Development of discontinuous Galerkin method for linear strain gradient elasticity

R Bala Chandran
Computation for Design and Optimizaton
Massachusetts Institute of Technology
Cambridge, MA

L. Noels*
Aerospace and Mechanical Engineering Department
University of Liège
Belgium

R. Radovitzky
Department of Aeronautics and Astronautics
Massachusetts Institute of Technology
Cambridge, MA

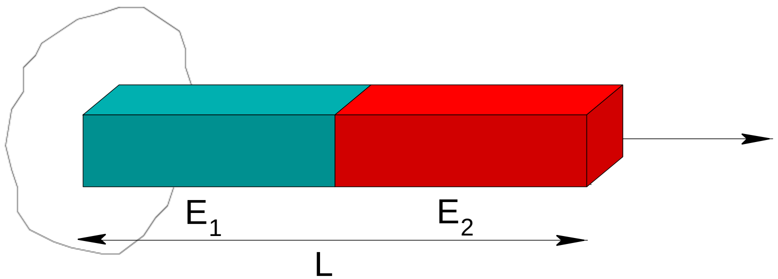
9th US National Congress on Computational Mechanics
San Francisco, California, USA, July 23-26, 2007

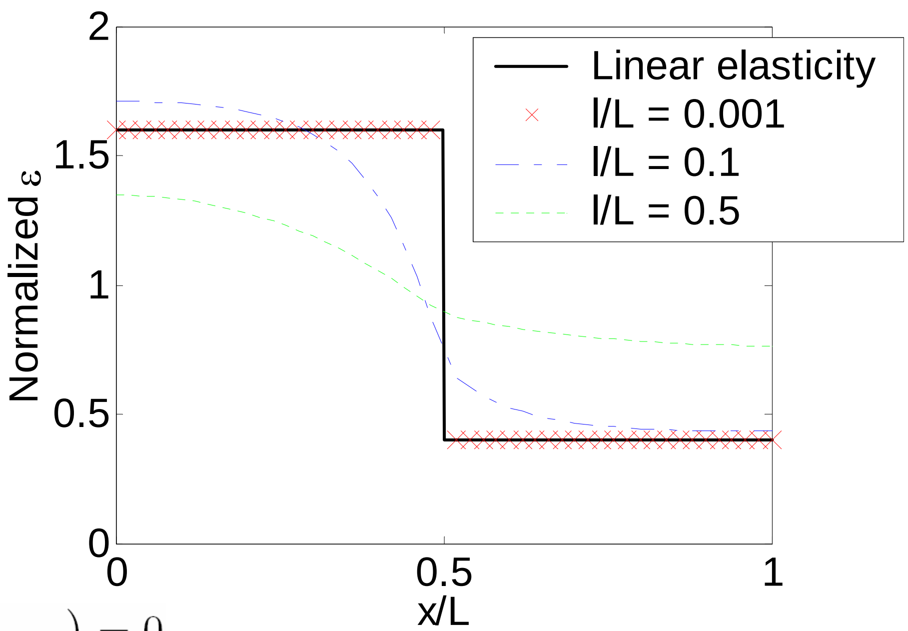
Massachusetts Institute of Technology

University of Liège

Introduction & Motivation

- Length scales in modern technology are now of the order of the micrometer or nanometer
- At these scales, material laws depend on strain but also on strain-gradient
- Example:
  - Bi-material tensile test:





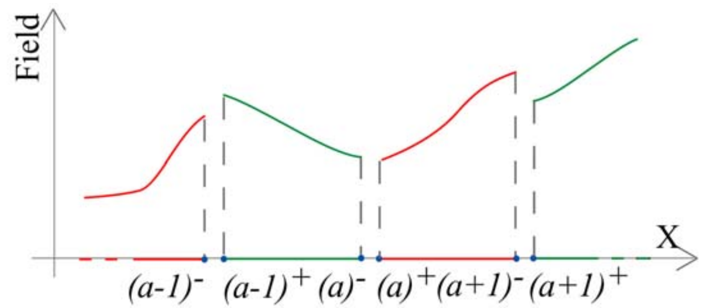
  - $E_1/E_2=4$
  - Characteristic length l
  - Differential equation: $E\left(u_{,xx} - l^2 u_{,xxxx}\right) = 0$

Introduction & Motivation

- Introduction of strain-gradient effect in numerical simulations
  - Domain of applications:
    - Stress concentrations (around hole, at crack tip, …)
    - Grain size effect on polycrystalline yield strength
    - Void growth
    - …
  - Finite elements framework
  - In the general 3D case, shape functions are not C^1, which prevents the direct evaluation of the strain gradients
- Idea: enforcing weakly the C^1 continuity by recourse to discontinuous Galerkin methods

Introduction & Motivation

- Discontinuous Galerkin methods
  - Finite element discretizations which allow for jump across elements
  - Compatibility of the field variable or its spatial derivative is imposed in a weak sense



  - Stability enforced with a quadratic interelement integrals
- Application of discontinuous Galerkin methods in solid mechanics
  - Allow weak enforcement of C^0 continuity:
    - Non-linear mechanics (Ten Eyck and Lew 2006, Noels and Radovitzky 2006)
    - Reduction of locking for shells (Güzey et al. 2006)
    - Beams and plates (Arnold et al. 2005, Celiker and Cockburn 2007)
  - Allow weak enforcement of C^1 continuity (strong enforcement of C^0):
    - Beams and plates (Engel et al. 2002)
    - Strain gradient (1D) (Molari et al. 2006)
    - Kirchhoff-Love shells (Noels and Radovitzky 2007)

Introduction & Motivation

- Purpose of the presentation is to develop a dG formulation for strain gradient elasticity, which
  - Is a single field formulation in displacement
  - Requires only the use of C^0 continuous interpolations
  - Is demonstrated to be consistent and stable
  - Is easy to integrate into a regular 3D finite-element code
  - Has C^1 continuity constrained in a weak sense
- Scope of this presentation
  - Strain gradient theory of elasticity
  - Discontinuous Galerkin formulation
  - Numerical properties
  - FEM 3D implementation
  - Numerical examples
  - Conclusions & Future work

Strain gradient theory of elasticity

- Strain gradient theory:
  - At a material point stress is a function of strain and of the gradient of strain *(Toupin 1962, Mindlin 1964)*
  - Strain energy $W = W(\epsilon_{ij}, \eta_{ijk})$ is assumed to be a function of strain and gradient of strain
  - Low and high order stresses introduced as the work conjugate of low and high order strains $\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = C_{ijkl}\,\epsilon_{kl}$ $\tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}} = J_{ijklmn}\,\eta_{lmn}$
  - Governing PDE obtained from satisfying the virtual work statement

$$\int_{B_0} (\sigma_{ij}\delta\epsilon_{ij} + \tau_{ijk}\delta\eta_{ijk})\, dV = \int_{B_0} \hat{b}_k \delta u_k dV + \int_{\partial_N B} \hat{t}_k \delta u_k dS + \int_{\partial_M B} \hat{r}_k \delta u_{k,l} n_l dS$$

Body forces

Low order tractions

Double stress tractions

Strain gradient theory of elasticity

- The boundary value problem:
  - Local equation

$$0 = b_k + \left(\sigma_{ik} - \tau_{jik,j}\right)_{,i} \text{ in } B_0$$

  - Natural boundary conditions

$$\hat{t}_k = n_i\left(\sigma_{ik} - \partial_j \tau_{ijk}\right) + n_i n_j \tau_{ijk}\left(D_p n_p\right) - D_i\left(n_j \tau_{ijk}\right) \text{ on } \partial_N B_0$$

$$\hat{r}_k = n_i n_j \tau_{ijk} \text{ on } \partial_M B_0$$

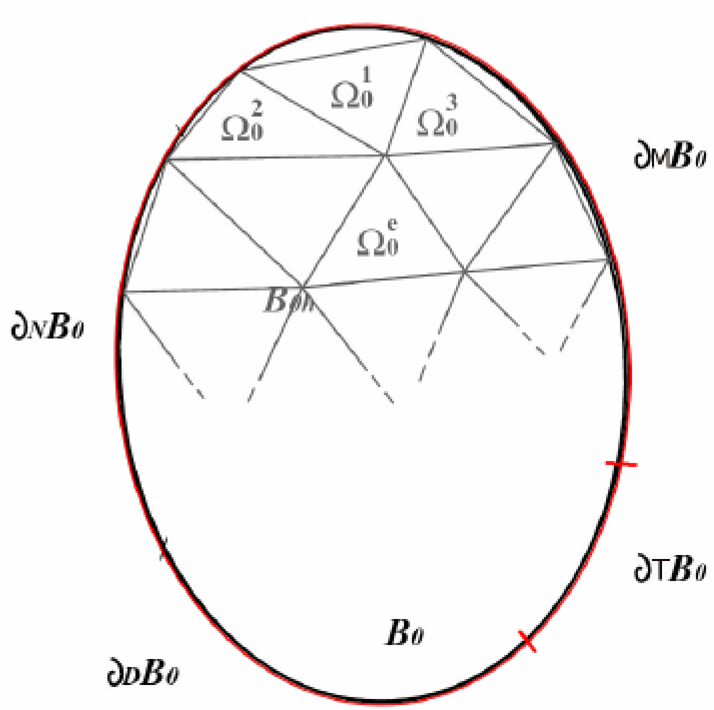
  - Essential boundary conditions

$$u_k = \overline{u}_k \text{on } \partial_D B_0$$

$$n_i u_{k,i} = \overline{Du}_{k,i} \text{ on } \partial_T B_0$$

  - Finite-elements discretization

$$\bigcup_{e=1}^{E} \Omega_e = B_{0h} \approx B_0$$



$$\partial_N B_0 \cup \partial_D B_0 = \partial B_0$$
$$\partial_M B_0 \cup \partial_T B_0 = \partial B_0$$
$$\partial_N B_0 \cap \partial_D B_0 = \emptyset$$
$$\partial_M B_0 \cap \partial_T B_0 = \emptyset$$

Discontinuous Galerkin formulation

- Derivation of the weak form:
  - Choose the appropriate space for the test (u_h) and trial functions (w), which:
    - Are C^0 on the whole domain
    - Are $\mathbb{P}^k$ in each element
    - Satisfy the essential BC's
  - Multiply the local equation by a test function

$$\sum_{e=1}^{E} \int_{\Omega_e} w_k \left(b_k + \left(\sigma_{ik} - \tau_{jik,j} \right)_{,i} \right) dV = 0$$

  - Integrate by parts and use divergence theorem

$$\sum_{e=1}^{E} \int_{\partial\Omega_e} w_k \left(\sigma_{ik} - \tau_{jik,j} \right) n_i dS - \sum_{e=1}^{E} \int_{\Omega_e} w_{k,i} \sigma_{ik} dV - \sum_{e=1}^{E} \int_{\Omega_e} w_{k,ij} \tau_{jik} dV$$

Introduces inter-element contributions

$$+ \sum_{e=1}^{E} \int_{\partial\Omega_e} w_{k,i} \tau_{jik} n_j dS + \sum_{e=1}^{E} \int_{\Omega_e} w_k b_k dV = 0$$

Discontinuous Galerkin formulation

- Introduction of the numerical fluxes:
  - On inter-element boundaries

$$\sum_{e=1}^{E}\int_{\partial\Omega_e\cap\partial_I\Omega} w_{k,i}\tau_{jik}n_j dS \approx -\int_{\partial_I\Omega} [\![w_{k,i}]\!]\,\widehat{\tau_{jik}n_j^-}\,dS$$

$$\widehat{\tau_{jik}n_j} = \langle\tau_{jik}\rangle\, n_j + n_j\left\langle\frac{\beta J_{jikqrp}}{h}\right\rangle [\![u_{p,q}]\!]\, n_r$$

Ensures consistency

Ensures stability (h = mesh size and β = parameter)

  - Extension to weak enforcement of high-order BC

$$\sum_{e=1}^{E}\int_{\partial\Omega_e\cap\partial_T\Omega} n_i w_{k,l} n_l \tau_{jik}n_j dS \approx \int_{\partial_T\Omega} n_i w_{k,l} n_l \widehat{\tau_{jik}n_j^-}\,dS$$

$$\widehat{\tau_{jik}n_j} = \tau_{jik}n_j + n_j\frac{\beta J_{jikqrp}}{h}\left(n_s u_{p,s} n_q - \overline{Du_p} n_q\right) n_r$$

Discontinuous Galerkin formulation

- Resulting bi-linear weak form:

$$a(\mathbf{u},\mathbf{w}) \;=\; b(\mathbf{w})$$

with

$$a(\mathbf{u},\mathbf{w}) = \sum_e \int_{\Omega_e} w_{k,i}\sigma_{ik} d\Omega + \sum_e \int_{\Omega_e} w_{k,ij}\tau_{jik} d\Omega$$

New inter-element contributions

$$\left\{ \begin{aligned} &+ \int_{\partial_I \Omega} [\![w_{k,i}]\!] \left[\langle \tau_{jik} \rangle n_j + n_j \left\langle \frac{\beta J_{jikqrp}}{h} \right\rangle [\![u_{p,q}]\!] n_r \right] dS \\ &- \int_{\partial_T \Omega} w_{k,l} n_l n_i \left[\tau_{jik} n_j + n_j \frac{\beta J_{jikqrp}}{h} n_s u_{p,s} n_q n_r \right] dS \end{aligned} \right.$$

$$b(\mathbf{w}) = \sum_e \int_{\Omega_e} w_k b_k d\Omega + \int_{\partial_N \Omega} w_k \hat{t}_k dS + \int_{\partial_M \Omega} w_{k,l} n_l \hat{r}_k dS$$

New inter-element contributions

$$\left\{ - \int_{\partial_T \Omega} w_{k,l} n_l n_i n_j \frac{\beta J_{jikqrp}}{h} \overline{D u_p} n_q n_r dS \right.$$

Numerical Properties

- Consistency
  - Exact solution ***u*** satisfies the DG formulation $a(\mathbf{u}, \mathbf{w}) = b(\mathbf{w})$
- Definition of a new energy norm

$$|||\mathbf{v}|||^2 = \sum_e \left\| \sqrt{C_{ijkl}} v_{k,l} \right\|^2_{L^2(\Omega^e)} + \sum_e \left\| \sqrt{J_{ijklmn}} v_{n,lm} \right\|^2_{L^2(\Omega^e)} + \sum_e \frac{1}{2} \left\| \sqrt{\frac{J_{ijklmn}}{h}} \; [\![v_{n,l}]\!] \; n_m^- \right\|^2_{L^2(\partial\Omega^e \cap \partial_I B_h)}$$

- Stability

$a(\mathbf{u}, \mathbf{u}) \geq C_2(\beta) \, |||\mathbf{u}|||^2 > 0$ with $C_2>0$ if $\beta > C^k$, C^k depends only on k.

- Convergence rate of the error with the mesh size:

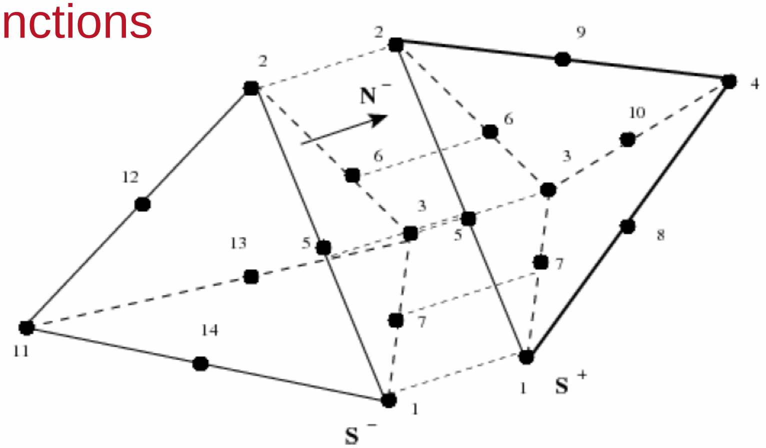
$$|||e||| = \sum_e C h^{k-1} |\mathbf{u}|_{\mathbf{H}^{k+1}(\partial\Omega^e)}$$

FEM 3D-implementation

- Linear system $K_{iakb}U_{kb} = f_{ia}$ with $K_{iakb} = \sum_e K^e_{iakb} + \sum_I K^I_{iakb} + \sum_B K^B_{iakb}$

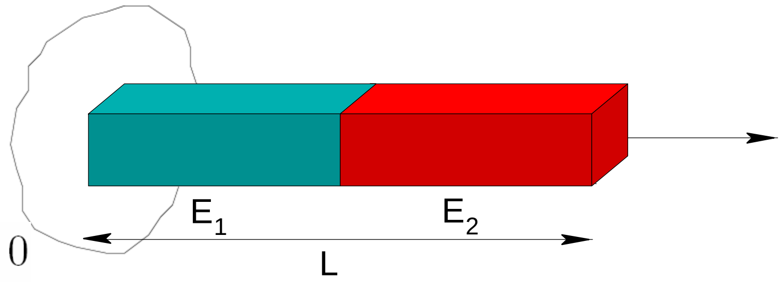
  Volume term Interface term Boundary term

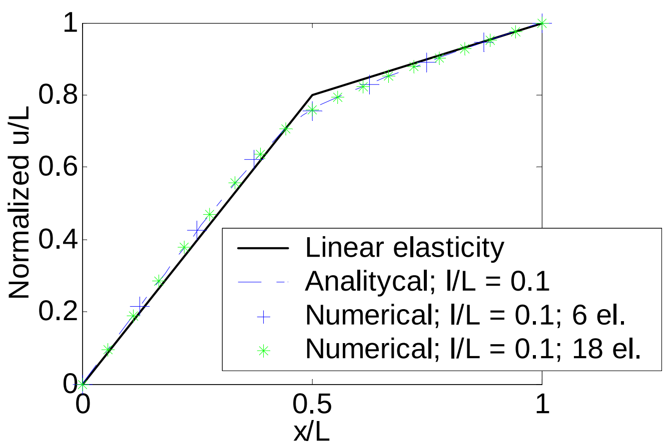
- Volume term $K^e_{akbl} = \int_{\Omega_e} \left(N_{a,ij} J_{jikmnl} N_{b,mn} + N_{a,i} C_{iklm} N_{b,m} \right) d\Omega$
  - 10-node isoparametric tetrahedra
  - 4 Gauss quadrature points
  - Needs up to second derivative of shape functions
- Interface & Boundary terms
  - No duplication of nodes (C^0 continuous)
  - Geometric data generated from *B-Rep* (Radovitzky 1999)
  - Derivatives of shape functions of adjacent tetrahedra stored on the facet
  - 6 quadrature points per interface

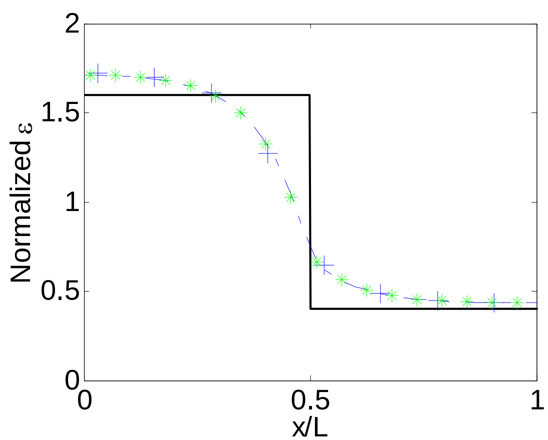


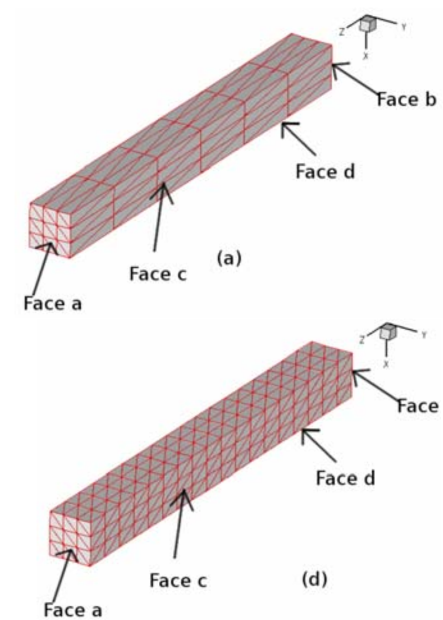
Numerical Examples

- Bi-material tensile test
  - $E_1/E_2=4$
  - Characteristic length l, with l/L=0.1
  - Differential equation: $E\left(u_{,xx} - l^2 u_{,xxxx}\right) = 0$
- 2 meshes are considered:
  - 6 & 18 tetrahedra on the length
  - Convergence toward analytical solution



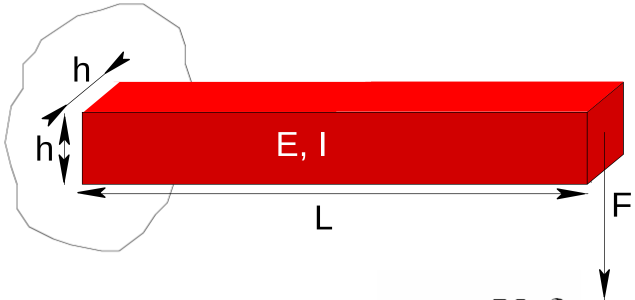




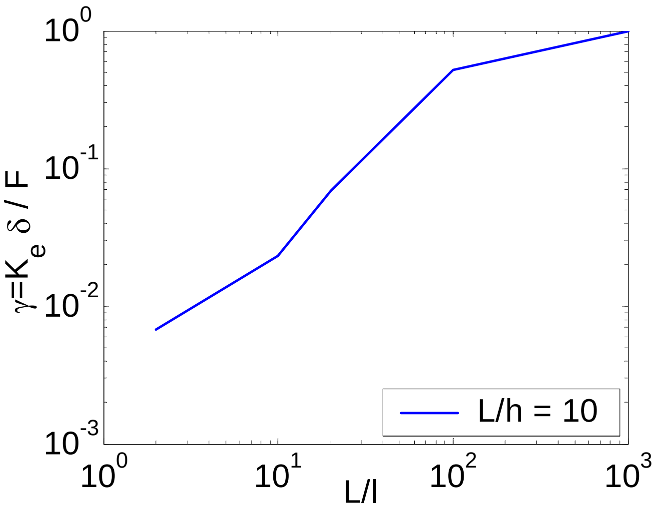


Numerical Examples

- Study of bending stiffness *K*
  - Elastic bending stiffness $K_e = \frac{F}{\delta} = \frac{3EI}{L^3}$

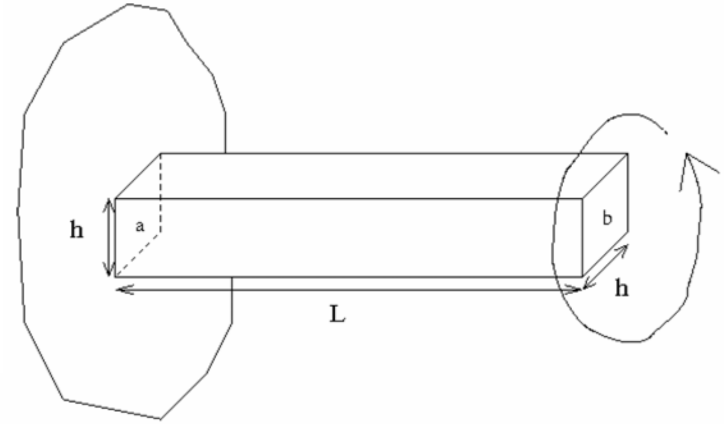


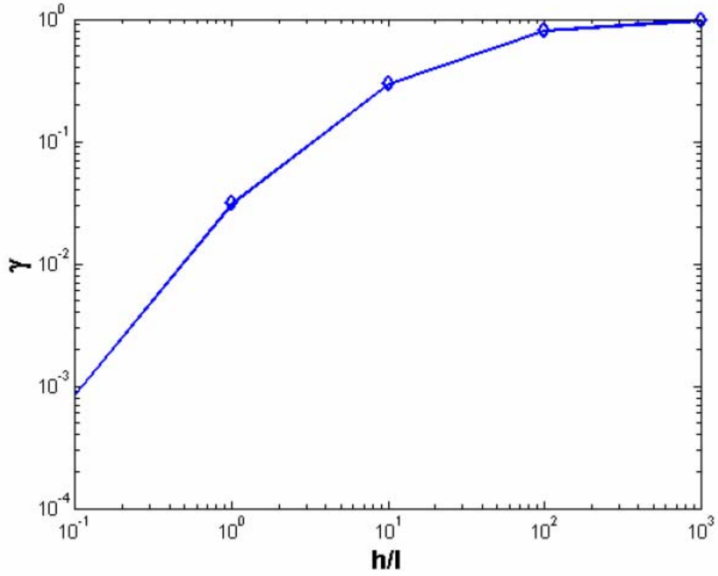
  - Influence of characteristic length l on the effective stiffness: $\gamma = \frac{K_e \delta}{F}$



Numerical Examples

- Study of torsion stiffness *K*
  - Elastic torsion stiffness K_e
  - Influence of characteristic length l on the effective stiffness: $\gamma = K_e/K$





Conclusions & Future work

- Conclusions:
  - Development of discontinuous Galerkin framework for linear strain gradient elasticity:
    - Single field formulation
    - Strong enforcement of C^0 continuity
    - No new degrees of freedom
    - Weak enforcement of C^1 continuity
    - Higher order Dirichlet condition enforced weakly
  - Implementation in a 3D finite-elements code
  - Passes standard patch tests
  - Size effects of gradient law demonstrated
- Future work
  - Consideration of the symmetrization term (super-convergence in L2-norm)
  - Application to crystal plasticity