An Adaptation of $\mathcal{S}^\nu$ Spaces

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Introduction

Let $\psi \in \mathcal{S}(\mathbb{R})$ and let

$$\psi_{j,k}(.) := \sum_{\ell \in \mathbb{Z}} 2^{j/2} \psi(2^j(. - \ell) - k), \quad j \in \mathbb{N}, \, k \in \{0, \ldots, 2^j - 1\}.$$

With the constant function 1, the functions $2^{j/2}\psi_{j,k}$ form an orthonormal basis of the space of the 1-periodic functions in $L^2([0,1])$. The function ψ is called **mother wavelet** and the functions $\psi_{j,k}$ simply **wavelets**.

If f is a 1-periodic function in $L^2([0,1])$, we denote by

$$c_{j,k} := 2^j \int_0^1 f(x) \overline{\psi_{j,k}(x)} \, dx$$

the **wavelet coefficients** of f (where we use a L^∞-normalization to simplify formulas and notations).

Introduction

S. Jaffard, *Beyond Besov Spaces, Part I : Distributions of Wavelet Coefficients*, The Journal of Fourier Analysis and its Applications **10** (3), 221-246, 2004.

Signals are very often stored by their wavelet coefficients because of the fast decomposition algorithms and the sparsity of the representation. In this setting, the use of Besov spaces is natural since they are expressed by simple conditions on wavelet coefficients : if $s \in \mathbb{R}$, $p \in]0,+\infty[$ and $q \in]0,+\infty[\cup \{\infty\}$, then

$$f \in B^s_{p,q}([0,1]) \quad \Leftrightarrow \quad \left(2^{(s-\frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^p \right)^{\frac{1}{p}} \right)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N}).$$

Remark

The information concerning the Besov spaces that contain a function f can be stored through the knowledge of the **scaling function** of f :

$$\eta_f(p) = \sup\{s \in \mathbb{R} : f \in B^{\frac{s}{p}}_{p,\infty}([0,1])\}, \quad p > 0.$$

However, the distribution of wavelet coefficients of a function can give more information than the Besov spaces.

Wavelet profile

S. Jaffard defined a function which contains the maximal information that can be derived from the distribution of the wavelet coefficients of a function.

Definition

The **wavelet profile** of a function f is the function ν_f defined by

$$\nu_f(\alpha) := \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \varepsilon))}{\ln(2^j)} \right) \right), \quad \alpha \in \mathbb{R}$$

where $E_j(C, \alpha) := \{k \in \{0, \dots, 2^j - 1\} : |c_{j,k}| \geq C 2^{-j\alpha}\}$ for $j \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $C > 0$.

<table>
<tr><td>2^j</td><td colspan="2">$k=0$</td><td colspan="2">$k=1$</td><td colspan="2">$k=2$</td><td>$\cdots$</td><td colspan="2">$k=2^j-2$</td><td colspan="2">$k=2^j-1$</td></tr>
<tr><td>2^{j+1}</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>$\cdots$</td><td></td><td></td><td></td><td>$2^{j+1}-1$</td></tr>
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- This definition formalizes the idea that there are about $2^{\nu_f(\alpha)j}$ wavelet coefficients in modulus larger than $2^{-j\alpha}$.
- By construction, ν_f is a non-decreasing and right-continuous function with values in $\{-\infty\} \cup [0, 1]$.
- The wavelet profile is robust, i.e. independent from the chosen basis of wavelets.

Wavelet profile

Example – Weierstrass function

If $\beta > 0$, let

$$W_\beta(x) = \sum_{\ell=0}^{+\infty} 2^{-\beta\ell} \sin(2\pi 2^\ell x), \quad x \in \mathbb{R}\,.$$

Its wavelet coefficients are $c_{j,k} = (-1)^k 2^{(1-j)\beta-\frac{1}{2}}$, $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j - 1\}$ and its wavelet profile is

$$\nu_{W_\beta}(\alpha) = \begin{cases} -\infty & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha \geq \beta \end{cases}.$$

Remark – Link between scaling function and wavelet profile

We have

$$\eta_f(p) = \inf_{\alpha\in\mathbb{R}} \{\alpha p - \nu_f(\alpha) + 1\}, \quad p > 0.$$

The information given by the Besov spaces which contain f only yields the concave hull of the wavelet profile. Thus, if ν_f is not concave, it contains strictly more information on f than η_f.

Hölder exponent

The regularity of large classes of signals may change from one point to another. Pointwise regularity is measured by the Hölder exponent.

Definition

Let $\alpha \geq 0$ and $x_0 \in \mathbb{R}$. A real-valued function F belongs to $C^\alpha(x_0)$ if there exist constants $C, R > 0$ and a polynomial P of degree less than α such that

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \quad \text{when} \quad |x - x_0| \leq R.$$

The **Hölder exponent** of F at x_0 is

$$h_F(x_0) := \sup\{\alpha \in \mathbb{R} : F \in C^\alpha(x_0)\}.$$

Spectrum of singularities

The purpose of multifractal analysis is to analyse the pointwise Hölder regularity of functions and to understand how this regularity fluctuates from point to point.

Characterize pointwise regularity for irregular signals is then useless. In this context, it is more interesting to determine the spectrum of singularities which allows to calculate the « size » of the set of points which have a same Hölder exponent.

Definition

The **spectrum of singularities** of a real-valued function F is the function d_f defined by

$$d_F(h) := \dim_{\mathcal{H}}\{x_0 \in \mathbb{R} : h_F(x_0) = h\}, \quad h \geq 0$$

(we use the convention $d_F(h) = -\infty$ if h is not a Hölder exponent of F).

Example

Proposition

Let H be a continuous function from $[0,1]$ to $[0,1]$ and set

$$H_{j,k} := \max\left\{\frac{1}{\log(j)}, H(k2^{-j})\right\}, \quad j \in \mathbb{N},\, k \in \{0,\dots,2^j - 1\}.$$

Let

$$s(x) := \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} 2^{-jH_{j,k}} \psi_{j,k}(x), \quad x \in [0,1].$$

Then, for all $x_0 \in [0,1]$, $H(x_0)$ is the Hölder exponent of s at x_0.

M. Clausel, S. Nicolay, *Wavelets Techniques for Pointwise Anti-Hölderian Irregularity*, Constructive Approximation **33**, 41-75, 2011.

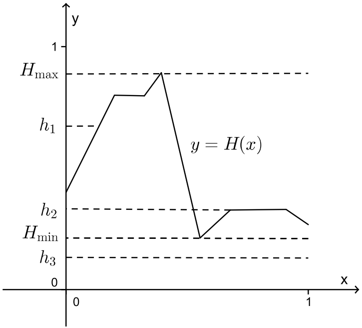
Example

Let's assume that H has a finite number of increasing and decreasing parts. We denote $H_{\min} := \inf_{t\in[0,1]} H(t)$ and $H_{\max} := \sup_{t\in[0,1]} H(t)$. Then,

$$\nu_s(\alpha) = \begin{cases} -\infty & \text{if } \alpha < H_{\min} \\ 1 & \text{if } \alpha \geq H_{\min} \end{cases}$$

and

$$d_s(h) = \begin{cases} 0 & \text{if } h \text{ is of type } h_1 \ (H \text{ is strictly monotone in a neighbourhood}) \\ 1 & \text{if } h \text{ is of type } h_2 \ (H \text{ is constant in a neighbourhood}) \\ -\infty & \text{if } h \text{ is of type } h_3 \ (h \notin [H_{\min}, H_{\max}]) \end{cases} .$$



Multifractal formalism

J-M. Aubry, S. Jaffard, *Random Wavelets Series*, Communications in Mathematical Physics **227**, 483-514, 2002.

In practice, it is difficult to calculate the spectrum of singularities of a signal. In general, we estimate the spectrum of singularities with some numerical formulas, called **multifractal formalisms**.

- The most famous multifractal formalism is the Frish-Parisi formalism, based on the scaling function :

$$d_1(h) := \min\left\{ \inf_{p>0}\{hp - \eta_f(p) + 1\}, 1 \right\}, \quad h > 0.$$

  The problem of this formalism is that d_1 is increasing and concave. However, the spectrum of singularities is not necessary increasing or concave.

- A formalism based on the wavelet profile allows to detect non-concave spectrum of singularities :

$$d_2(h) := \begin{cases} h \sup_{\alpha\in\,]0,h]} \frac{\nu_f(\alpha)}{\alpha} & \text{if } h \in]0, h_{\max}] \\ 1 & \text{otherwise} \end{cases}$$

  where $h_{\max} := \inf_{h\geq\alpha_{\min}} \frac{h}{\nu_f(h)}$ and $\alpha_{\min} := \inf\{\alpha \in \mathbb{R} : \nu_f(\alpha) \geq 0\}$.

$\mathcal{S}^\nu$ space

The distribution of wavelet coefficients can be used to define new spaces of functions and a fortiori new spaces of sequences.

Definition

Let ν be a non-decreasing and right-continuous function which takes values in $\{-\infty\} \cup [0,1]$. A function f belongs to the space $\mathcal{S}^\nu$ if its wavelet coefficients satisfy the following property :

$$\forall \alpha \in \mathbb{R},\ \forall \varepsilon > 0,\ \forall C > 0,\ \exists J \in \mathbb{N} \, : \, \# E_j(C,\alpha) \leq 2^{(\nu(\alpha)+\varepsilon)j} \ \forall j \geq J$$

where $E_j(C,\alpha) := \{k \in \{0,\dots,2^j-1\} : |c_{j,k}| \geq C 2^{-j\alpha}\}$ for $j \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $C > 0$.

These spaces are robust.

Then, we can also define sequence spaces of type $\mathcal{S}^\nu$. The sequences play in a way the role of wavelet coefficients of a signal.

$\mathcal{S}^\nu$ space

J-M. Aubry, F. Bastin, S. Dispa, S. Jaffard, *Topological properties of the sequence spaces $\mathcal{S}^\nu$*, Journal of Mathematical Analysis and Applications **321**, 364-387, 2006.

Definition

Let ν be an non-decreasing and right-continuous function which takes values in $\{-\infty\} \cup [0,1]$. A sequence $\vec{c} := (c_{j,k})_{j\in\mathbb{N}, k\in\{0\ldots,2^j-1\}}$ belongs to the space $\mathcal{S}^\nu$ if

$$\forall \alpha \in \mathbb{R}, \forall \varepsilon > 0, \forall C > 0, \exists J \in \mathbb{N} \,:\, \# E_j(C,\alpha) \leq 2^{(\nu(\alpha)+\varepsilon)j} \; \forall j \geq J$$

where $E_j(C,\alpha) := \{k \in \{0,\ldots,2^j-1\} : |c_{j,k}| \geq C2^{-j\alpha}\}$ for $j \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $C > 0$.

We can define a distance δ on $\mathcal{S}^\nu$.

Theorem

The space $(\mathcal{S}^\nu, \delta)$ is a complete metric space.

Other properties have been studied : the locally p-convexity, the strong topological dual of $\mathcal{S}^\nu$,...

$\mathcal{S}^\nu$ space

For $s \in \mathbb{R}$ and $p \in]0, +\infty[$, we define the Besov sequence space $b^s_{p,\infty}$ as the set of sequences $\vec{c} := (c_{j,k})_{j\in\mathbb{N}, k\in\{0\dots,2^j-1\}}$ such that

$$\sup_{j\in\mathbb{N}} \left(2^{(s-\frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^p \right)^{\frac{1}{p}} \right) < +\infty.$$

Moreover, we define the Hölder sequence space $c^s := b^s_{\infty,\infty}$ as the set of sequences $\vec{c} := (c_{j,k})_{j\in\mathbb{N}, k\in\{0\dots,2^j-1\}}$ such that

$$\sup_{j\in\mathbb{N}} \sup_{k\in\{0,\dots,2^j-1\}} 2^{sj}|c_{j,k}| < +\infty.$$

Example 1

Let $a \in \mathbb{R}$. If

$$\nu(\alpha) = \begin{cases} -\infty & \text{if } \alpha < a \\ 1 & \text{if } \alpha \geq a \end{cases},$$

then $\mathcal{S}^\nu = \bigcap_{\varepsilon>0} c^{a-\varepsilon}$.

$\mathcal{S}^\nu$ space

Example 2

Let $a > 0$. If

$$\nu(\alpha) = \begin{cases} -\infty & \text{if } \alpha < 0 \\ a\alpha & \text{if } \alpha \in [0, \frac{1}{a}[\\ 1 & \text{if } \alpha \geq \frac{1}{a} \end{cases},$$

then $\mathcal{S}^\nu = \bigcap_{\varepsilon > 0} b^{\frac{1}{a} - \varepsilon}_{a,\infty}$.

C. Esser, *Les espaces de suites $\mathcal{S}^\nu$: propriétés topologiques, localement convexes et de prévalence*, Mémoire de fin d'études, Université de Liège, 2011.

If ν is not concave, then $\mathcal{S}^\nu$ contains strictly more information than the one given by the knowledge of Besov spaces (and the concave conjugate of ν).

Towards a « continuous » version of $\mathcal{S}^\nu$ space

Question

What do classical $\mathcal{S}^\nu$ spaces become when the wavelet coefficients of a signal f are replaced with the coefficients of the continuous wavelet transform of f ?

Let $\psi \in \mathcal{S}(\mathbb{R})$ and let

$$\psi_{a,b}(.) := \frac{1}{a} \sum_{\ell \in \mathbb{Z}} \psi\left(\frac{. - b + k}{a}\right), \quad a > 0,\ b \in \ [0,1[.$$

If f is a 1-periodic function in $L^2([0,1])$, we denote by

$$Wf(a,b) := \int_0^1 f(x)\overline{\psi_{a,b}(x)}\, dx$$

the **continuous wavelet coefficients** of f (in fact, it is the continuous wavelet transform of f).

We can reconstruct f from its continuous wavelet coefficients $Wf(a,b)$ with $a, b \in]0,1[$ if ψ satisfies the admissible condition :

$$C_\psi := \int_0^{+\infty} \frac{|\hat{\psi}(\xi)|^2}{\xi}\, d\xi = \int_0^{+\infty} \frac{|\hat{\psi}(-\xi)|^2}{\xi}\, d\xi < +\infty.$$

Towards a « continuous » version of $\mathcal{S}^\nu$ space

Let

$$C_0 := 0, \quad C_k := \int_0^{2k\pi} \frac{|\hat{\psi}(\xi)|^2}{\xi}\, d\xi \quad \text{and} \quad C_{-k} := \int_0^{2k\pi} \frac{|\hat{\psi}(-\xi)|^2}{\xi}\, d\xi$$

for $k \in \mathbb{N}_0$ and let $D_k := C_\psi - C_k$ for $k \in \mathbb{Z}$.

Proposition – Reconstruction formula

If f and g are 1-periodic in $L^2([0,1])$, then

$$\iint_{]0,1[\times[0,1[} Wf(a,b)\,\overline{Wg(a,b)}\,\frac{dadb}{a} = C_\psi \langle f, g\rangle - \sum_{k\in\mathbb{Z}} D_k \langle f, e_k\rangle\,\overline{\langle g, e_k\rangle}$$

where $e_k(x) = e^{2i\pi kx}$, $x \in [0,1]$, for $k \in \mathbb{Z}$. In particular,

$$\lim_{\substack{A_1,B_1\to 0\\ A_2,B_2\to 1}} \left\| f(\cdot) - \frac{1}{C_\psi}\left(\iint_{]A_1,A_2[\times]B_1,B_2[} Wf(a,b)\,\psi_{a,b}(\cdot)\,\frac{dadb}{a^2} + \sum_{k\in\mathbb{Z}} D_k \langle f, e_k\rangle\, e_k(\cdot)\right)\right\|_{L^2([0,1])} = 0.$$

Towards a « continuous » version of $\mathcal{S}^\nu$ space

- An idea to define the « continuous » wavelet profile of a signal could be

$$\nu_f^\star(\alpha) := \lim_{\varepsilon\to 0^+}\left(\limsup_{a\to 0^+}\left(\frac{\ln(\mathrm{mes}(E_a(C,\alpha)))}{-\ln(a)}\right)\right), \quad \alpha\in\mathbb{R}\,.$$

where $E_a^\star(C,\alpha) := \{\ell\in[0,1/a[\colon |Wf(a,a\ell)|\geq a^{\alpha+\varepsilon}\}$ for $a\in\,]0,1[$, $\alpha\in\mathbb{R}$ and $C>0$.

Is there a link between $\nu_f^\star$ and ν_f ?

- An idea to define the « continuous » $\mathcal{S}^\nu$ space (if ν is a non-decreasing and right-continuous function which takes values in $\{-\infty\}\cup[0,1]$) could be the set of functions f such that its continuous wavelet coefficients satisfy the following property :

$$\forall\alpha\in\mathbb{R},\ \forall\varepsilon>0,\ \forall C>0,\ \exists A\in\,]0,1[\,\colon\ \mathrm{mes}(E_a^\star(C,\alpha))\leq\frac{1}{a^{\nu(\alpha)+\varepsilon}}\ \forall a\in\,]0,A[.$$

Is this space robust ?