Development of discontinuous Galerkin method for linear strain gradient elasticity

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Introduction & Motivation

- Length scales in modern technology are now of the order of the micrometer or nanometer.
- At these scales, material laws depend on strain but also on strain-gradient.
- Example:
  - Bi-material tensile test:
    - $E_1/E_2=4$
    - Characteristic length $l$
    - Differential equation: $E \left( u_{,xx} - l^2 u_{,xxx} \right) = 0$
Introduction & Motivation

- Introduction of strain-gradient effect in numerical simulations
  - Domain of applications:
    - Stress concentrations (around hole, at crack tip, …)
    - Grain size effect on polycrystalline yield strength
    - Void growth
    - …
  - Finite elements framework
  - In the general 3D case, shape functions are not $C^1$, which prevents the direct evaluation of the strain gradients

- Idea: enforcing weakly the $C^1$ continuity by recourse to discontinuous Galerkin methods
Introduction & Motivation

• Discontinuous Galerkin methods
  – Finite element discretizations which allow for jump across elements
  – Compatibility of the field variable or its spatial derivative is imposed in a weak sense
  – Stability enforced with a quadratic interelement integrals

• Application of discontinuous Galerkin methods in solid mechanics
  – Allow weak enforcement of $C^0$ continuity:
    • Non-linear mechanics (Ten Eyck and Lew 2006, Noels and Radovitzky 2006)
    • Reduction of locking for shells (Güzey et al. 2006)
    • Beams and plates (Arnold et al. 2005, Celiker and Cockburn 2007)
  – Allow weak enforcement of $C^1$ continuity (strong enforcement of $C^0$):
    • Beams and plates (Engel et al. 2002)
    • Strain gradient (1D) (Molari et al. 2006)
    • Kirchhoff-Love shells (Noels and Radovitzky 2007)
Introduction & Motivation

• Purpose of the presentation is to develop a dG formulation for strain gradient elasticity, which
  – Is a single field formulation in displacement
  – Requires only the use of $C^0$ continuous interpolations
  – Is demonstrated to be consistent and stable
  – Is easy to integrate into a regular 3D finite-element code
  – Has $C^1$ continuity constrained in a weak sense

• Scope of this presentation
  – Strain gradient theory of elasticity
  – Discontinuous Galerkin formulation
  – Numerical properties
  – FEM 3D implementation
  – Numerical examples
  – Conclusions & Future work
Strain gradient theory of elasticity

- **Strain gradient theory:**
  - At a material point stress is a function of strain and of the gradient of strain \((\text{Toupin 1962, Mindlin 1964})\)
  - Strain energy \(W = W(\varepsilon_{ij}, \eta_{ijk})\) is assumed to be a function of strain and gradient of strain
  - Low and high order stresses introduced as the work conjugate of low and high order strains
    \[
    \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} \quad \tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}} = J_{ijklmn} \eta_{lmn}
    \]
  - Governing PDE obtained from satisfying the virtual work statement
    \[
    \int_{B_0} \left( \sigma_{ij} \delta \varepsilon_{ij} + \tau_{ijk} \delta \eta_{ijk} \right) dV = \int_{B_0} \hat{b}_k \delta u_k dV + \int_{\partial_N B} \hat{t}_k \delta u_k dS + \int_{\partial_M B} \hat{r}_{k,i} \delta u_{k,i} n_i dS
    \]
    - Body forces
    - Low order tractions
    - Double stress tractions
Strain gradient theory of elasticity

- The boundary value problem:
  - Local equation
    \[ 0 = b_k + (\sigma_{ik} - \tau_{ijk,j})_i \text{ in } B_0 \]
  - Natural boundary conditions
    \[ \hat{t}_k = n_i (\sigma_{ik} - \partial_j \tau_{ijk}) + n_i n_j \tau_{ijk} (D_p n_p) - D_i (n_j \tau_{ijk}) \text{ on } \partial_N B_0 \]
    \[ \hat{r}_k = n_i n_j \tau_{ijk} \text{ on } \partial_M B_0 \]
  - Essential boundary conditions
    \[ u_k = \overline{u}_k \text{ on } \partial_D B_0 \]
    \[ n_i u_{k,i} = \overline{D} u_{k,i} \text{ on } \partial_T B_0 \]
  - Finite-elements discretization
    \[ \bigcup_{e=1}^{E} \Omega_e = B_{0h} \approx B_0 \]

\[ \partial_N B_0 \cup \partial_D B_0 = \partial B_0 \]
\[ \partial_M B_0 \cup \partial_T B_0 = \partial B_0 \]
\[ \partial_N B_0 \cap \partial_D B_0 = \emptyset \]
\[ \partial_M B_0 \cap \partial_T B_0 = \emptyset \]
Discontinuous Galerkin formulation

• Derivation of the weak form:
  – Choose the appropriate space for the test \( (u_h) \) and trial functions \( (w) \), which:
    • Are \( C^0 \) on the whole domain
    • Are \( P^k \) in each element
    • Satisfy the essential BC’s
  – Multiply the local equation by a test function
    \[
    \sum_{e=1}^{E} \int_{\Omega_e} w_k \left( b_k + (\sigma_{ik} - \tau_{jik,j,i}) \right) dV = 0
    \]
  – Integrate by parts and use divergence theorem
    \[
    \sum_{e=1}^{E} \int_{\partial\Omega_e} w_k (\sigma_{ik} - \tau_{jik,j}) n_i dS - \sum_{e=1}^{E} \int_{\Omega_e} w_{k,i} \sigma_{ik} dV - \sum_{e=1}^{E} \int_{\Omega_e} w_{k,ij} \tau_{jik} dV + \sum_{e=1}^{E} \int_{\partial\Omega_e} w_{k,i} \tau_{jik} n_j dS + \sum_{e=1}^{E} \int_{\Omega_e} w_k b_k dV = 0
    \]
    Introduces inter-element contributions
Discontinuous Galerkin formulation

- Introduction of the numerical fluxes:
  - On inter-element boundaries
    \[ \sum_{e=1}^{E} \int_{\partial \Omega_e \cap \partial \Omega} w_{k,i} \tau_{jik} n_j dS \approx - \int_{\partial \Omega} [w_{k,i}] \tau_{jik} n_j^- dS \]
    \[ \tau_{jik}^- n_j = \langle \tau_{jik} \rangle n_j + n_j \left( \frac{\beta J_{jikqr}}{h} \right) [u_{p,q}] n_r \]
    Ensures consistency
    Ensures stability ($h$ = mesh size and $\beta$ = parameter)

- Extension to weak enforcement of high-order BC
    \[ \sum_{e=1}^{E} \int_{\partial \Omega_e \cap \partial \Omega} n_i w_{k,l} n_i \tau_{jik} n_j dS \approx \int_{\partial \Omega} n_i w_{k,l} n_i \tau_{jik} n_j^- dS \]
    \[ \tau_{jik}^- n_j = \tau_{jik} n_j + n_j \frac{\beta J_{jikqr}}{h} \left( n_s u_{p,q} n_q - Du_p n_q \right) n_r \]
Discontinuous Galerkin formulation

- Resulting bi-linear weak form:

\[ a(\mathbf{u}, \mathbf{w}) = b(\mathbf{w}) \]

with

\[
a(\mathbf{u}, \mathbf{w}) = \sum_e \int_{\Omega_e} w_{k,i} \sigma_{ik} d\Omega + \sum_e \int_{\Omega_e} w_{k,ij} \tau_{jik} d\Omega \\
+ \int_{\partial I \Omega} \left[ w_{k,i} \right] \left[ \langle \tau_{jik} \rangle n_j + n_j \left( \frac{\beta J_{jikqp}}{h} \right) [u_{p,q}] n_r \right] dS \\
- \int_{\partial T \Omega} w_{k,in_i n_i} \left[ \tau_{jik} n_j + n_j \frac{\beta J_{jikqp}}{h} n_s u_{p,s} n_q n_r \right] dS
\]

\[
b(\mathbf{w}) = \sum_e \int_{\Omega_e} w_k b_k d\Omega + \int_{\partial N \Omega} w_k \hat{t}_k dS + \int_{\partial M \Omega} w_{k,in_i \hat{r}_k} dS \\
- \int_{\partial T \Omega} w_{k,in_i n_i n_j} \frac{\beta J_{jikqp}}{h} D u_{p,n_q n_r} dS
\]
Numerical Properties

- **Consistency**
  - Exact solution \( u \) satisfies the DG formulation \( a(u, w) = b(w) \)

- **Definition of a new energy norm**
  
  \[
  \|v\|^2 = \sum_e \left\| \sqrt{C_{ijkl}^e} u_{k,l} \right\|_{L^2(\Omega^e)}^2 + \sum_e \left\| \sqrt{J_{ijklmn}^e} v_{n,lm} \right\|_{L^2(\Omega^e)}^2 + \sum_e \frac{1}{2} \left\| \sqrt{J_{ijklmn}^e} \left[ v_{n,l} \right] n_m \right\|_{L^2(\partial\Omega^e \cap \partial B_h)}^2
  \]

- **Stability**
  
  \[ a(u, u) \geq C_2(\beta) \|u\|^2 > 0 \] with \( C_2 > 0 \) if \( \beta > C^k \), \( C^k \) depends only on \( k \).

- **Convergence rate of the error with the mesh size**:
  
  \[ \|e\| = \sum_e C h^{k-1} |u|_{H^{k+1}(\partial \Omega^e)} \]
FEM 3D-implementation

- Linear system $K_{iakb}U_{kb} = f_{ia}$ with $K_{iakb} = \sum_{e} K_{iakb}^{e} + \sum_{I} K_{iakb}^{I} + \sum_{B} K_{iakb}^{B}$
  - Volume term
  - Interface term
  - Boundary term

- Volume term $K_{akbl}^{e} = \int_{\Omega_{e}} \left( N_{a,ij} J_{jikmn} N_{b,mn} + N_{a,i} C_{iklm} N_{b,m} \right) d\Omega$
  - 10-node isoparametric tetrahedra
  - 4 Gauss quadrature points
  - Needs up to second derivative of shape functions

- Interface & Boundary terms
  - No duplication of nodes ($C^0$ continuous)
  - Geometric data generated from B-Rep (Radovitzky 1999)
  - Derivatives of shape functions of adjacent tetrahedra stored on the facet
  - 6 quadrature points per interface
Numerical Examples

- Bi-material tensile test
  - $E_1/E_2=4$
  - Characteristic length $l$, with $l/L=0.1$
  - Differential equation: $E \left( u_{xx} - l^2 u_{xxxx} \right) = 0$

- 2 meshes are considered:
  - 6 & 18 tetrahedra on the length
  - Convergence toward analytical solution

[Graphs showing normalized displacement and strain for linear elasticity and numerical solutions with different mesh sizes and characteristic lengths.]
Numerical Examples

- Study of bending stiffness $K$
  - Elastic bending stiffness
    $$K_e = \frac{F}{\delta} = \frac{3EI}{L^3}$$
  - Influence of characteristic length $l$ on the effective stiffness:
    $$\gamma = \frac{K_e \delta}{F}$$

![Graph showing the relationship between $\gamma$ and $L/l$ for $L/h = 10$.]
Numerical Examples

- Study of torsion stiffness $K$
  - Elastic torsion stiffness $K_e$
  - Influence of characteristic length $l$ on the effective stiffness: $\gamma = K_e/K$
Conclusions & Future work

• Conclusions:
  – Development of discontinuous Galerkin framework for linear strain gradient elasticity:
    • Single field formulation
    • Strong enforcement of $C^0$ continuity
    • No new degrees of freedom
    • Weak enforcement of $C^1$ continuity
    • Higher order Dirichlet condition enforced weakly
  – Implementation in a 3D finite-elements code
  – Passes standard patch tests
  – Size effects of gradient law demonstrated

• Future work
  – Consideration of the symmetrization term (super-convergence in L2-norm)
  – Application to crystal plasticity