A New Discontinuous Galerkin Formulation for Kirchhoff-Love Shells

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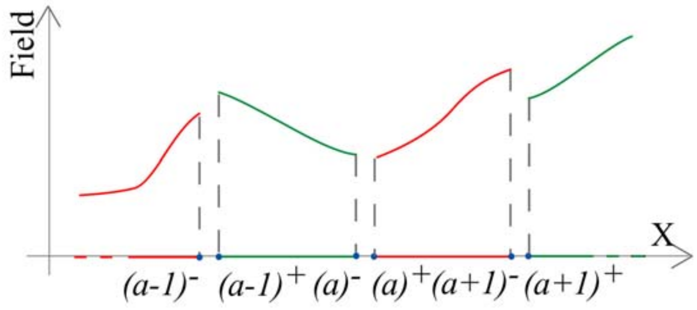
Introduction

- Discontinuous Galerkin methods
  – Finite-element discretization allowing for inter-elements discontinuities
  – Weak enforcement of compatibility equations and continuity (C^0 or C^1, …) through interelement integrals called numerical fluxes
  – Stability is ensured with quadratic interelement integrals



- Applications of DG to solid mechanics
  – Allowing weak enforcement of C^0 continuity:
    - Non-linear mechanics (Noels and Radovitzky 2006; Ten Eyck and Lew 2006)
    - Reduction of locking for shells (Güzey et al. 2006)
    - Beams and plates (Arnold et al. 2005, Celiker and Cockburn 2007)
  – Allowing weak enforcement of C^1 continuity (strong enforcement of C^0):
    - Beams and plates (Engel et al. 2002)
    - Strain gradient continuity (Molari et al. 2006)

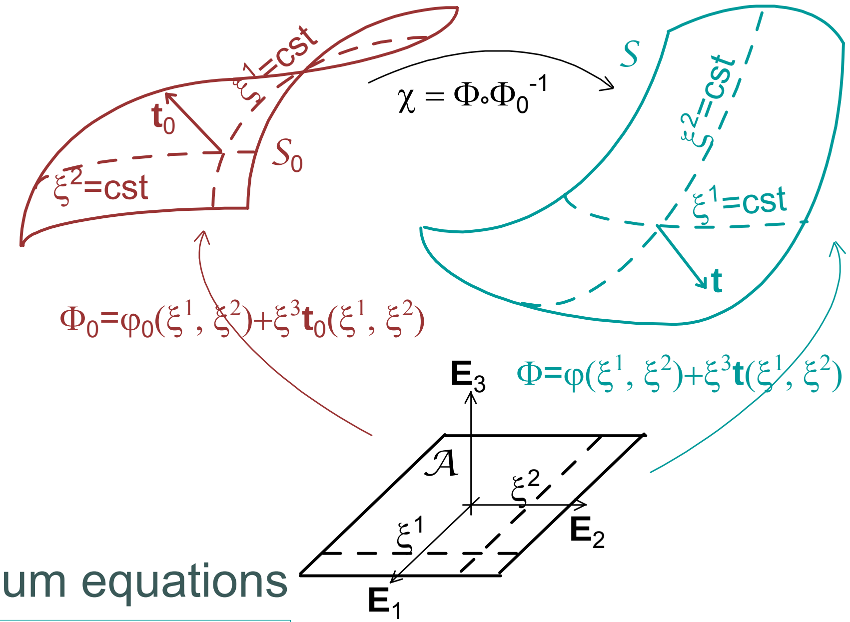
Introduction

- Purpose of the presentation: to develop a DG formulation
  - for Kirchhoff-Love shells,
  - which is a C^0 displacement formulation,
  - without addition of degrees of freedom,
  - where C^1 continuity is enforced by DG interface terms,
  - which leads to an easy implementation of the shell elements in the reduced coordinates,
  - without locking in bending
- Scope of the presentation
  - Kirchhoff-Love shells
  - DG formulation
  - Numerical properties
  - Implementation
  - Numerical examples

Kirchhoff-Love shells

- Kinematics of the shell
  - Shearing is neglected

$$\boldsymbol{t} = \frac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|} \quad \Longrightarrow \quad \begin{cases} \boldsymbol{t}_{,\alpha} = \lambda_{\alpha}^{\mu} \boldsymbol{\varphi}_{,\mu} \\ \bar{j} = \|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\| \end{cases}$$



  - Small displacements formulation

$$\boldsymbol{\varphi}_{,\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \boldsymbol{u}_{,\alpha} \quad \Longrightarrow \quad \boldsymbol{t}\left(\boldsymbol{u}\right) = \boldsymbol{t}_0 + \boldsymbol{\Delta t}\left(\boldsymbol{u}\right)$$

  - Resultant linear and angular equilibrium equations

$$\boxed{\frac{1}{\bar{j}_0}\left(\bar{j}_0 \boldsymbol{n}^{\alpha}\right)_{,\alpha} + \boldsymbol{n}^{\mathcal{A}} = 0} \quad \text{and} \quad \boxed{\frac{1}{\bar{j}_0}\left(\bar{j}_0 \tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha} - \boldsymbol{l} + \lambda \boldsymbol{t}_0 + \tilde{\boldsymbol{m}}^{\mathcal{A}} = 0},$$

in terms of the resultant stress components $\tilde{n}^{\alpha\beta}$ and $\tilde{m}^{\alpha\beta}$, with

$$\boxed{\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{0,\beta} + \boxed{\lambda_{\mu}^{\beta}\tilde{m}^{\alpha\mu}}\boldsymbol{\varphi}_{0,\beta}}, \quad \boxed{\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\alpha\beta}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0} \quad \text{and} \quad \boldsymbol{l} = \lambda \boldsymbol{t}_0 + \lambda_{\mu}^{\alpha}\tilde{m}^{3\mu}\boldsymbol{\varphi}_{0,\alpha}$$

$\boldsymbol{n}^{\mathcal{A}}$ and $\tilde{\boldsymbol{m}}^{\mathcal{A}}$ are the resultant applied tension and torque

Kirchhoff-Love shells

- Constitutive behavior and BC
  - Resultant strain components

$$\begin{cases} \boxed{\varepsilon_{\alpha\beta} = \frac{1}{2}\boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{\varphi}_{,\beta} - \frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} = \frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{u}_{,\beta} + \frac{1}{2}\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta}} \\ \boxed{\begin{aligned}\rho_{\alpha\beta} &= \boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{t}_{,\beta} - \boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{t}_{0,\beta} \\ &= \boldsymbol{\varphi}_{0,\alpha\beta}\cdot\boldsymbol{t}_0\frac{e_{\mu\eta3}}{\bar{j}_0}\boldsymbol{u}_{,\mu}\cdot\left(\boldsymbol{\varphi}_{0,\eta}\wedge\boldsymbol{t}_0\right) + \frac{e_{\mu\eta3}}{\bar{j}_0}\boldsymbol{u}_{,\mu}\cdot\left(\boldsymbol{\varphi}_{0,\alpha\beta}\wedge\boldsymbol{\varphi}_{0,\eta}\right) - \boldsymbol{u}_{,\alpha\beta}\cdot\boldsymbol{t}_0\end{aligned}} \end{cases} \longrightarrow \text{high order}$$

  - Linear constitutive relations

$$\begin{cases} \boxed{\tilde{n}^{\alpha\beta} = \frac{E\left(h_{\max}-h_{\min}\right)}{1-\nu^2}\mathcal{H}^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}} \\ \boxed{\tilde{m}^{\alpha\beta} = \frac{E\left(h_{\max}-h_{\min}\right)^3}{12\left(1-\nu^2\right)}\mathcal{H}^{\alpha\beta\gamma\delta}\rho_{\gamma\delta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}} \end{cases}$$

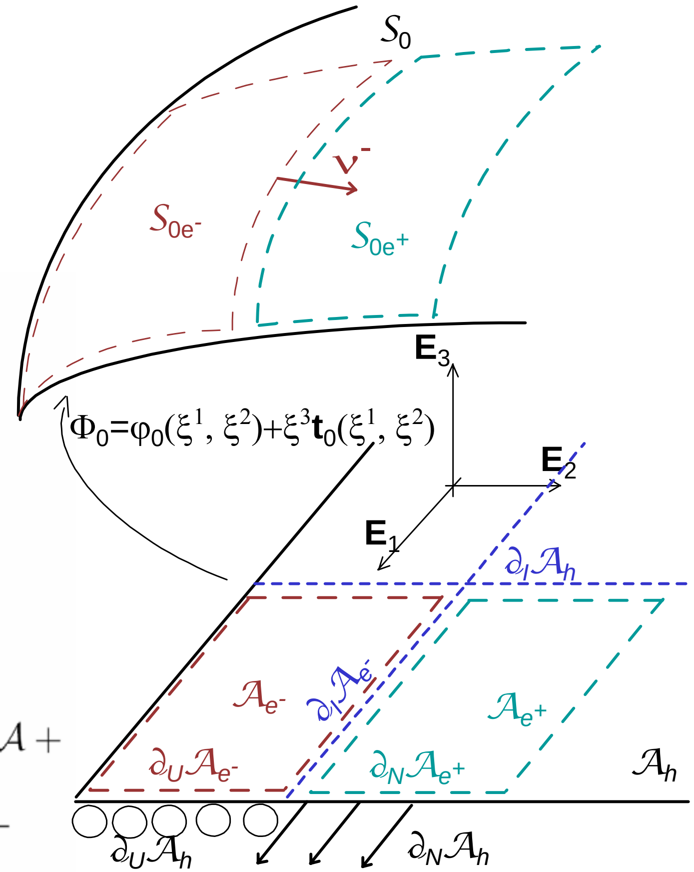
  - Boundary conditions

$$\boxed{\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\varphi}_0 + \bar{\boldsymbol{u}} & \forall\left(\xi^1,\,\xi^2\right) \in \partial_U\mathcal{A} \\ \boldsymbol{n}^\alpha\nu_\alpha &= \bar{\boldsymbol{n}} & \forall\left(\xi^1,\,\xi^2\right) \in \partial_N\mathcal{A}\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned}\boldsymbol{t} &= \boldsymbol{t}_0 + \bar{\boldsymbol{\Delta t}} & \forall\left(\xi^1,\,\xi^2\right) \in \partial_T\mathcal{A} \\ \tilde{\boldsymbol{m}}^\alpha\nu_\alpha &= \bar{\tilde{\boldsymbol{m}}} & \forall\left(\xi^1,\,\xi^2\right) \in \partial_M\mathcal{A}\end{aligned}}$$

Discontinuous Galerkin formulation

- Hu-Washizu-de Veubeke functional
  - Polynomial approximation $\boldsymbol{u}_h \in \mathbb{P}^k \subset C^0$
  - New inter-elements term accounting for discontinuities in the derivatives

$$
\begin{aligned}
&I_h\left(\boldsymbol{u}_h, \varepsilon_{h\alpha\beta}, \rho_{h\alpha\beta}, \tilde{n}_h^{\alpha\beta}, \tilde{m}_h^{I\beta}, \lambda\right) = \\
&\int_{\mathcal{A}_h}\left(\frac{1}{2}\varepsilon_{h\alpha\beta}\mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{h\gamma\delta} + \frac{1}{2}\rho_{h\alpha\beta}\mathcal{H}_m^{\alpha\beta\gamma\delta}\rho_{h\gamma\delta}\right)\bar{j}_0 d\mathcal{A} + \\
&\int_{\mathcal{A}_h}\tilde{n}_h^{\alpha\beta}\left(\frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{u}_{h,\beta} + \frac{1}{2}\boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} - \varepsilon_{h\alpha\beta}\right)\bar{j}_0 d\mathcal{A} + \\
&\int_{\mathcal{A}_h}\tilde{m}_h^{\alpha\beta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta} - \rho_{h\alpha\beta}\right)\bar{j}_0 d\mathcal{A} - \\
&\int_{\mathcal{A}_h}\left(\boldsymbol{n}^{\mathcal{A}}\cdot\boldsymbol{u}_h + \tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\right)\bar{j}_0 d\mathcal{A} - \\
&\int_{\partial_U\mathcal{A}_h}\left(\boldsymbol{u}_h - \bar{\boldsymbol{u}}\right)\cdot\left(\tilde{n}_h^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \lambda_{0\mu}^{\beta}\tilde{m}_h^{\alpha\mu}\boldsymbol{\varphi}_{0,\beta} + \lambda\boldsymbol{t}_0\right)\nu_\alpha\bar{j}_0 d\partial\mathcal{A} + \\
&\int_{\partial_T\mathcal{A}_h \cup \partial_I\mathcal{A}_h}[\![\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)]\!]\cdot\left\langle\bar{j}_0\tilde{m}_h^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \bar{j}_0\tilde{m}_h^{3\alpha}\boldsymbol{t}_0\right\rangle\nu_\alpha d\partial\mathcal{A} - \\
&\int_{\partial_N\mathcal{A}_h}\bar{\boldsymbol{n}}\cdot\boldsymbol{u}_h\bar{j}_0 d\partial\mathcal{A} - \int_{\partial_M\mathcal{A}}\bar{\tilde{\boldsymbol{m}}}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\bar{j}_0 d\partial\mathcal{A}.
\end{aligned}
$$



Discontinuous Galerkin formulation

- Minimization of the functional (1/2)
  - With respect to the resultant strains ${\varepsilon_h}^{\alpha\beta}$ and ${\rho_h}^{\alpha\beta}$

$\boxed{\tilde{n}_h^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{h\gamma\delta}}$ and $\boxed{\tilde{m}_h^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\rho_{h\gamma\delta}}$

  - With respect to the resultant stresses $\tilde{n}^{\alpha\beta}$ and $\tilde{m}^{\alpha\beta}$

$$\left\{\begin{array}{l} \varepsilon_{h\alpha\beta} = \frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{u}_{h,\beta} + \frac{1}{2}\boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta}\,, \text{ in } \mathcal{A}_e \\ \rho_{h\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta} + \sum_{s\in\partial\mathcal{A}_e} r^s_{\alpha\beta}\left([\![\boldsymbol{\Delta t}]\!]\right) \text{ in } \bar{\mathcal{A}}_e \\ {\rho_h}^s_{\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta} + \beta r^s_{\alpha\beta}\left([\![\boldsymbol{\Delta t}]\!]\right) \text{ on } s\in\partial\mathcal{A}_e \end{array}\right.$$

Discontinuities result in new terms (lifting operators)

$$\boxed{\int_{\mathcal{A}_h} r^s_{\alpha\beta}\left([\![\boldsymbol{\Delta t}]\!]\right)\delta\tilde{m}^{\alpha\beta}\bar{j}_0 d\mathcal{A} = \int_s [\![\boldsymbol{\Delta t}]\!]\cdot\left\langle\boldsymbol{\varphi}_{0,\beta}\delta\tilde{m}^{\alpha\beta}\bar{j}_0\right\rangle\nu^-_\alpha d\partial\mathcal{A}}$$

and in the introduction of a stabilization parameter β.

  - With respect to the displacement field → balance equation (next slide)

Discontinuous Galerkin formulation

- Minimization of the functional (2/2)
  - With respect to the displacement field $\boldsymbol{u}_h$
  - Reduction to a one-field formulation

$\rightarrow$ $a(\boldsymbol{u}_h, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u})$ with

$$
\begin{aligned}
a(\boldsymbol{u}_h, \delta\boldsymbol{u}) = & \int_{\mathcal{A}_h} \left(\frac{1}{2}\boldsymbol{\varphi}_{0,\gamma}\cdot\boldsymbol{u}_{h,\delta} + \frac{1}{2}\boldsymbol{u}_{h,\gamma}\cdot\boldsymbol{\varphi}_{0,\delta}\right)\mathcal{H}_n^{\alpha\beta\gamma\delta}\left(\frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{u}_{,\beta} + \frac{1}{2}\boldsymbol{\varphi}_{0,\beta}\cdot\delta\boldsymbol{u}_{,\alpha}\right)\bar{j}_0 d\mathcal{A} + \\
& \int_{\mathcal{A}_h} \left(\boldsymbol{\varphi}_{0,\gamma}\cdot\boldsymbol{\Delta t}(\boldsymbol{u}_h)_{,\delta} + \boldsymbol{u}_{h,\gamma}\cdot\boldsymbol{t}_{0,\delta}\right)\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{\Delta t}(\boldsymbol{u})_{,\beta} + \delta\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0 d\mathcal{A} + \\
& \int_{\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} [\![\boldsymbol{\Delta t}(\boldsymbol{u}_h)]\!]\cdot\left\langle\boldsymbol{\varphi}_{0,\gamma}\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{\Delta t}(\boldsymbol{u})_{,\beta} + \delta\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0\right\rangle\nu_\delta^- d\partial\mathcal{A} + \\
& \int_{\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} [\![\delta\boldsymbol{\Delta t}(\boldsymbol{u})]\!]\cdot\left\langle\boldsymbol{\varphi}_{0,\gamma}\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}(\boldsymbol{u}_h)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0\right\rangle\nu_\delta^- d\partial\mathcal{A} + \\
& \int_{s\in\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} \frac{\beta}{h^s}[\![\delta\boldsymbol{\Delta t}]\!]\cdot\boldsymbol{\varphi}_{0,\gamma}\nu_\delta^-\left\langle\mathcal{H}_m^{\alpha\beta\gamma\delta}\bar{j}_0\right\rangle[\![\boldsymbol{\Delta t}(\boldsymbol{u}_h)]\!]\cdot\boldsymbol{\varphi}_{0,\alpha}\nu_\beta^- d\partial\mathcal{A}
\end{aligned}
$$

Mesh size

$$
b(\delta\boldsymbol{u}) = \int_{\mathcal{A}_h}\left(\boldsymbol{n}^{\mathcal{A}}\cdot\delta\boldsymbol{u} + \tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\delta\boldsymbol{\Delta t}(\boldsymbol{u})\right)\bar{j}_0 d\mathcal{A} + \int_{\partial_N\mathcal{A}_h}\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}\,\bar{j}_0 d\partial\mathcal{A} + \int_{\partial_M\mathcal{A}}\bar{\tilde{\boldsymbol{m}}}\cdot\delta\boldsymbol{\Delta t}(\boldsymbol{u})\,\bar{j}_0 d\partial\mathcal{A}
$$

Numerical properties

- Consistency
  - Exact solution ***u*** satisfies the DG formulation $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u})$
- Definition of an energy norm

$$|||\boldsymbol{u}|||^2 = \sum_e \left\| \sqrt{\mathcal{H}_n \bar{j}_0}^{\alpha\beta} \frac{1}{2}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot \boldsymbol{u}_{,\beta} + \boldsymbol{u}_{,\alpha}\cdot \boldsymbol{\varphi}_{0,\beta}\right) \right\|^2_{\mathrm{L}^2(\mathcal{A}_e)} + \sum_e \left\| \sqrt{\mathcal{H}_m \bar{j}_0}^{\alpha\beta} \left(\boldsymbol{\varphi}_{0,\alpha}\cdot \boldsymbol{\Delta t}_{,\beta} + \boldsymbol{u}_{,\alpha}\cdot \boldsymbol{t}_{0,\beta}\right) \right\|^2_{\mathrm{L}^2(\mathcal{A}_e)} + \sum_{s\in\partial_I\mathcal{A}_h} \left\| \sqrt{\frac{\beta\mathcal{H}_m \bar{j}_0}{h_s}}^{\alpha\beta} \boldsymbol{\varphi}_{0,\alpha}\cdot [\![\boldsymbol{\Delta t}]\!]\, \nu^-_\beta \right\|^2_{\mathrm{L}^2(s)}$$

- Stability

$a(\boldsymbol{u}_h,\, \boldsymbol{u}_h) \geq C(\beta)\, |||\boldsymbol{u}_h|||^2$ with $C>0$ if $\beta > C^k$, C^k depends only on k.

- Convergence rate of the error $\boldsymbol{e}^k = \boldsymbol{u}_h - \boldsymbol{u}^k$ in the mesh size h^s

		General	Pure bending	Pure membrane
– Energy norm:	$\lvert\lvert\lvert \boldsymbol{e}^k \rvert\rvert\rvert$	k-1	k-1	k
– L^2 norm:	$\lVert \boldsymbol{e}^k \rVert_{\mathrm{L}^2(\mathcal{A}_h)}$	≥ k-1	k+1 (if k>2)	k+1 (if k>0)

Motivates the use of quadratic elements

Implementation of 8-node bi-quadratic quadrangles

- Membrane equations

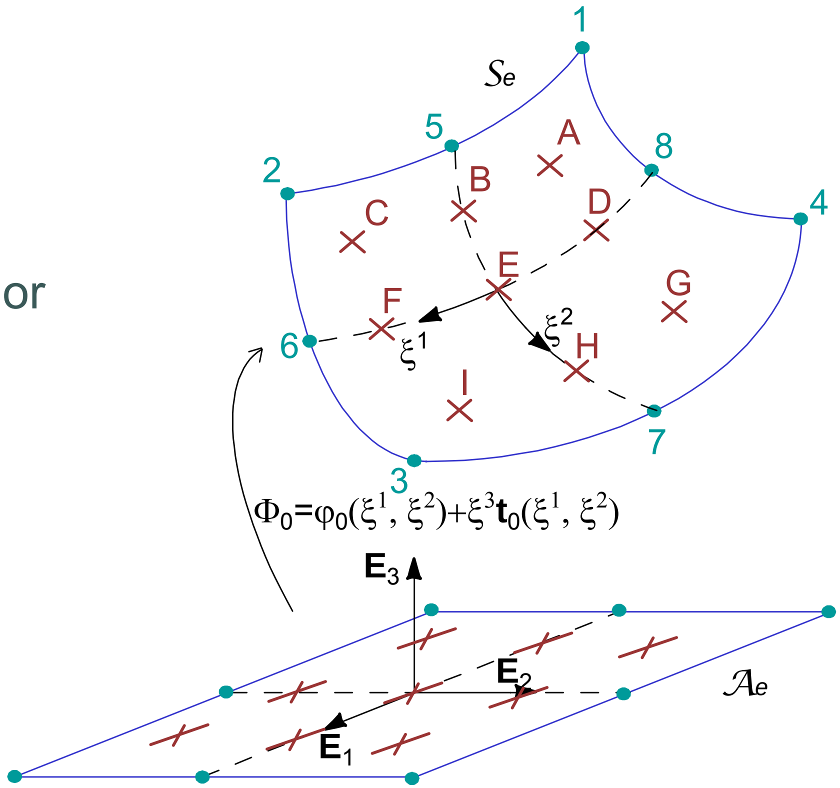
$$a_n^e\left(\boldsymbol{u}_h,\,\delta\boldsymbol{u}\right)=\int_{\mathcal{A}_e}\varepsilon_{\gamma\delta}\left(\boldsymbol{u}_h\right)\mathcal{H}_n^{\alpha\beta\gamma\delta}\delta\varepsilon_{\alpha\beta}\left(\boldsymbol{u}\right)\bar{j}_0 d\mathcal{A}$$

  – Solved in (ξ^1, ξ^2) system
  – 3 X 3 Gauss points with EAS method or 2 X 2 Gauss points

- Bending equations

$$a_m^e\left(\boldsymbol{u}_h,\,\delta\boldsymbol{u}\right)=\int_{\mathcal{A}_e}\rho_{\gamma\delta}\left(\boldsymbol{u}_h\right)\mathcal{H}_m^{\alpha\beta\gamma\delta}\delta\rho_{\alpha\beta}\left(\boldsymbol{u}\right)\bar{j}_0 d\mathcal{A}$$

  – Solved in (ξ^1, ξ^2) system
  – 3 X 3 or 2 X 2 Gauss points
  – Locking taken care of by the DG formulation



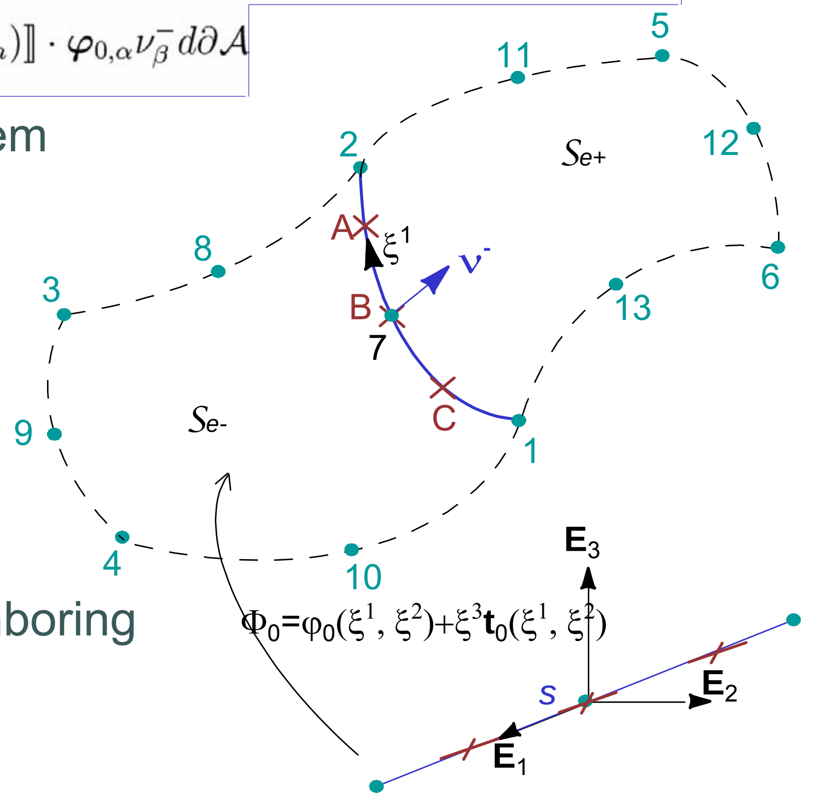
→ Straightforward implementation of the equations

Implementation of 8-node bi-quadratic quadrangles

- Interface equations

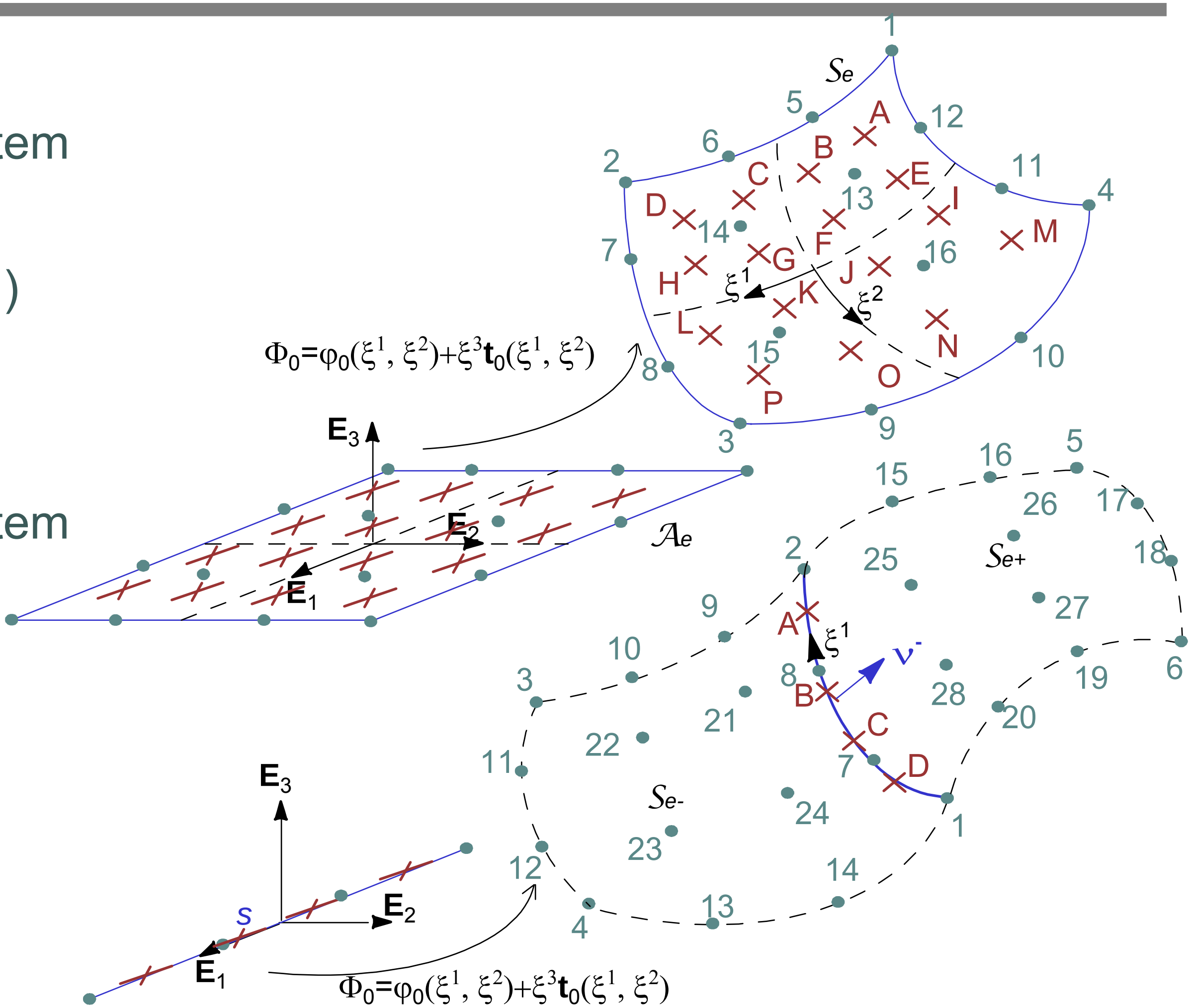
$$a_I^s\left(\boldsymbol{u}_h,\, \delta\boldsymbol{u}\right) = \int_s \left[\!\left[\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\right]\!\right] \cdot \left\langle \varphi_{0,\gamma} \mathcal{H}_m^{\alpha\beta\gamma\delta} \delta\rho_{\alpha\beta}\left(\boldsymbol{u}\right) \bar{j}_0 \right\rangle \nu_\delta^- \, d\partial\mathcal{A} + \int_s \left[\!\left[\delta\boldsymbol{\Delta t}\left(\boldsymbol{u}\right)\right]\!\right] \cdot \left\langle \varphi_{0,\gamma} \mathcal{H}_m^{\alpha\beta\gamma\delta} \rho_{\alpha\beta}\left(\boldsymbol{u}_h\right) \bar{j}_0 \right\rangle \nu_\delta^- \, d\partial\mathcal{A} + \int_s \left[\!\left[\delta\boldsymbol{\Delta t}\left(\boldsymbol{u}\right)\right]\!\right] \cdot \varphi_{0,\gamma} \nu_\delta^- \left\langle \frac{\beta \mathcal{H}_m^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle \left[\!\left[\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\right]\!\right] \cdot \varphi_{0,\alpha} \nu_\beta^- \, d\partial\mathcal{A}$$

  - Interface element *s* solved in ξ^1 system
  - 3 or 2 Gauss points
  - Neighboring elements $\mathcal{S}_{e^-}$ and $\mathcal{S}_{e^+}$ evaluate values ($\Delta\boldsymbol{t}$, $\delta\Delta\boldsymbol{t}$, ρ, $\delta\rho$, $\mathcal{H}_m$) on the interface Gauss points and send them to the interface element *s*
  - Local frame ($\varphi_{0,1}$, $\varphi_{0,2}$, $\boldsymbol{t}_0$) of interface element *s* is the average of the neighboring elements' frames



Implementation of 16-node bi-cubic quadrangles

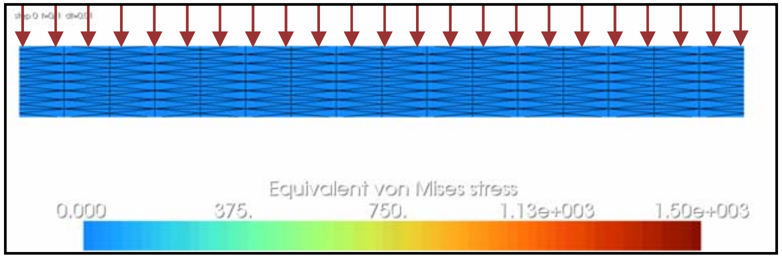
- Membrane equations
  - Solved in (ξ^1, ξ^2) system
  - 4 X 4 Gauss points
    (without EAS method)
- Bending equations
  - Solved in (ξ^1, ξ^2) system
  - 4 X 4 Gauss points
- Interface equations
  - Interface element *s* solved in ξ^1 system
  - 4 Gauss points



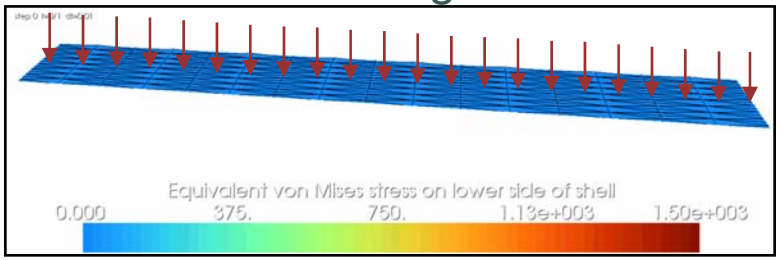
Numerical example: Cantilever beam (L/t = 10)

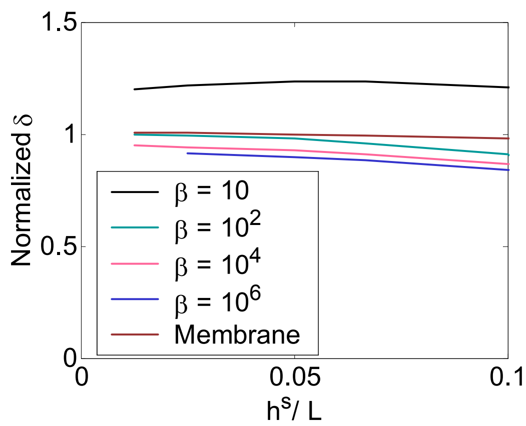
- 8-node bi-quadratic quadrangles

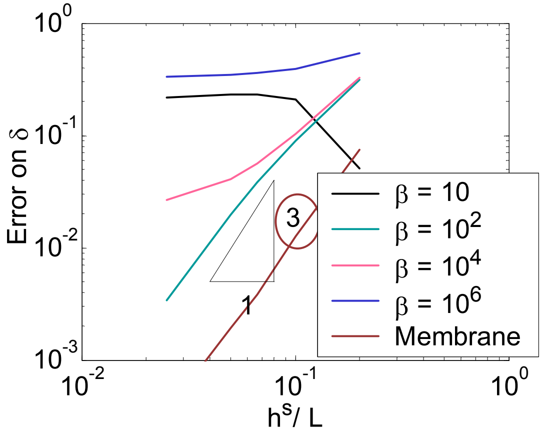
Membrane test

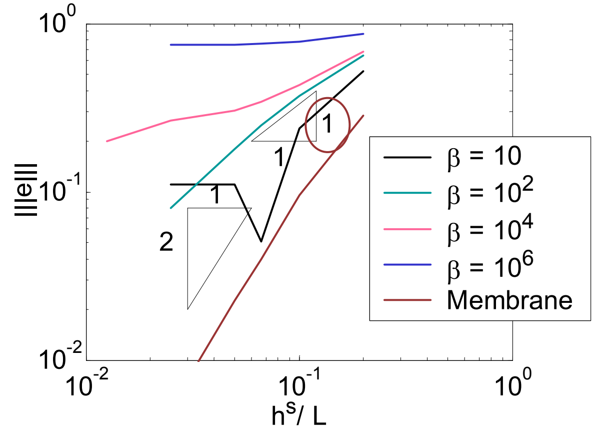


Bending test







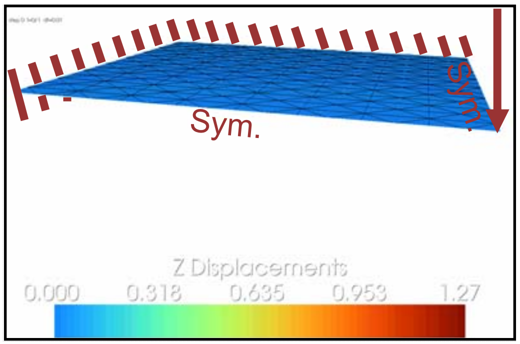


- Bending test:
  - Instability if $\beta \leq 10$ and locking if $\beta > 1000$
  - Convergence rate *k-1* in the energy-norm and *k+1* in the $\mathbf{L}^2$-norm

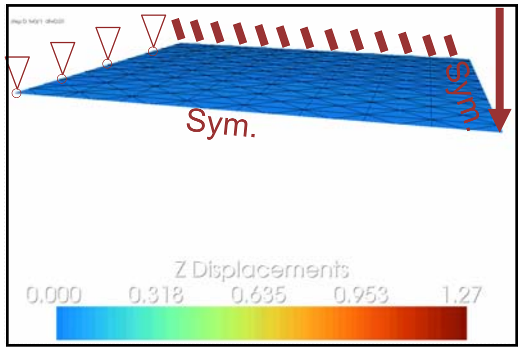
Numerical example: Plate bending (L/t = 100)

- 8-node bi-quadratic quadrangles

Clamped/Clamped



Supported/Clamped



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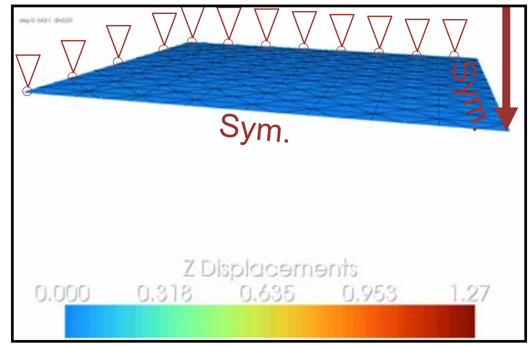
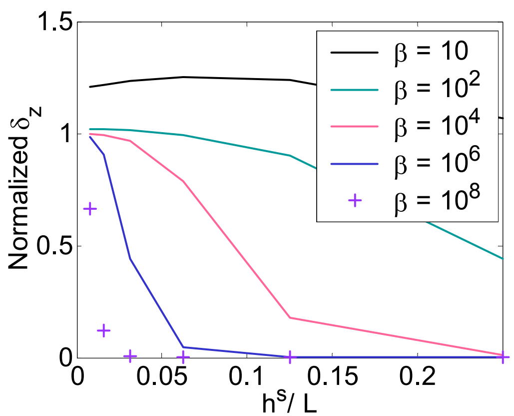
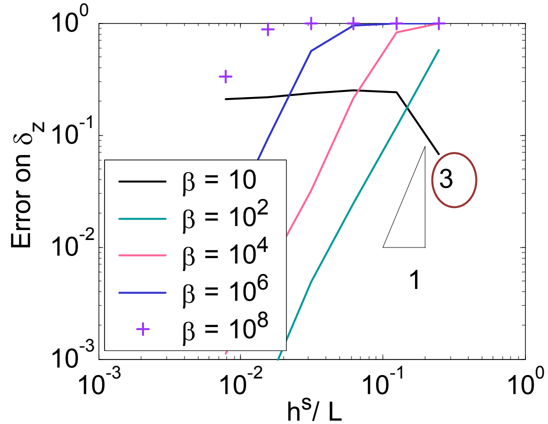
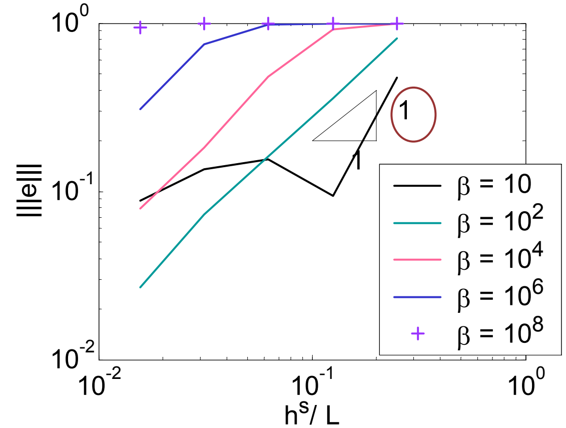




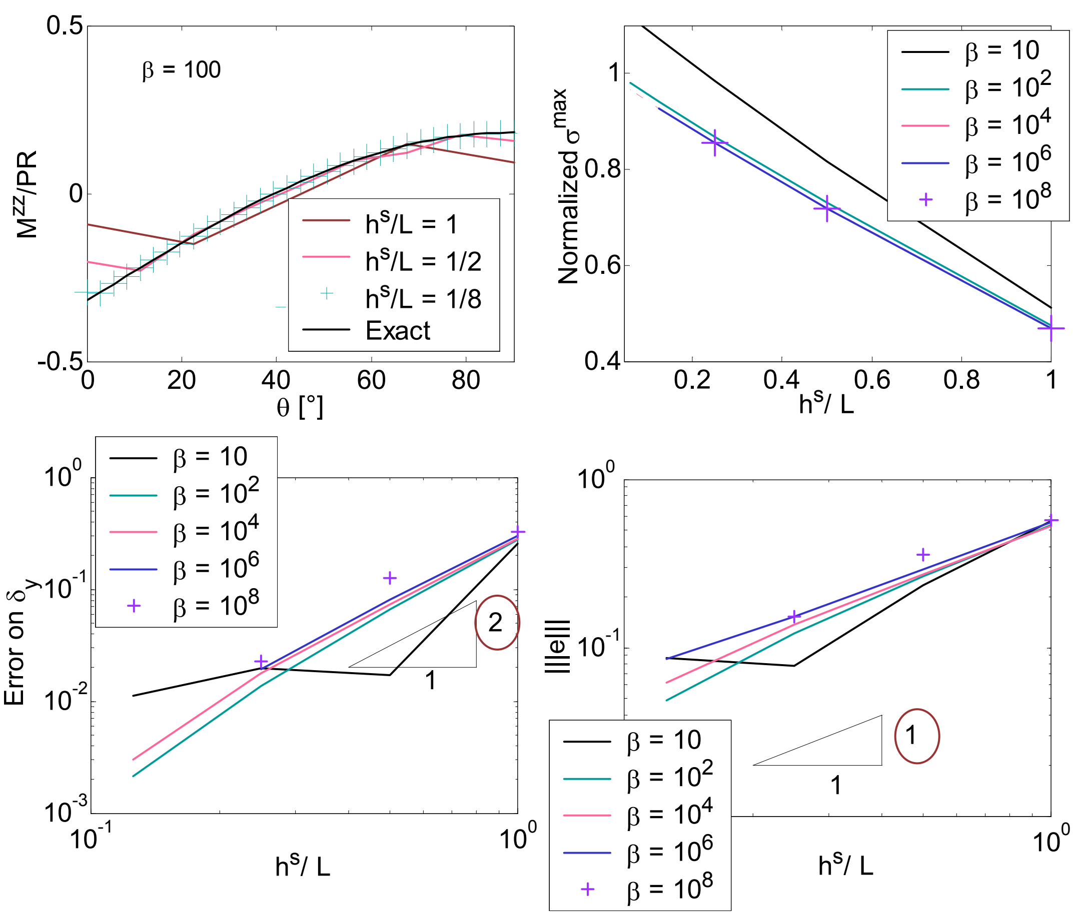


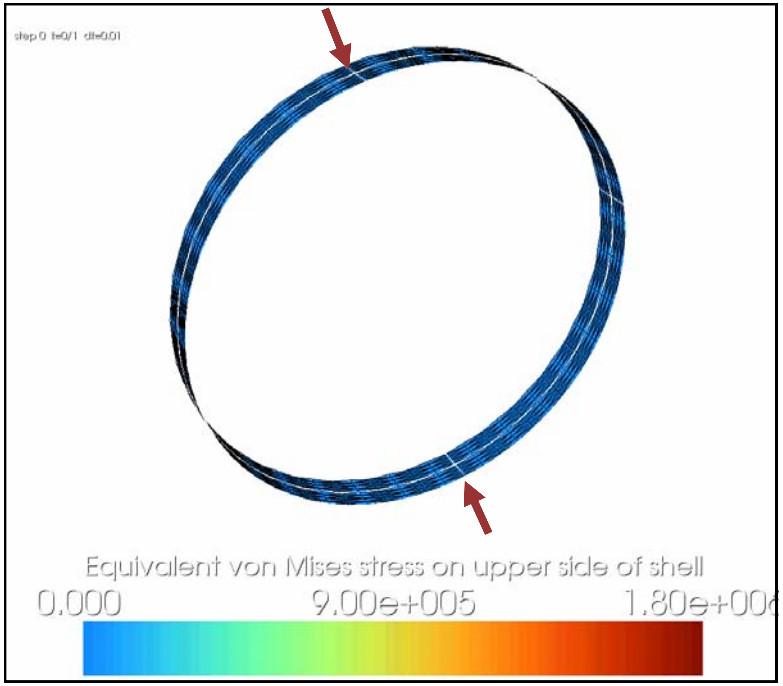


- Instability if $\beta \leq 10$ and locking if $\beta > 1000$

Numerical example: Pinched ring (R/t = 10)

- 8-node bi-quadratic quadrangles



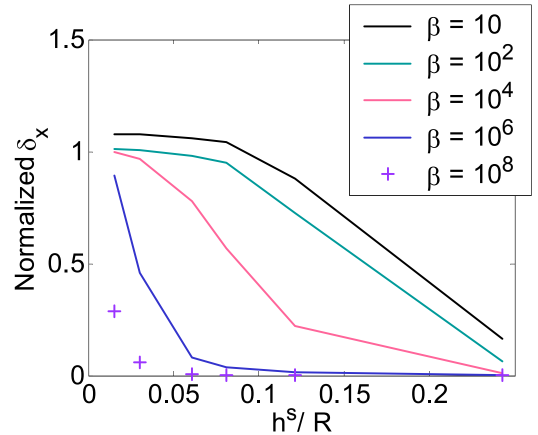


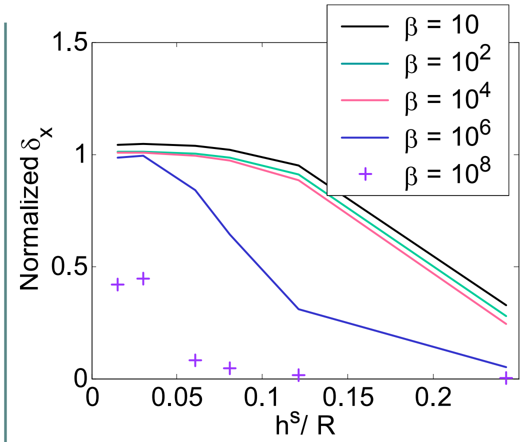
Bending and membrane coupling

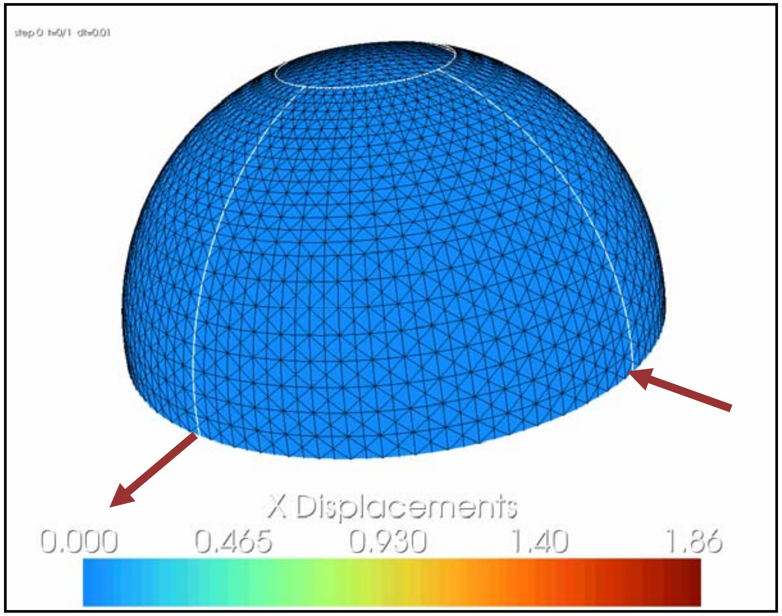
- Instability if $\beta \leq 10$
- Convergence: *k-1* in the energy-norm and *k* in the $\mathbf{L}^2$-norm

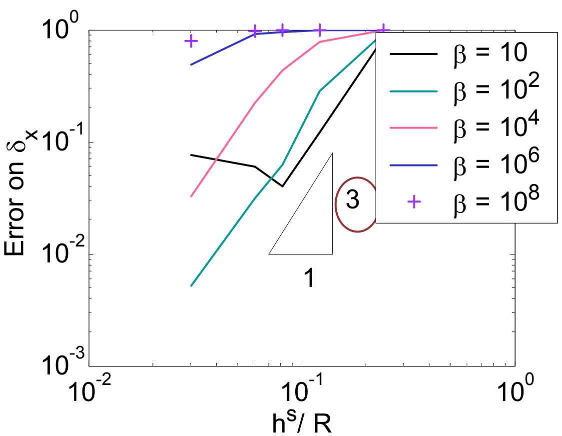
Numerical example: Pinched open-hemisphere (R/t = 250)

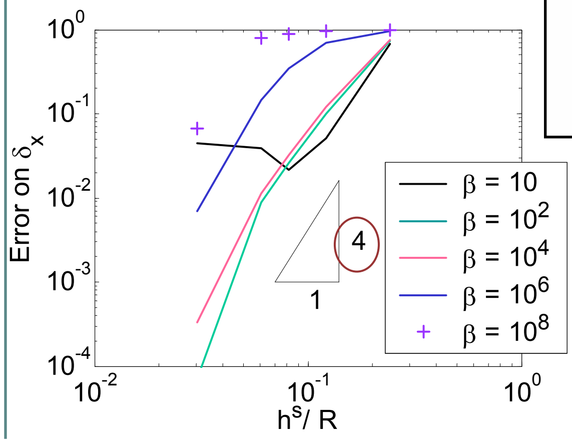
8-node bi-quad. 16-node bi-cub.









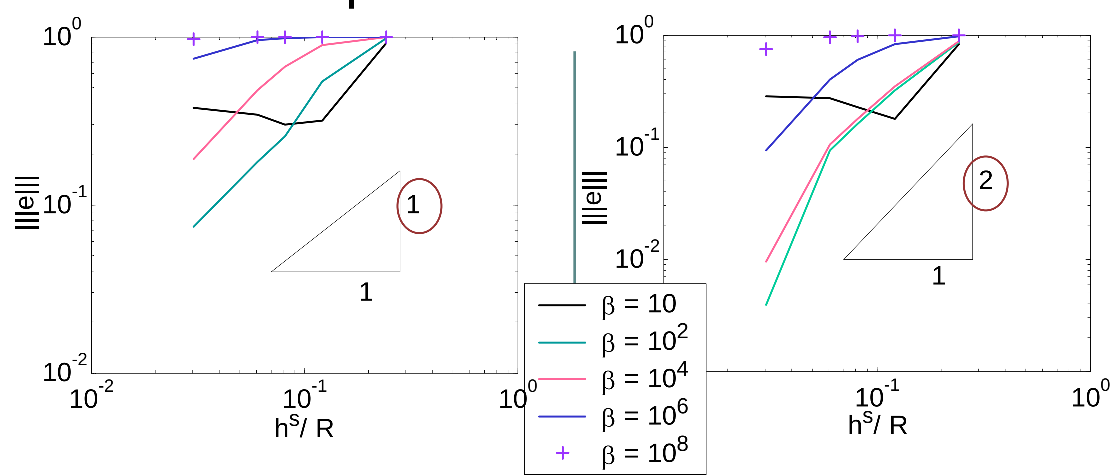


Double curvature

- Instability if $\beta \leq 10$
- Locking if $\beta > 1000$ (quad.) and if $\beta > 100000$ (cubic)
- Convergence in $\mathbf{L}^2$ norm: k+1

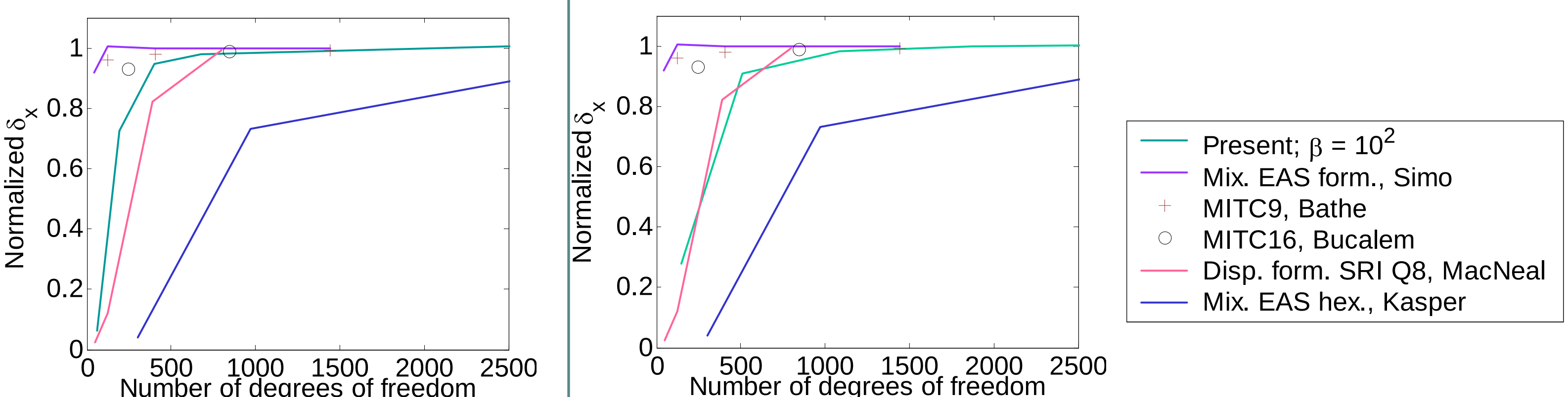
Numerical example: Pinched open-hemisphere (R/t = 250)

8-node bi-quad. 16-node bi-cub.



Double curvature

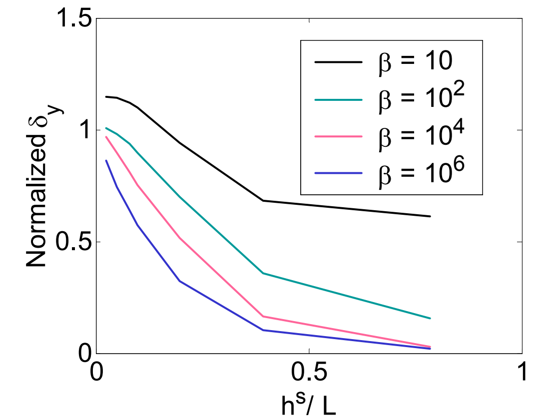
- Convergence in energy-norm: *k*-1

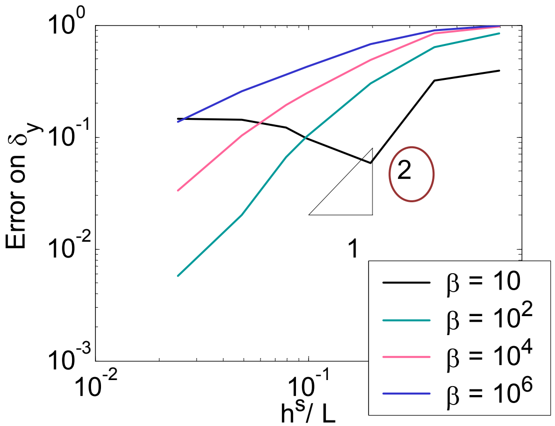


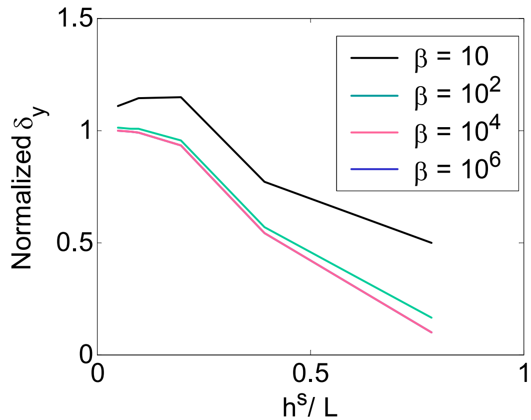
Numerical example: Pinched cylinder (R/t = 100)

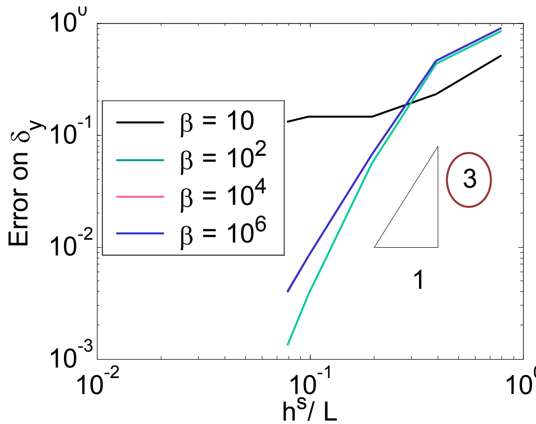
8-node bi-quad.

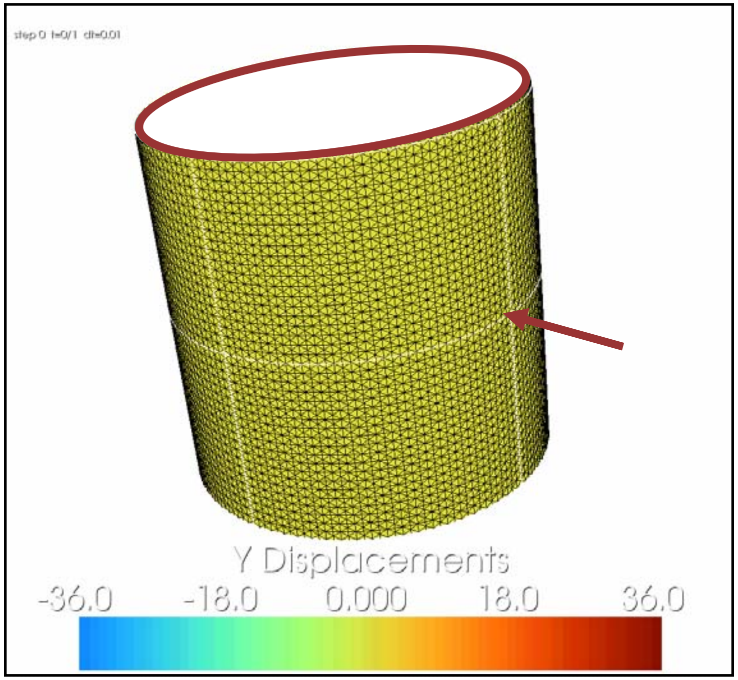
16-node bi-cub.









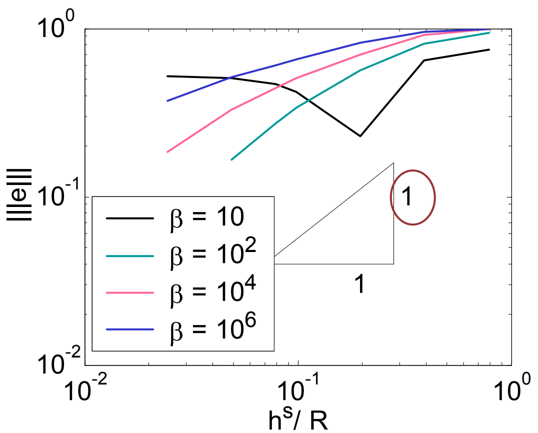


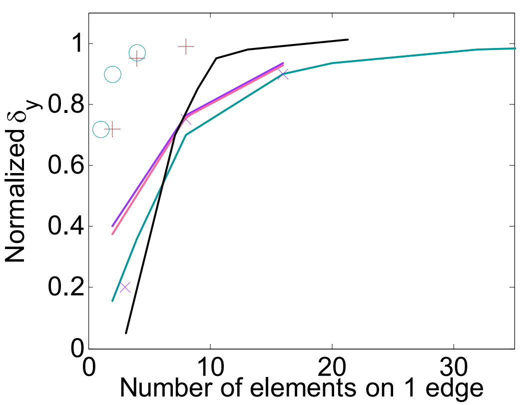
Complex membrane state

- Instability if $\beta \leq 10$
- Locking if $\beta > 10000$ (quad.) and if $\beta > 100000$ (cubic)
- Convergence in $\mathbf{L}^2$-norm: k

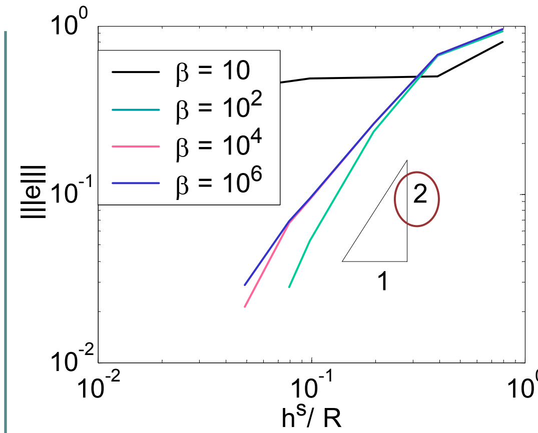
Numerical example: Pinched cylinder (R/t = 100)

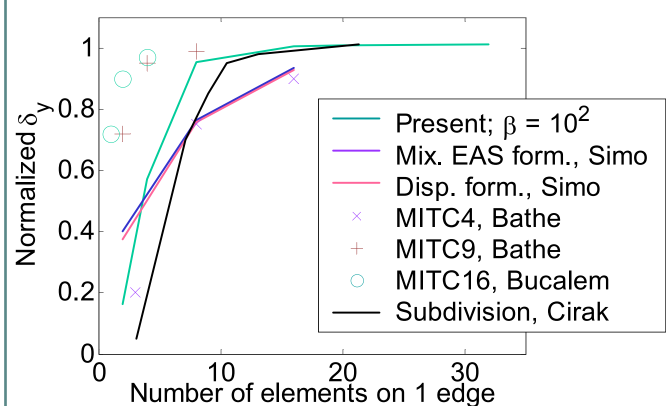
8-node bi-quad.





16-node bi-cub.





Complex membrane state

- Convergence rate *k-1* in the energy-norm
- Convergence improved with *k* increasing

Conclusions

- Development of a discontinuous Galerkin framework for Kirchhoff-Love shells:
  - Displacement formulation (no additional degree of freedom)
  - Strong enforcement of C^0 continuity
  - Weak enforcement of C^1 continuity
- Quadratic elements:
  - Method is stable if $\beta \geq 100$
  - Bending locking avoided if $\beta \leq 1000$
  - Membrane equations integrated with EAS or Reduced integration
- Cubic elements:
  - Method is stable if $\beta \geq 100$
  - Bending locking avoided if $\beta \leq 100000$
  - Full Gauss integration
- Convergence rate:
  - k-1 in the energy norm
  - k *or* k+1 in the $\mathbf{L}^2$-norm