text{int}}_{n+1/2} \neq \int_{V_0} \mathbf{F}^{-T}_{n+1/2} \mathbf{S}_{n+1/2} \vec{D} dV_0$ and $\vec{F}^{\text{ext}}_{n+1/2}$ designed to verify conserving equations

    F: deformation gradient; **C**: right Cauchy-Green strain; **S**: 2nd Piola-Kirchhoff stress; φ: shape functions; $\vec{D} = \partial\varphi / \partial\vec{x}_0$
  - Energy-Dissipation Momentum-Conserving (Armero & Romero [CMAME, 2001]):
    - Same internal and external forces as in the EMCA
    - With $\vec{F}^{\text{diss}}_{n+1/2}$ and $\dot{\vec{x}}^{\text{diss}}_{n+1}$ designed to achieve positive numerical dissipation without spectral bifurcation

2. Conserving scheme in the non-linear range
The mass-spring system

- Forces of the spring for any potential V
  - Without numerical dissipation (EMCA) (Gonzalez & Simo [CMAME, 1996])

$$\vec{F}^{\text{int}}_{n+1/2} = \frac{V(l_{n+1}) - V(l_n)}{l^2_{n+1} - l^2_n} \left[\vec{x}_{n+1} + \vec{x}_n\right]$$



EMCA, Δt=1s EMCA, Δt=1.5s

  - The consistency of the EMCA solution does not depend on Δt
  - The Newmark solution does not conserve the angular momentum

- Elastic formulation:
  - Saint Venant-Kirchhoff hyperelastic model (Simo et al. [ZAMP, 1992])
  - General formulation for hyperelasticity (stress derived from a potential V) (Gonzalez [CMAME, 2000]):

$$\vec{F}^{\text{int}}_{n+1/2} = \int_{V_0} \frac{\mathbf{F}_n + \mathbf{F}_{n+1}}{2} \left[2\frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) + 2 \overbrace{\left[\frac{V(\mathbf{C}_{n+1}) - V(\mathbf{C}_n) - \frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n+\mathbf{C}_{n+1}}{2}\right) : \Delta\mathbf{C}}{\left\|\Delta\mathbf{C}\right\|^2} \right] \Delta\mathbf{C}}^{O\left(\left\|\mathbf{C}_{n+1}-\mathbf{C}_n\right\|^2\right)} \right] \vec{D}\, dV_0$$

$\mathbf{F}$: deformation gradient; $\mathbf{C}$: right Cauchy-Green strain; V: potential; φ: shape functions; $\vec{D} = \partial\varphi / \partial\vec{x}_0$

  - Classical formulation: $\vec{F}^{\text{int}} = 2\int_{V_0} \mathbf{F}\frac{\partial V}{\partial \mathbf{C}} \vec{D}\, dV_0$

- Penalty contact formulation (Armero & Petöcz [CMAME, 1998-1999]):

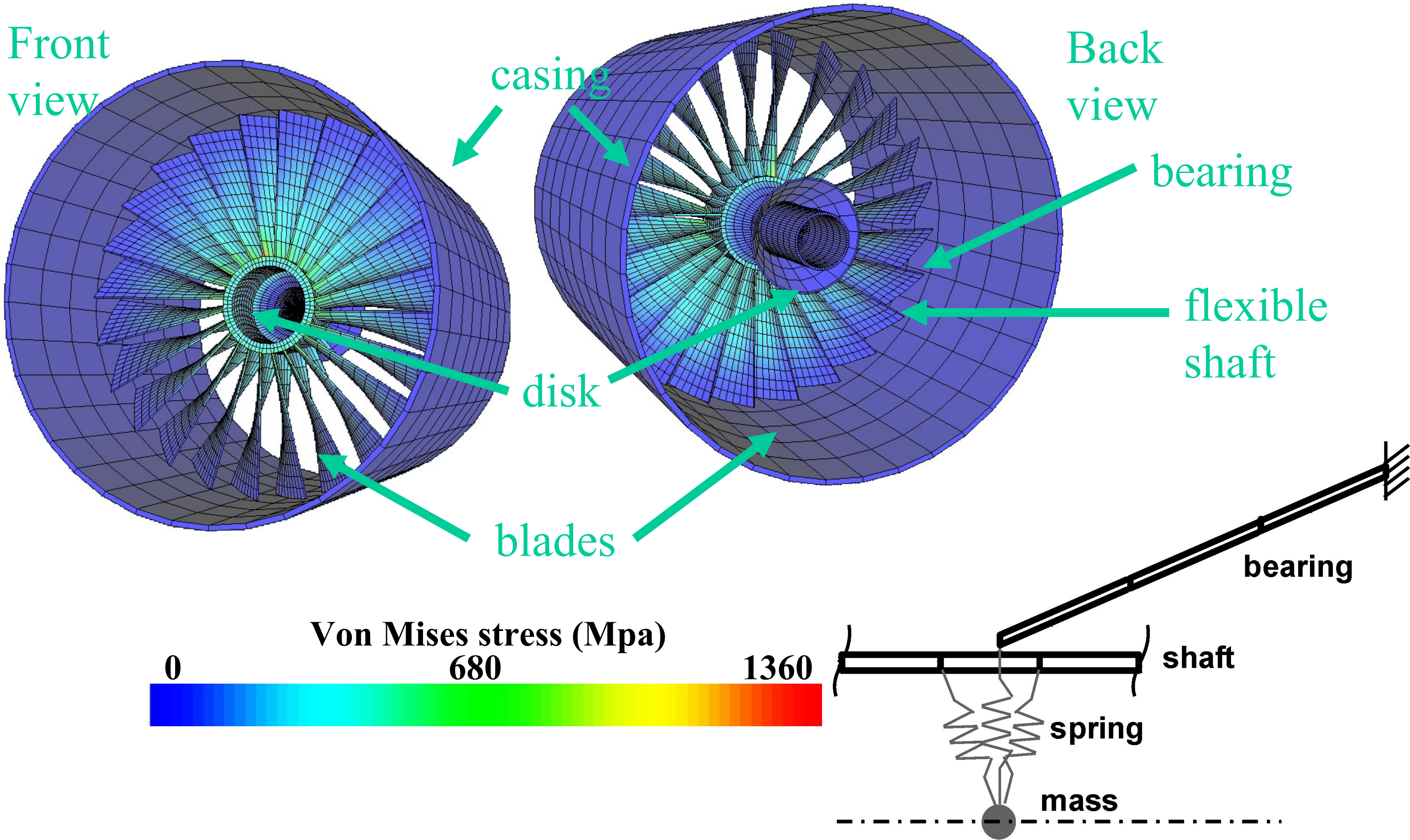
$$g^{\mathrm{d}}_{n+1} = g^{\mathrm{d}}_n + \vec{n}_{n+1/2} \bullet \left[\vec{x}_{n+1} - \vec{x}_n - \vec{y}_{n+1}(u_{n+1/2}) + \vec{y}_n(u_{n+1/2})\right]$$

$$\vec{F}^{\text{cont}}_{n+1/2} = \frac{V\left(g^{\mathrm{d}}_{n+1}\right) - V\left(g^{\mathrm{d}}_n\right)}{g^{\mathrm{d}}_{n+1} - g^{\mathrm{d}}_n}\vec{n}_{n+1/2}$$

2. Conserving scheme in the non-linear range Formulations in the literature

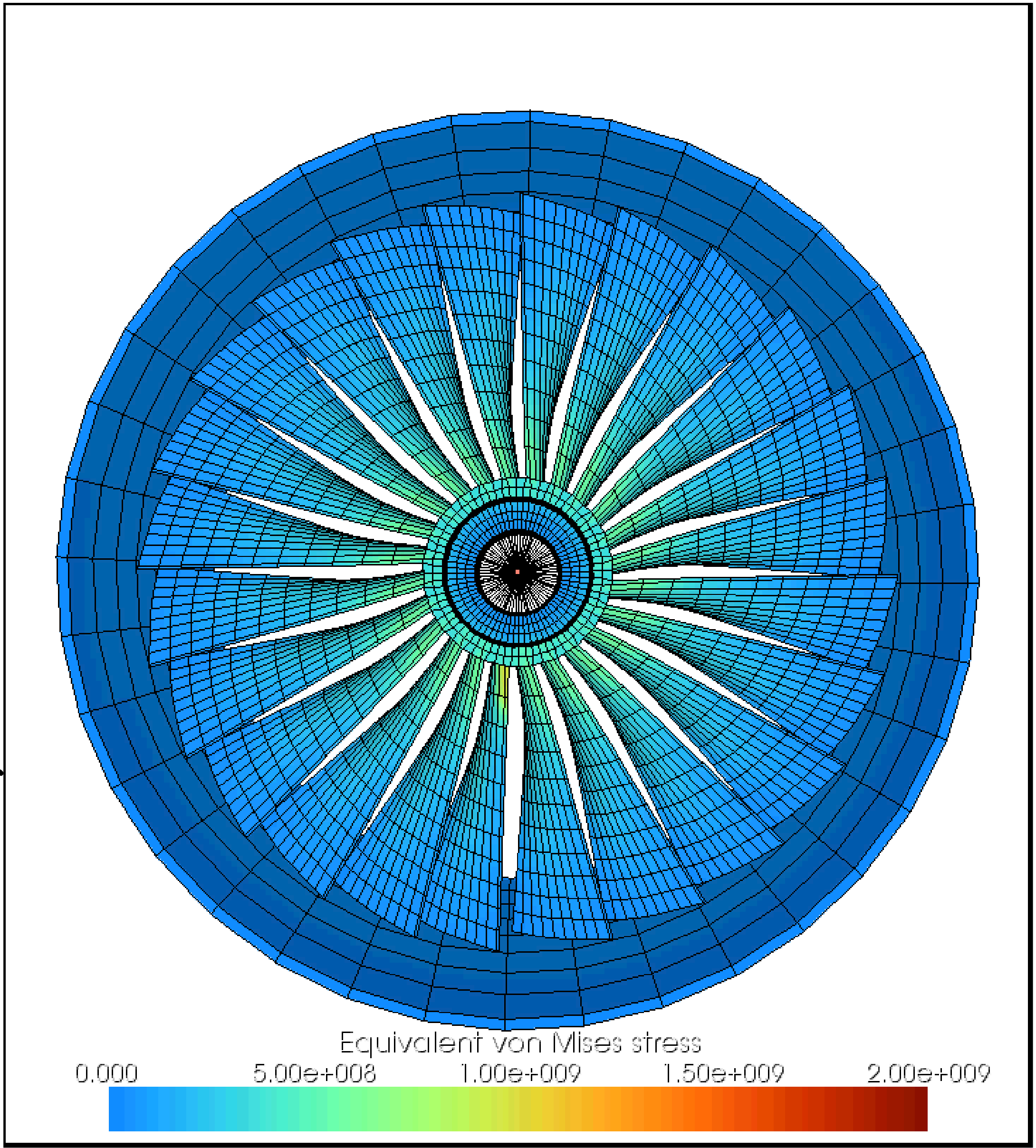
- Elasto-plastic materials:
  - Hyperelasticity with elasto-plastic behavior (Meng & Laursen [CMAME, 2001]):
    - energy dissipation of the algorithm corresponds to the internal dissipation of the material
    - Isotropic hardening only
  - Hyperelasticity with elasto-plastic behavior (Armero [CMAME, 2006]):
    - Energy dissipation from the internal forces corresponds to the plastic dissipation
    - Modification of the radial return mapping
    - Yield criterion satisfied at the end of the time-step
  - Hypoelastic formulation:
    - Stress obtained incrementally from a hardening law
    - No possible definition of an internal potential!
    - Idea: the internal forces are established to be consistent on a loading/unloading cycle
    - Assumption made on the Hooke tensor
    - Energy dissipation from the internal forces corresponds to the plastic dissipation

- Numerical simulation of a blade loss in an aero engine



2. Conserving scheme in the non-linear range
Numerical results

- Blade off:
  - Rotation velocity 5,000rpm
  - EDMC algorithm
  - Hypoelastic formulation
  - 29,000 dof's
  - One revolution simulation
  - 9,000 time steps
  - 50,000 iterations (only 9,000 with stiffness matrix updating)
- Demonstrates the robustness and efficiency of the conserving schemes



3. Variational update approach
Purpose of the work

- Development of a general approach leading to conserving algorithm for any material behavior!
- What we want:
  - No assumption on the material behavior
  - Material model unchanged compared to the standard approach:
    - From a given strain tensor, the outputs of the model are the same
    - Use of the same material libraries
  - Expression of the internal forces for the conserving algorithm remains the same as in the elastic case
  - Yield criterion satisfied at the end of the time step
- Solution derives from the variational formulation of visco-plastic updates [Ortiz & Stainier, CMAME 1999] that allows the definition of an energy, even for complex material behaviors

$$\mathbf{S}_{n+1} = 2\frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}_{n+1}}\left(\mathbf{C}_{n+1},\mathbf{C}_{n}\right)$$

C: right Cauchy-Green strain

S: second Piola-Kirchhoff stress

ΔD^{eff}: incremental potential

3. Variational update approach
Use of an incremental potential

- Description of the variational update for elasto-plasticity
  - Multiplicative plasticity: $\mathbf{F} = \mathbf{F}^{\mathrm{el}}\mathbf{F}^{\mathrm{pl}}$
  - Plastic flow: $\mathbf{F}^{\mathrm{pl}}_{n+1} = \exp(\Delta\varepsilon^{\mathrm{pl}}\mathbf{N})\mathbf{F}^{\mathrm{pl}}_{n}$ **N**: flow direction
  - Functional increment:

$$\Delta D(\mathbf{F}_{n+1}, \mathbf{F}_n, \varepsilon^{\mathrm{pl}}_{n+1}, \varepsilon^{\mathrm{pl}}_{n}, \mathbf{N}) = W^{\mathrm{el}}\left(\mathbf{F}_{n+1}\mathbf{F}^{\mathrm{pl}\,-1}_{n+1}(\varepsilon^{\mathrm{pl}}_{n+1}, \mathbf{N})\right) - W^{\mathrm{el}}\left(\mathbf{F}_{n}\mathbf{F}^{\mathrm{pl}-1}_{n}(\varepsilon^{\mathrm{pl}}_{n}, \mathbf{N})\right) + W^{\mathrm{pl}}\left(\varepsilon^{\mathrm{pl}}_{n+1}\right) - W^{\mathrm{pl}}\left(\varepsilon^{\mathrm{pl}}_{n}\right) + \Delta t\Psi^{*}\left(\frac{\varepsilon^{\mathrm{pl}}_{n+1} - \varepsilon^{\mathrm{pl}}_{n}}{\Delta t}\right)$$

W^{el}: reversible potential;

W^{pl}: dissipation by plasticity;

Ψ^{*}: dissipation by viscosity

  - Effective potential: $\Delta D^{\mathrm{eff}}\left(\mathbf{F}_{n+1}\right) = \min_{\varepsilon^{\mathrm{pl}}_{n+1}, \mathbf{N}} \Delta D(\mathbf{F}_{n+1}, \mathbf{F}_n, \varepsilon^{\mathrm{pl}}_{n+1}, \varepsilon^{\mathrm{pl}}_{n+1}, \mathbf{N})$
    - Minimization with respect to $\varepsilon^{\mathrm{pl}}$ satisfies yield criterion
    - Minimization with respect to **N** satisfies radial return mapping
  - Stress derivation: $\mathbf{S}_{n+1} = 2\frac{\partial \Delta D^{\mathrm{eff}}\left(\mathbf{F}_{n+1}\right)}{\partial \mathbf{C}_{n+1}}$

3. Variational update approach
Use of an incremental potential

- Conserving internal forces directly obtained from Gonzalez elastic formulation [CMAME 2000]

$$\vec{F}^{\text{int}}_{n+1/2} = \int_{V_0} \frac{\mathbf{F}_n + \mathbf{F}_{n+1}}{2} \left[2\frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n+\mathbf{C}_{n+1}}{2}\right) + 2\overbrace{\left[\frac{V(\mathbf{C}_{n+1}) - V(\mathbf{C}_n) - \frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n+\mathbf{C}_{n+1}}{2}\right):\Delta\mathbf{C}}{\|\Delta\mathbf{C}\|^2}\right]}^{O\left(\|\mathbf{C}_{n+1}-\mathbf{C}_n\|^2\right)} \Delta\mathbf{C} \right] \vec{D}\, dV_0$$

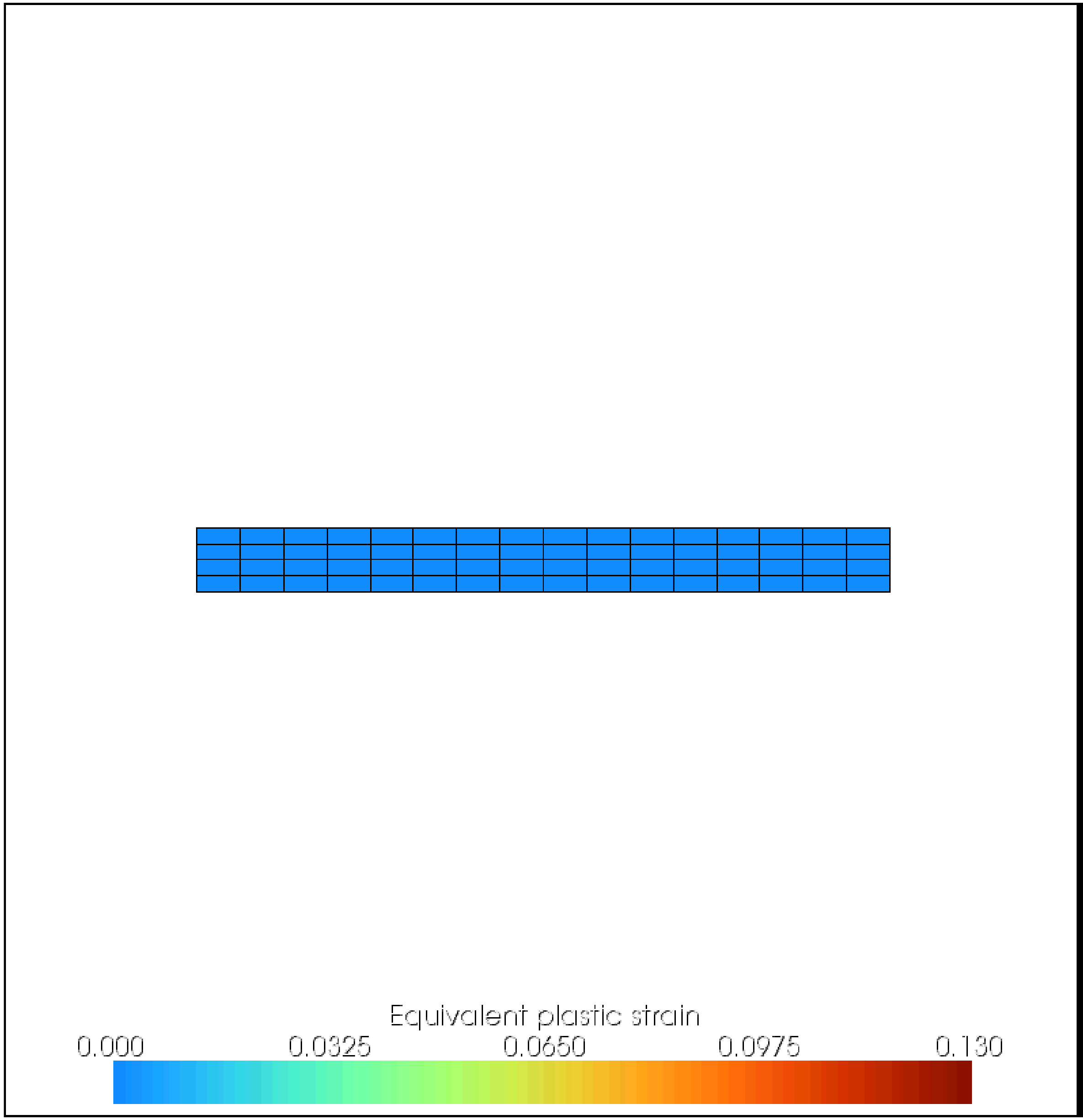
$$\vec{F}^{\text{int}}_{n+1/2} = \int_{V_0} \frac{\mathbf{F}_n + \mathbf{F}_{n+1}}{2} \left[2\frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n+\mathbf{C}_{n+1}}{2}\right) + 2\overbrace{\left[\frac{\Delta D^{\text{eff}}(\mathbf{C}_{n+1}, \mathbf{C}_n) - \frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n+\mathbf{C}_{n+1}}{2}\right):\Delta\mathbf{C}}{\|\Delta\mathbf{C}\|^2}\right]}^{O\left(\|\mathbf{C}_{n+1}-\mathbf{C}_n\|^2\right)} \Delta\mathbf{C} \right] \vec{D}\, dV_0$$

3. Variational update approach
Use of an incremental potential

- Properties:
  - 2 material configurations computed:
    - Mid configuration trough $\frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_{n+1}+\mathbf{C}_n}{2}\right)$
    - Final configuration trough $\Delta D^{\text{eff}}\left(\mathbf{C}_{n+1},\mathbf{C}_n\right)$
  - Material model unchanged
  - Yield criterion verified at configuration *n+1*
  - Conservation of linear and angular momentum
  - Conservation of energy: $\sum_{nodes} \vec{F}^{\text{int}}_{n+1/2} \bullet [\vec{x}_{n+1} - \vec{x}_n] = W^{\text{el}}_{n+1} - W^{\text{el}}_n + W^{\text{pl}}_{n+1} - W^{\text{pl}}_n + \Delta t \Psi^*$

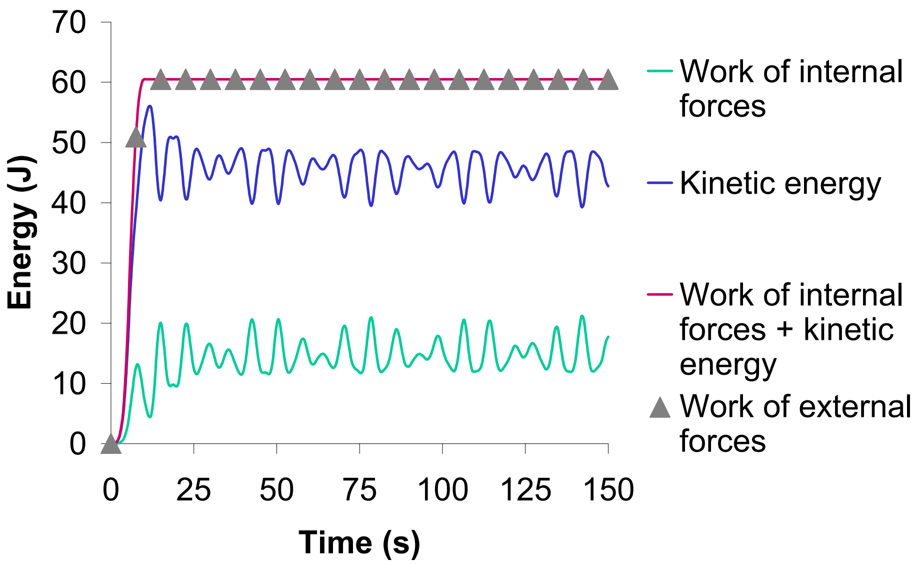
Simulation of a tumbling beam

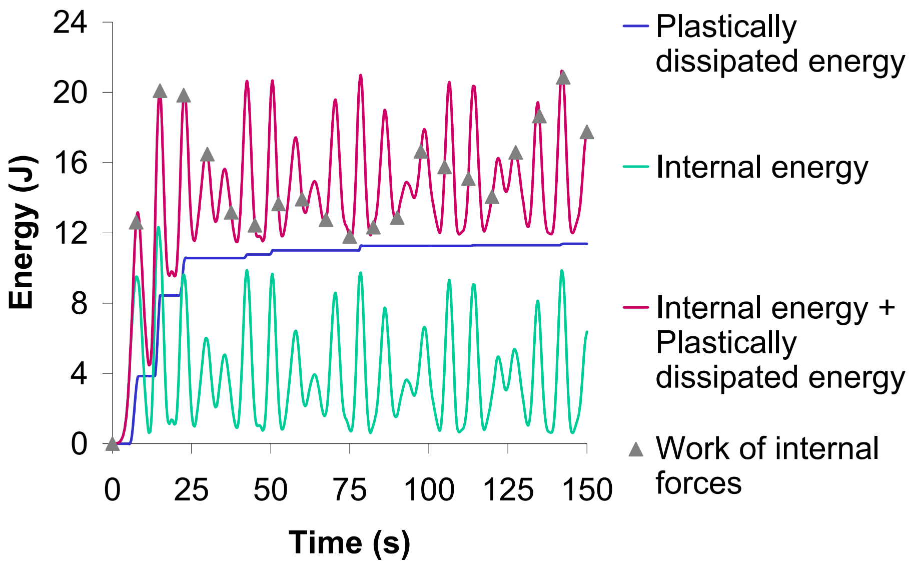
- Tumbling beam:
  - Initial symmetrical loads ($t < 10s$)
  - Elasto-perfectly-plastic hyperelastic material



3. Variational update approach
Simulation of a tumbling beam

- Time evolution of the results:

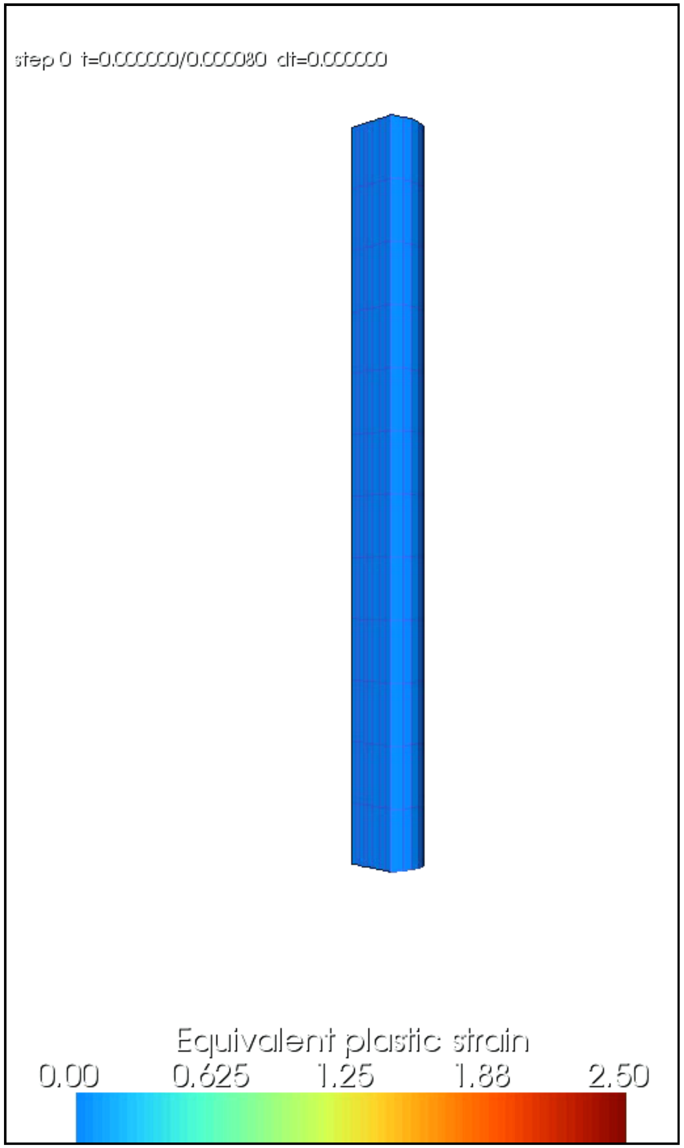




3. Variational update approach
Simulation of the Taylor impact

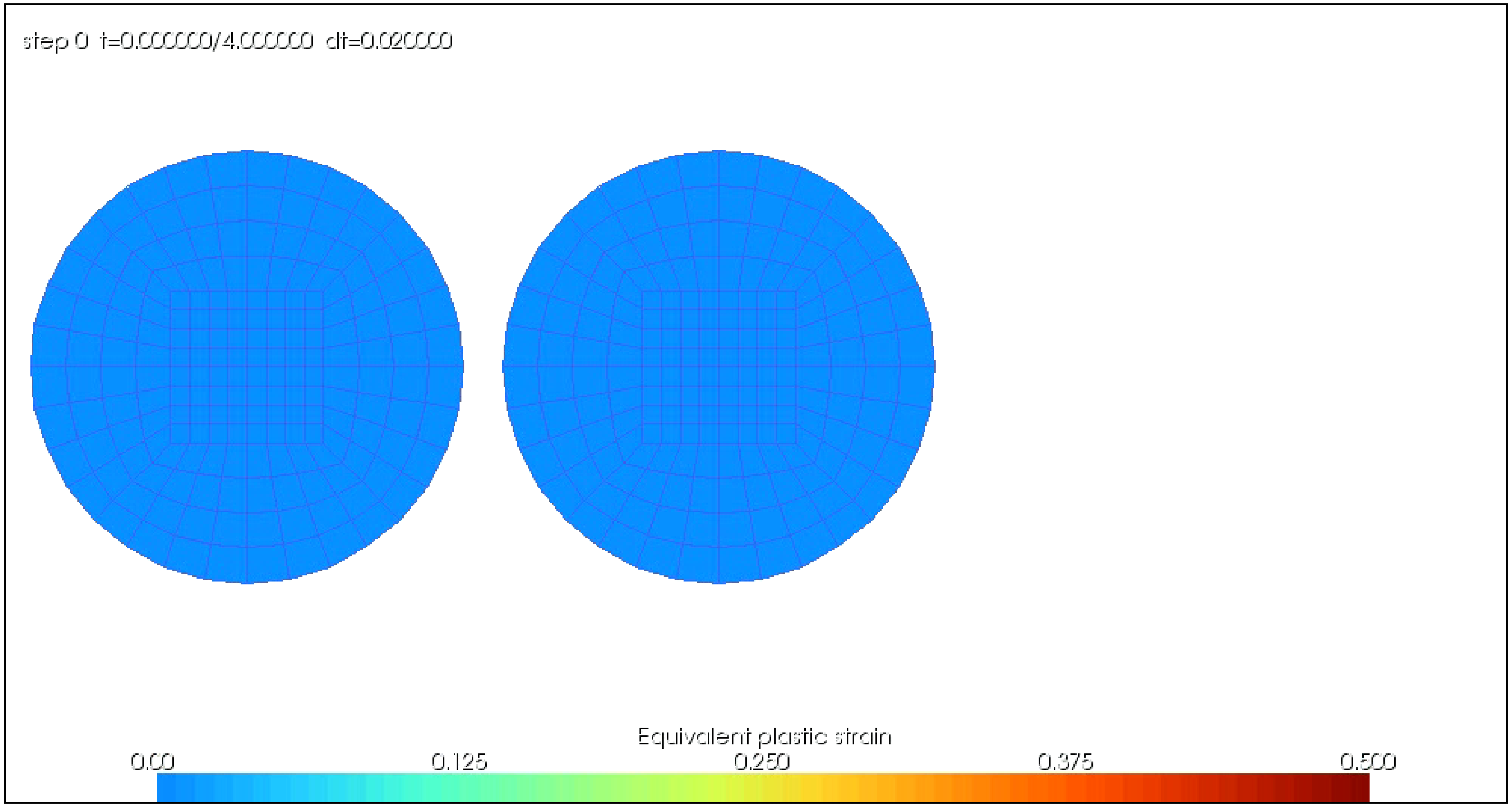
- Impact of a cylinder:
  - Hyperelastic model
  - Elasto-plastic hardening law
  - Simulation during 80 μs

Method	Final length [mm]	Final radius [mm]	Max ε^{pl}
EMCA; Δt = 25 ns	21.4	6.77	2.61
EMCA; Δt = 400 ns	21.4	6.81	2.61
Newmark; Δt = 25 ns	21.4	6.77	2.61
Newmark; Δt = 400 ns	21.5	6.87	2.81
EMCA; Meng & Laursen	21.6	6.78	2.62
Newmark; Simo	-	6.97	-



3. Variational update approach
Impact of two 2D-cylinders

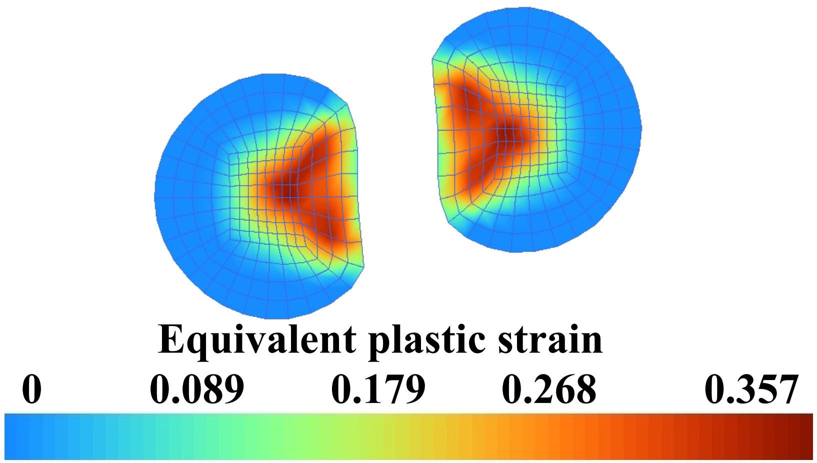
- Two cylinders (Meng & Laursen):
  - Left one has an initial velocity (initial kinetic energy 14J)
  - Elasto-perfectly-plastic hyperelastic material



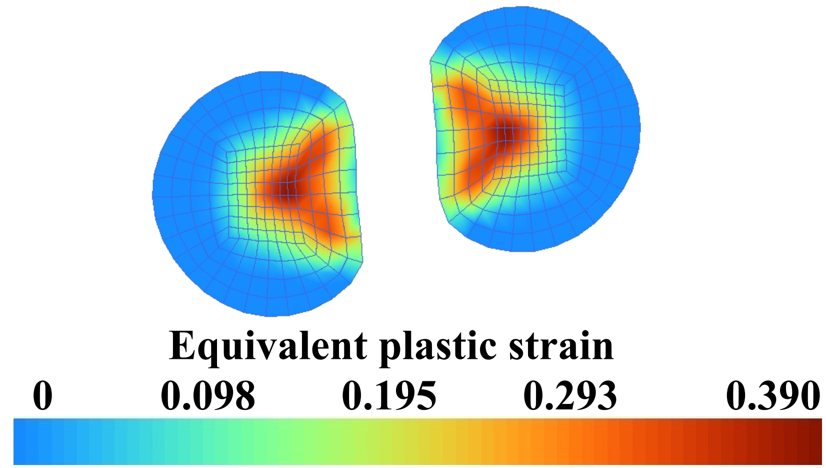
3. Variational update approach
Impact of two 2D-cylinders

- Results comparison at the end of the simulation

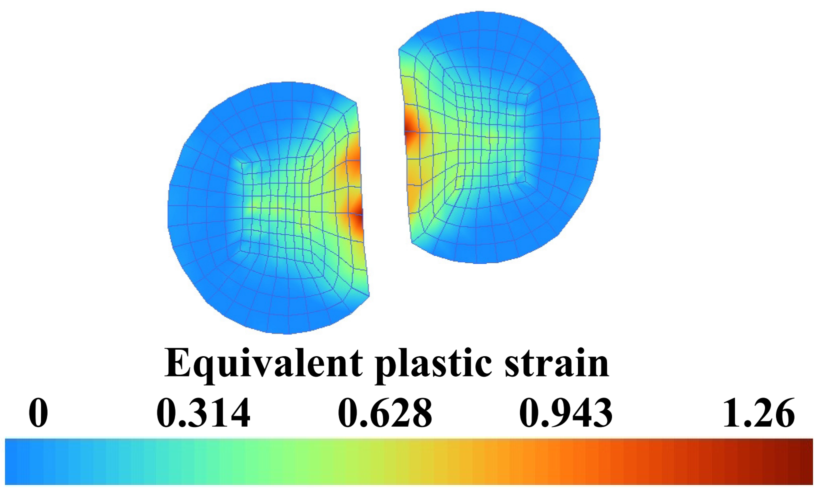
Newmark; Δt=2 ms



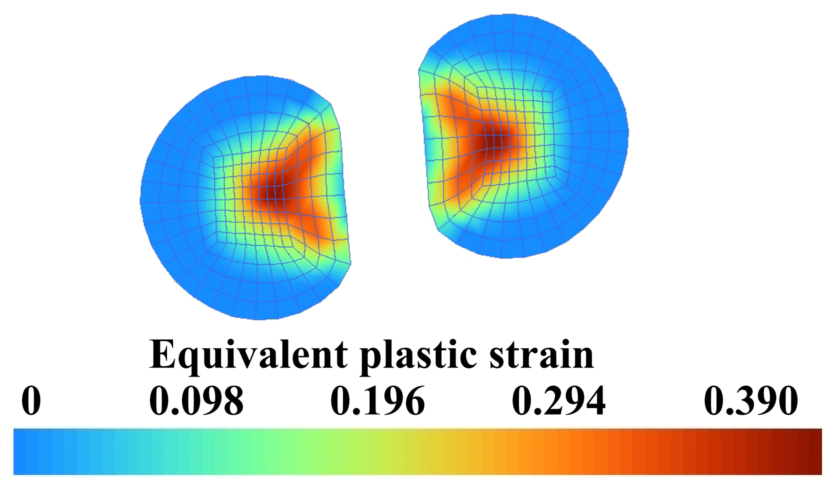
EMCA; Δt=2 ms



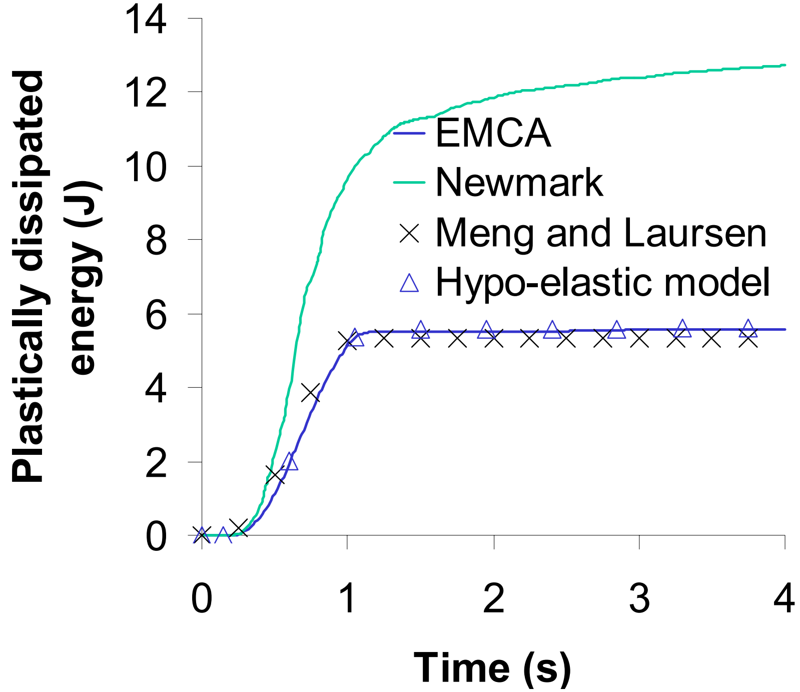
Newmark; Δt=20 ms

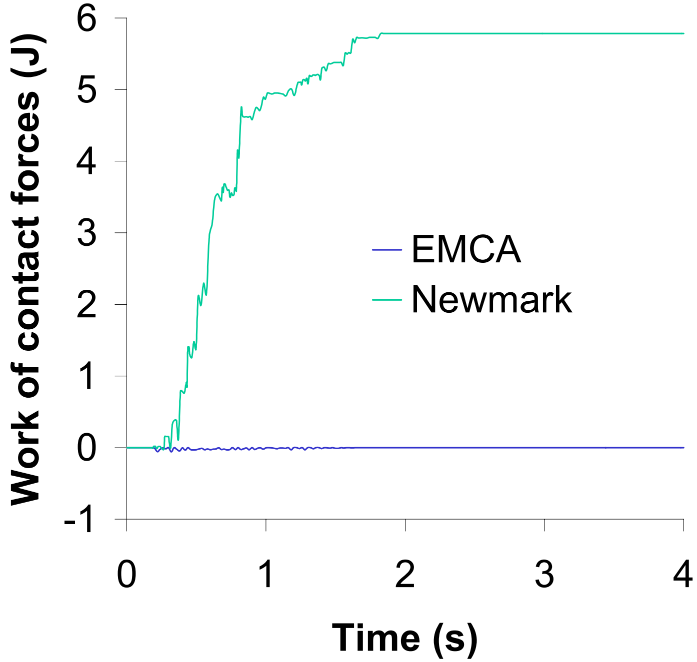


EMCA; Δt=20 ms



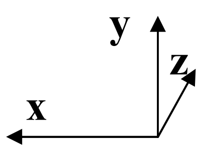
- Results evolution comparison for Δt = 20 ms

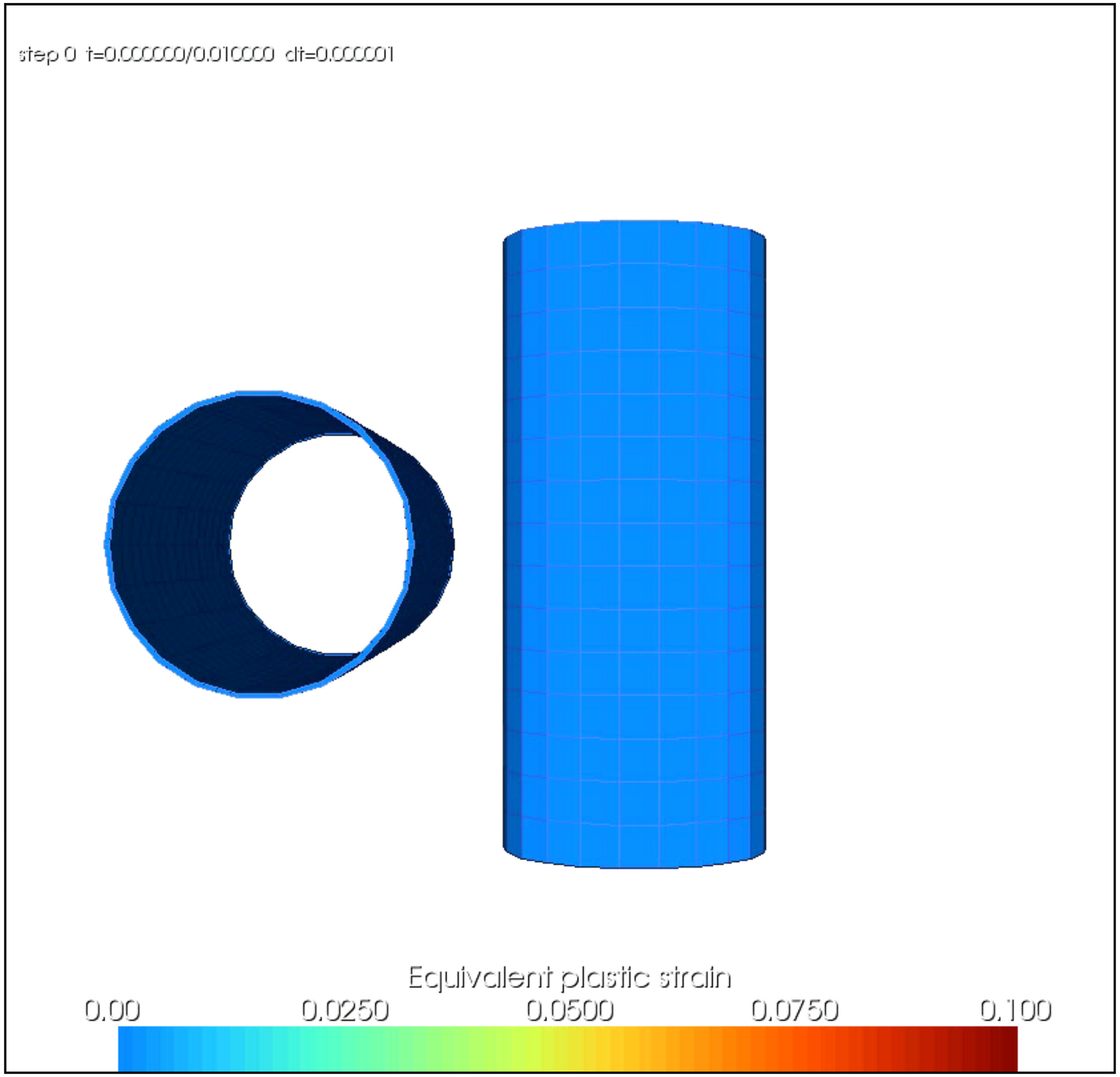




Impact of two 3D-cylinders

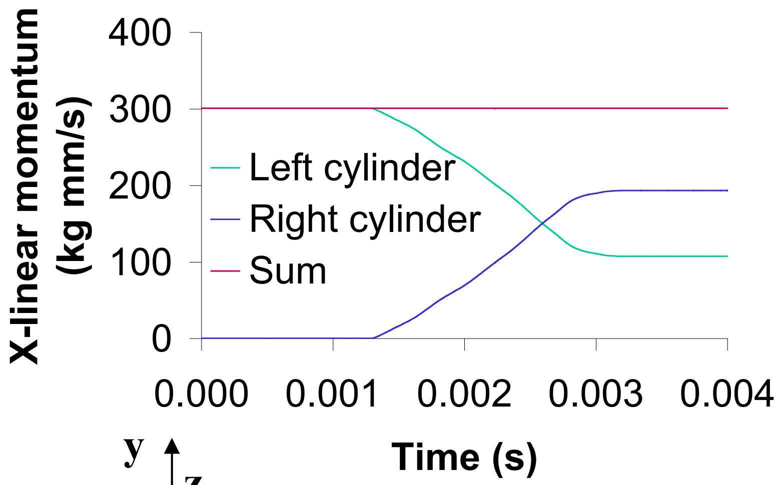
- Impact of 2 hollow 3D-cylinders:
  - Right one has a initial velocity ($\dot{\vec{x}}_{0X} = 10\dot{\vec{x}}_{0Y}$)
  - Elasto-plastic hyperelastic material (steel)
  - Frictional contact

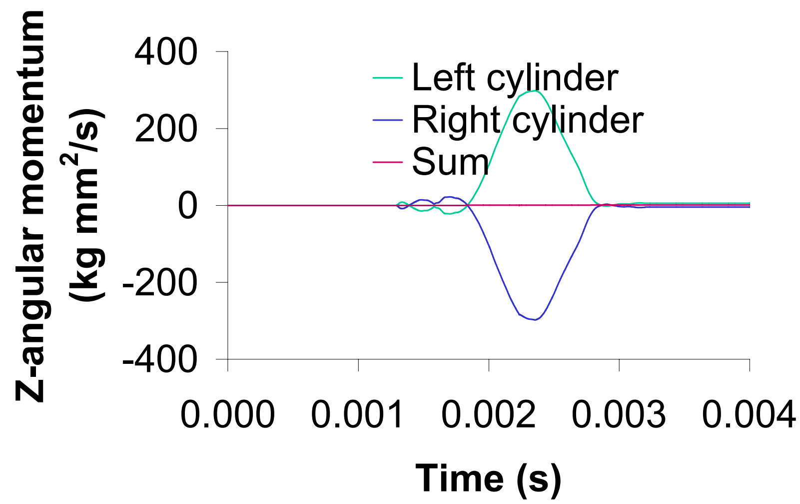


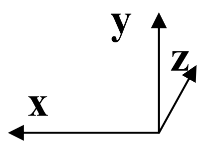


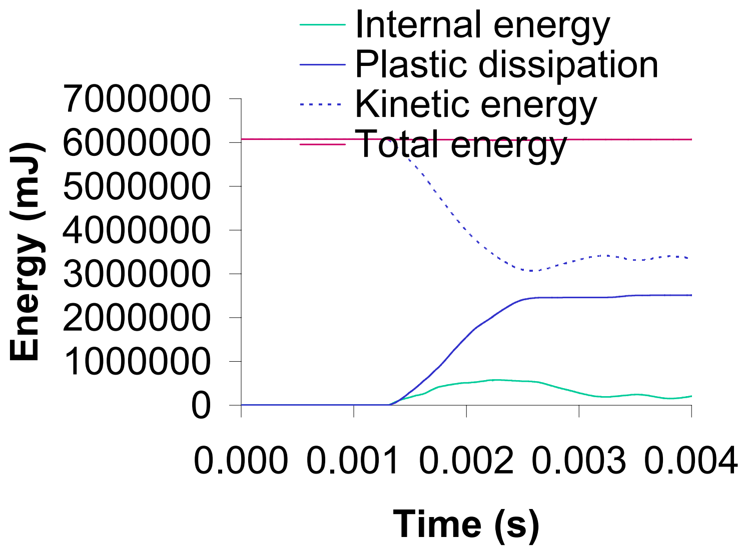
3. Variational update approach
Impact of two 3D-cylinders

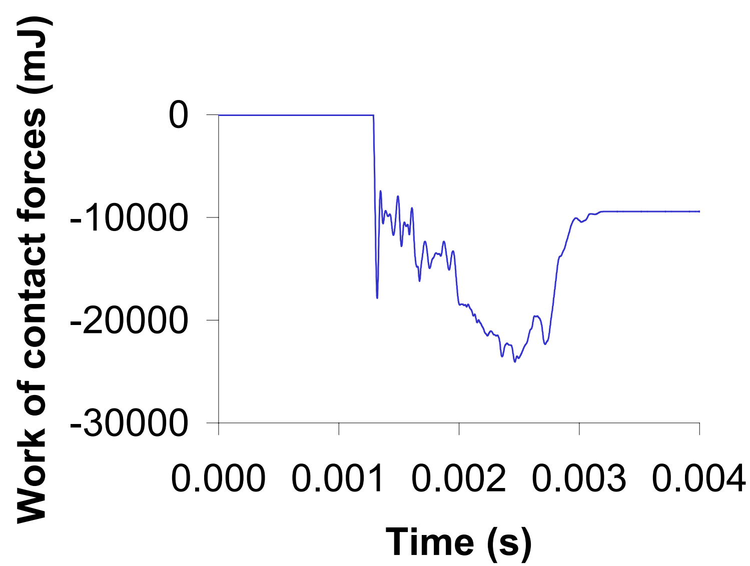
- Time evolution of the results:











5. Conclusions

- Developed a visco-elastic formulation leading to a conserving time integration scheme
- Use of the variational update formulation:
  - The formulation derives from a energy potential
  - The formulation is general for any material behavior
- The internal force expression remains the same as for elasticity
- The momentum and the energy are conserved
- The yield criterion is satisfied at the end of the time step
- Numerical examples demonstrate the robustness