Importance of modal cross-correlation on wind loaded structures

V. Denoël, H. Degée
University of Liège,Department of Mechanics of materials and Structures, Liège, Belgium

V. de Ville-de Goyet
Bureau d'Etudes GREISCH, Angleur, Belgium

ABSTRACT: In civil engineering applications the dynamic analysis of large structure is often performed in a modal space. This method is known to offer an interesting decrease of the number of degrees of freedom as well as a decomposition of the structure's response in "uncoupled" components. Even if the response in each mode can be computed independently from the responses in the other ones, in the context of stochastic loading, the coherence of these modal responses must be accounted for in the determination of the structural response. These quantities are known as the modal cross-correlations. In this paper we will show that these cross-correlations can be important (contrarily to what is sometimes thought) even in case of well-separated natural frequencies. This will be illustrated on the analysis of the famous Viaduct of Millau during an erection stage.

1 INTRODUCTION

Because of their high flexibility, long span bridges can exhibit unacceptable dynamic behaviours. More exactly the very low natural frequencies (~0.5 Hz to 2 Hz) of suspended and cable-stayed bridges make them very sensitive to the wind loading, whose frequency content is also in the very low frequency range. During erection stages, the final restraints to which the bridge will be subjected aren't present yet. This reduction of stiffness results generally in even lower natural frequencies. In this case, the stability conditions at erection stages consist in critical criteria. It is thus important to cautiously study the fulfilment of these conditions.

As an important point attention has to be dedicated to a special dynamic phenomenon: the modal coupling, which is frequently neglected. It is often thought that this kind of coupling just occurs in case of closely spaced natural frequencies, which is wrong. After a short recall of the main stages of a stochastic analysis, this paper will present the conditions under which modal coupling can be significant. Thanks to usual assumptions that are made in the context of wind loaded structures, it will indeed be shown that the modal coupling can occur in other contexts. As a final point the effects of modal coupling will be illustrated on the dynamic response of a bridge. These effects will be enlightened by comparing the results obtained with both methods (with or without neglecting the modal coupling).

2 STOCHASTIC ANALYSIS

The dynamic analysis of large structures is commonly performed in a modal space. This projection into a subspace is beneficial since it allows (REF CLOUGH):

- a reduction of the number of unknowns (from the *n* of dofs of the structure to the *m* generalized coordinates);
- the uncoupling of the equations of motion (provided the damping is proportional)

The basic idea of this modal decomposition lies in the fact that the displacements of the structure $\{x\}$ are expressed by a linear combination of mode shapes $\{x\} = [\Phi]\{\eta\}$ in which a few

mode shapes ($m<<n$) are collected in the mode shape matrix $[\Phi]$ and $\{\eta\}$ represents the vector of modal amplitudes (or generalized coordinates).
This subspace reduction can be useful in a deterministic analysis since the equation of motion in each mode can be solved independently from the others. In stochastic analysis, this subspace projection is also interesting since it gives a diagonal transfer matrix (see Table 1).

Table 1 summarizes the main stages of a stochastic analysis. At this stage, it is supposed that the power spectral density (PSD) matrix of the applied forces is known. In case of a wind loaded structure subjected to buffeting forces this quantity can be expressed as a function of the PSD of the wind velocity and of the aerodynamic characteristics of the structure (REF, REF). By pre- and post-multiplication by the mode shapes $[\Phi]$, the PSD matrix of the generalized forces can be obtained (see Table 1, line 1).

Then the equations motion can be solved very easily by pre- and post-multiplying by the diagonal transfer matrix (see Table 1, line 2). Each component of the PSD matrix of the generalized coordinates can be expressed as a function of the corresponding element of the PSD matrix of the generalized forces only.

As a final step the PSD matrix of the structural displacements can be estimated by pre- and post-multiplying again by the mode shapes. The computation of this matrix is the main objective of a stochastic analysis.

Table 1. Summary of the progress of a stochastic analysis.

Stage	Operation
Projection of the forces in the modal space	$\left[S_{F^*}(\omega)\right]=[\Phi]^T\left[S_F(\omega)\right][\Phi]$
Resolution in the modal space	$\left[S_\eta(\omega)\right]=[H]\left[S_{F^*}(\omega)\right][H]^{T,conj}$
Come back to displacements of structure	$\left[S_x(\omega)\right]=[\Phi]\left[S_\eta(\omega)\right][\Phi]^T$

T,conj stands for transposed conjuguate. Note that *[H]* is a diagonal matrix.

3 STOCHASTIC ANALYSIS OF LARGE STRUCTURES

The design of structures is based on extreme displacements and internal forces. In this context, “extreme values” means maximum values that can be expected on a certain life time or duration of observation. For Gaussian processes the extreme value of a quantity can be estimated by the product of its variance and a peak factor (REF REF). Even if a rigorous value for the peak factor can be computed, for reasons of simplicity and computation time, the peak factor is often estimated by simplified relations (REF REF). This indicates that the main objective of a stochastic analysis can be reached once the covariance matrices of the structural displacements and internal forces are computed.

If structural displacements are considered, the extreme values, needed for the design, can thus be estimated thanks to the variances of the displacements.

The computation of the elements of the PSD matrix of the structural displacements consists in the main objective of a stochastic analysis. For large structures, the estimation of the whole PSD matrix of the displacements is however too expensive in terms of computation time. Two solutions are thus generally considered:

- to compute the diagonal elements of this matrix only; this can be also realized for the most important degrees of freedom only
- to integrate the last expression of Table 1 along frequencies and to estimate directly the covariance matrix of the displacements as a function of the covariance matrix of the modal amplitudes :

$$\left[\mathrm{cov}_x\right]=[\Phi]\left[\mathrm{cov}_\eta\right][\Phi]^T \qquad (1)$$

4 MODAL COUPLING

There are several kinds of modal coupling depending on the matrix whose off-diagonal terms are considered. In this paper we will just focus on the influence of the cross-correlations between the generalized coordinated, i.e. the off-diagonal terms of the covariance matrix of the modal amplitudes. Since these quantities come from the off-diagonal terms of the corresponding PSD matrix it should rather be spoken of *modal coherence* instead of *modal coupling*.

The covariance matrix of the modal amplitudes $\left[\text{cov}_\eta\right]$ is a full matrix obtained by integration (along frequencies) of the corresponding PSD matrix:

$$\left[\text{cov}_\eta\right] = \int_{-\infty}^{+\infty} \left[S_\eta(\omega)\right] d\omega \tag{2}$$

Its diagonal elements represent the variance of the modal amplitudes and the off-diagonal terms represent the cross-correlations between the amplitudes in different modes.

In civil engineering application it is often supposed that the modal amplitudes are independent random processes and thus that these off-diagonal terms are all equal to zero. It is generally accepted that this condition is valid in case of well-separated natural frequencies. It is however less commonly known that this condition is not sufficient. In the following, we will see that a second condition has to be fulfilled for the assumption of modal uncoupling.

The variance of the displacement at degree-of-freedom *i* (see Equation 1) is expressed by:

$$\begin{aligned} \sigma_{x_i}^2 &= \sum_m \sum_n \Phi_{im} \Phi_{in} \rho_{\eta_{mn}} \sigma_{\eta_m} \sigma_{\eta_n} \\ &= \sum_m \Phi_{im}^2 \sigma_{\eta_m}^2 + \sum_m \sum_{n \neq m} \Phi_{im} \Phi_{in} \rho_{\eta_{mn}} \sigma_{\eta_m} \sigma_{\eta_n} \end{aligned} \tag{3}$$

in which the first double summation has been rewritten in such a way to introduce a first term corresponding to the diagonal elements of the covariance matrix (of the modal amplitudes) and another double summation based on off-diagonal terms only. An interesting way to judge the importance of modal cross-correlations is thus to estimate the relative importance of the second double summation compared to the result of the single summation.

A first sight shows quickly that the second term can be negligible if the correlation coefficients $\rho_{\eta_{mn}}$ between different modes are small. In the next paragraph this condition will be studied in the context of wind loading. Note that discussions about Equation 3 have to be related to the differences between the famous SRSS and CQC combinations that are commonly used in seismic engineering (REF).

5 MODAL COUPLING AND WIND LOADING

Even if modal coupling has already been studied in many references (REF ?) it is interesting to understand what happens in case of structures subjected to turbulent wind loading. Indeed in this domain some hypotheses (concerning the integration along frequencies leading to Equation 1) that are often formulated allow giving an interesting physical meaning to modal coupling.

The aim of this paragraph is present the most useful hypothesis and to establish the conditions under which the cross-correlations can be neglected or not. As exposed before the discussion will concern the estimation of the correlation coefficients of the modal amplitudes:

$$\rho_{\eta_{mn}} = \frac{v_{\eta_{mn}}}{\sigma_{\eta_m} \sigma_{\eta_n}} \tag{4}$$

Because of the very low frequency content of the wind loading, the covariance matrix of the modal amplitudes is rarely computed by numerical integration (see Equation 1). The frequency content of modal amplitudes of a structure subjected to buffeting forces is composed of two important bands corresponding to the very low frequency of the loading on the one hand, and to the natural frequency of the considered mode on the other hand. It can thus generally be assumed quite accurately that:

$$\sigma_{\eta_m}^2 = B_m + R_m = \left(\frac{\sigma_{F_m}^*}{K_m}\right)^2 + \frac{\pi\omega_m}{2\xi_m}\frac{S_{F_m}^*(\omega_m)}{K_m} \tag{5}$$

where B_m and R_m stand respectively for the background (quasi-static) and resonant contributions.

Concerning the cross-correlations, in the most general case the covariance between modal amplitudes in different modes $v_{\eta_{mn}}$ presents a quasi-static component and two frequency bands corresponding to resonance in each mode. It is thus difficult to justify the use of the same simplified procedure for the estimation of the covariances. We propose however to continue using one single term for both resonance peaks and thus:

$$v_{\eta_m} = B_{mn} + R_{mn} = \frac{v_{F_{mn}}^*}{K_m K_n} + \alpha(\omega_m, \omega_n)\sqrt{\frac{S_{F_{mn}}^*(\omega_m)}{K_m}\frac{S_{F_{mn}}^*(\omega_n)}{K_n}} \tag{6}$$

where

$$\alpha(\omega_m, \omega_n) = \frac{4\pi\omega_m^2\omega_n^2(\xi_m\omega_m + \xi_n\omega_n)}{(\omega_m^2 - \omega_n^2)^2 + 4\omega_m\omega_n(\xi_m\omega_m + \xi_n\omega_n)(\xi_m\omega_n + \xi_n\omega_m)} \tag{7}$$

If the natural frequencies are well-separated, $\alpha(\omega_m, \omega_n)$ is very small and the estimation of the second factor of R_{mn}. is useless. In this case, there is no coupling due to the resonant term. The estimation of the second factor just has to be realized accurately when $\alpha(\omega_m, \omega_n)$ is approaching unity, i.e. for close natural frequencies. Supposing that:

$$S_{F_m}^*(\omega_m) \approx S_{F_m}^*(\omega_n),\ S_{F_n}^*(\omega_m) \approx S_{F_n}^*(\omega_n),\ \Gamma_{F_n}^*(\omega_m) \approx \Gamma_{F_n}^*(\omega_n) \tag{8}$$

(which is true in the limit case) the resonant contribution can be written as:

$$R_{mn} = \alpha(\omega_m, \omega_n)\Gamma_{F_{mn}}^*(\omega_m)\sqrt{\frac{S_{F_m}^*(\omega_m)}{K_m}\frac{S_{F_n}^*(\omega_n)}{K_n}} \tag{9}$$

Following the arguments presented before and despite its lack of rigor, Equation 6 gives very good approximations of the covariance, no matter the proximity of the natural frequencies. By introducing Equations 5, 6 and 9 into Equation, an approximation of the correlation coefficient can be obtained:

$$\rho_{\eta_{mn}} = \frac{B_{mn} + R_{mn}}{\sqrt{B_m + R_m}\sqrt{B_n + R_n}} = \frac{\dfrac{B_{mn}}{\sqrt{B_m B_n}}}{\sqrt{1 + R_m/B_m}\sqrt{1 + R_n/B_n}} + \frac{\dfrac{R_{mn}}{\sqrt{R_m R_n}}}{\sqrt{1 + B_m/R_m}\sqrt{1 + B_n/R_n}} \tag{10}$$

which gives after simplification

$$\rho_{\eta_{mn}} = \gamma_F \rho_{F_{mn}^*} + \gamma_D \alpha(\omega_m, \omega_n)\Gamma(\omega_m, \omega_n) \tag{11}$$

where $\rho_{F_{mn}^*}$ is the correlation coefficient of the generalized forces and

$$\gamma_F = \frac{1}{\sqrt{1 + b_m^{-1}}\sqrt{1 + b_n^{-1}}},\ \gamma_D = \frac{1}{\sqrt{1 + b_m}\sqrt{1 + b_n}} \text{ and } b_m = B_m/R_n \tag{12}$$

This relation shows that the correlation coefficient between the modal amplitudes can be approached by a weighted combination of *(i)* the correlation coefficient of the generalized forces and *(ii)* a "dynamic" correlation coefficient expressed as a function of the proximity of the natural frequencies (in $\alpha(\omega_m, \omega_n)$) and of the coherence of the generalized forces. Figure 1 gives a representation of γ_F and γ_D. It can be seen that γ_F, γ_D and $\gamma_F + \gamma_D$ are always smaller than unity which indicates that the actual correlation of the generalized coordinates is smaller than the maximum value of $\rho_{F_{mn}^*}$ and $\alpha(\omega_m, \omega_n)\Gamma(\omega_m, \omega_n)$. Furthermore two interesting limit case can be considered:

- b_m and b_n are both small compared to unity; γ_F is thus very small and the correlation coefficient of the modal amplitude is governed by $\alpha\left(\omega_m, \omega_n\right)\Gamma\left(\omega_m, \omega_n\right)$. This is understandable since the response is mainly resonant.
- b_m and b_n are both much larger than unity; the response is mainly quasi-static and the correlation coefficient of the modal amplitude is equal to the corresponding coefficient associated to generalized forces $\rho_{F^*_{mn}}$, which could also be expected since the response is quasi-static in this case.

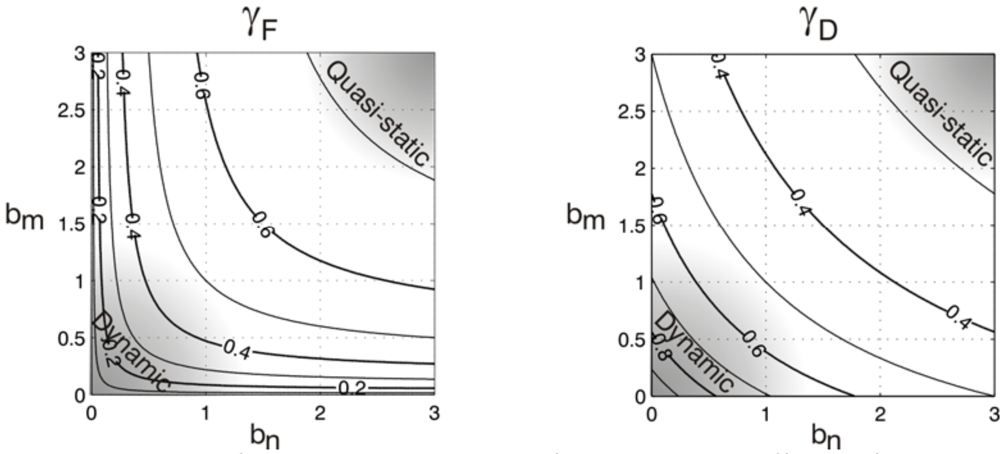

Figure 1. Weighting functions in the expression of the correlation coefficient of the generalized coordinates. They are expressed as a function of b_m and b_m which represent the sharing out of the total energy in modes m and n between background and resonant components.

As a conclusion Equation 11 shows clearly that the condition of proximity of the natural frequencies is not enough to ensure modal coupling. These findings can be established:

- If the natural frequencies are close but the generalized forces are not coherent in the vicinity of these frequencies there is no reason to account for coupling;
- If the natural frequencies are close and the generalized forces are coherent in the vicinity of these frequencies, coupling effects could have to be taken into account. They only have to be so if the dynamic contribution to the response is important. Otherwise if the response is mainly quasi-static, there is no reason to account for the cross-correlation;
- If the response is rather quasi-static, this importance of the coupling terms depends on the correlation of the generalized forces.

6 APPLICATION

As an example, the influence of the modal cross-correlations will be illustrated on the famous Viaduct of Millau (France) during an erection stage. With its pylons dominating the Tarn valley about 350 m above the ground, this exceptional 7-span cable-stayed bridge (approx. 2.5 km long) is the highest bridge ever built. In order to limit the risks undertaken during its construction, it was decided to opt for a deck launching on temporary piers. Amongst more than six hundred erection stages considered by the design office (REF…), just one of them will be considered for this paper: the tip of the deck is reaching the second pile and the pylon (already installed for the deck launching) is located approximately at mid-span.

For this particular structure, Table 2 lists the first eigen modes and the corresponding natural frequencies. The loading is a gusty wind with a 34.2 m/s mean wind velocity and three turbulence components with a 5.4 m/s standard deviation. Thanks to PSD and spatial coherence of wind velocities (measured on site) and aerodynamic coefficients (measured in a wind tunnel), the PSD matrix of the wind forces can be established (REF XXXXX). The progress presented in Table 1 can thus be followed. It should be noted that in the applications of this paper all PSD matrices have been numerically integrated (and not using the simplified procedure of § 4). This simplified procedure was thus just dedicated to establish a easily understandable formulation.

Figure 2 (a) represents the aerodynamic damping ratios inherent to any buffeting loading. This kind of damping must be added to the classical structural damping that is present in any structure. The aerodynamic damping depends on the mean wind velocity and on the aerodynamic characteristics of the structure. It will be constant and equal to values given in Figure 2 for the following applications.

Figure 2 (b) is a graphical representation of the correlation matrix between the generalized forces ($\rho_{F^*_{mn}}$). This correlation matrix could seem to be messy but this has to be linked together with the complexity of the structure. With modal descriptions given in Table 2 it can be seen that similar vibration modes are more correlated than pairs of modes vibrating in different directions.

Table 2. Natural frequencies and brief description of the modes of the studied structure.

No.	Frequency [Hz]	Description of mode
1	0.238	Out-of plane bending of the deck (I)
2	0.324	In plane bending of the deck (I)
3	0.446	Torsion of the deck (I) (+ out-of-plane vibration of the pylon)
4	0.531	In plane bending of the deck (II)
5	0.650	Bending of pile 1
6	0.708	In plane bending of the deck (III)
7	0.736	Out-of plane bending of the deck (II)
8	0.926	Axial vibration of the deck (I)
9	1.129	Out-of plane bending of the deck (III)
10	1.327	In plane bending of the deck (IV)

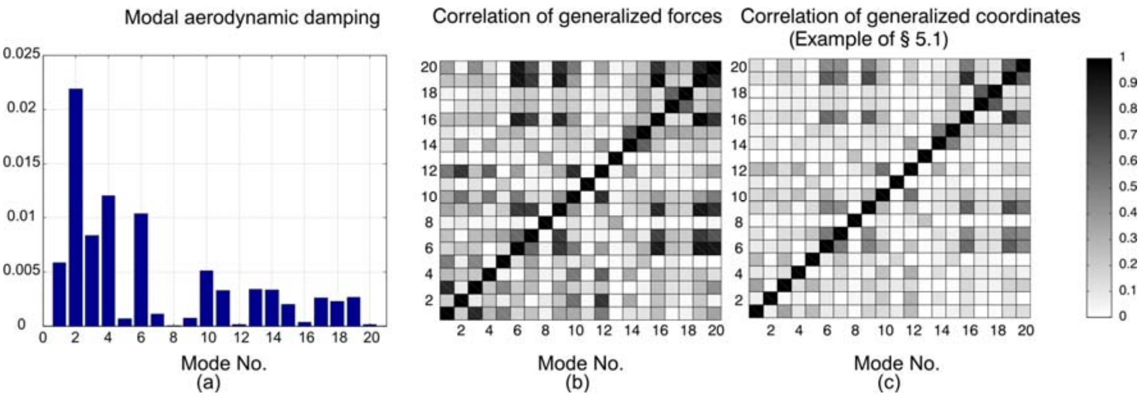

Figure 2. (a) Aerodynamic damping ratios and (b) Graphical representation of the correlation matrix of the generalized forces (c) Graphical representation of the correlation matrix of the generalized coordinates

6.1 *First example*

In this first example, let us consider the structural damping is equal to ξ=0.03 for all modes, which is slightly more than could be expected in this kind of structure. In this case the correlation coefficients of the modal amplitudes (numerically integrated) are given in Figure 2 (c). As stated in § 3 it can be seen that the correlation of the generalized forces is a maximum limit for the correlation of the generalized coordinates. Furthermore – and also as stated before – it could be seen that there is less difference in Figures 2 (b) and 2 (c) for modes having a dominant background response (e.g. (14,15) or (17,18)). A closer look at modal coefficients b_m would show that the coefficients are in the interval *[0.6,1]* for the first four modes and in *[1,3]* for the others. This means that the dynamic part of the response is important and thus that the correlation coefficient of the generalized coordinates tends to be close to the product $\alpha\Gamma$ which is very small since natural frequencies are well separated (except mode 6 and 7).

The numerical computation of the PSD's of the displacements of the structure (Figure 3) have been realized with and without considering the cross-correlation. Noting that axes are presented with logarithmic scales, it can be seen that the main differences (i.e. the difference giving the most important differences) are in the very-low frequency range. This shows again that the effects of cross-correlation terms are the most important for the background component of the response. The modal coupling could be considered as *a background coupling* and not from *a resonant coupling*. According to developments of § 3, this kind of coupling comes from the important correlation of the generalized forces.

As an endpoint, Figure 4 shows the standard deviations of the bending moments in the deck and in the piles. The coupling terms mainly affect the out-of-plane bending with a maximum effect of approx. 20% in the deck under the pylon.

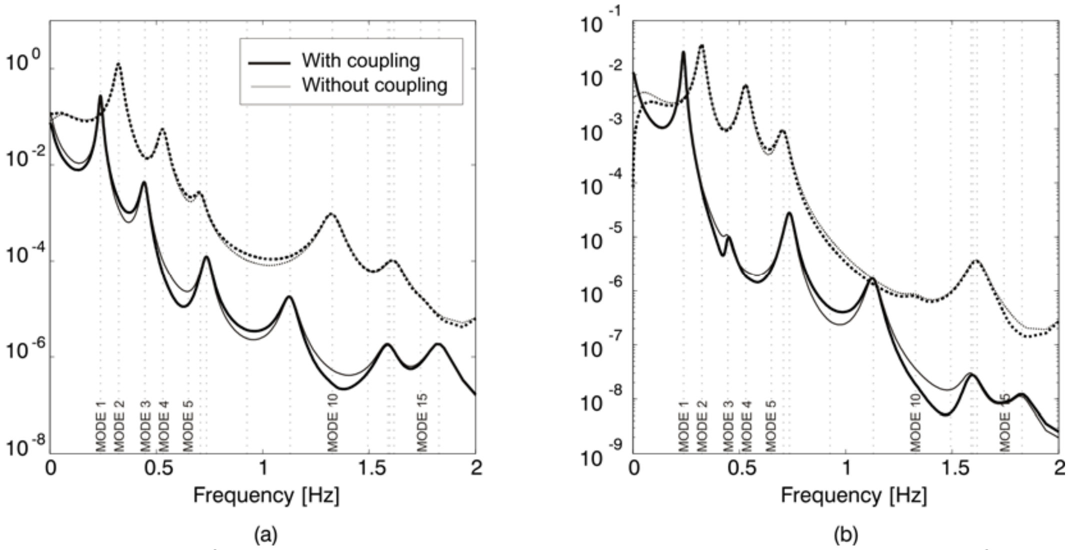

Figure 3 : PSD's of the vertical (dashed lines) and horizontal (solid lines) displacements of the deck : (a) left-side end and (b) under the pylon

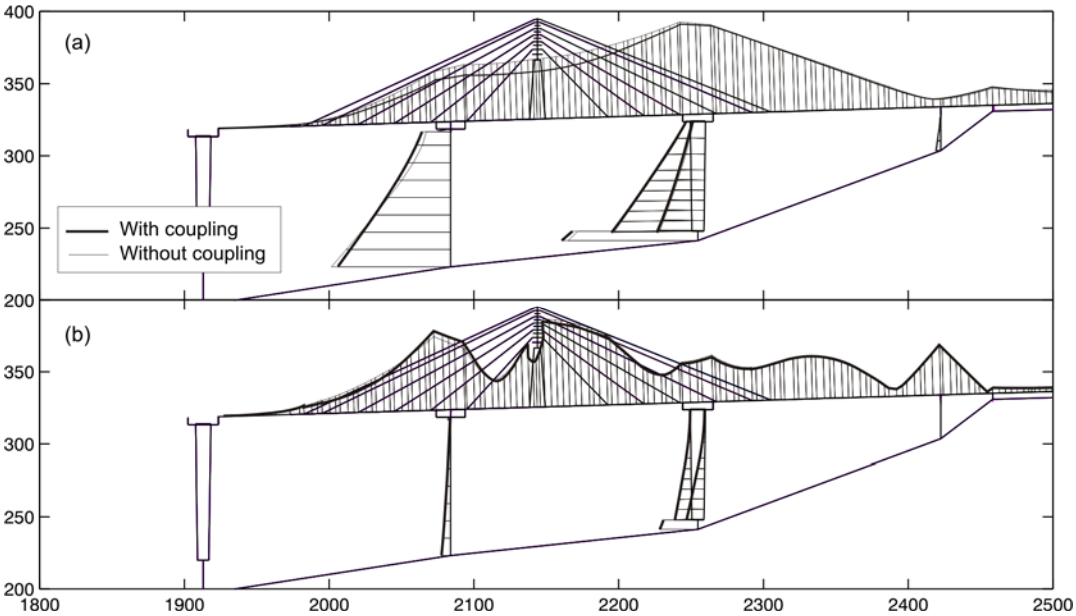

Figure 4: Standard deviation of the out-of-plane (a) and in-plane (b) bending moments

6.2 *Parametric study*

In this paragraph we will show the influence of the structural damping coefficient on the modal coupling effect. If this coefficient is smaller the modal dynamic contributions are larger, coefficients b_m are smaller and the correlation coefficients are moving away from the correlation coefficient of the generalized forces; they are thus decreasing and the effects of the cross-correlation terms are thus reduced. As a conclusion, for slightly damped structures, the results obtained with and without modal coupling should be in a better agreement.

This basic reasoning based on the very simple developments of § 3 can be illustrated with a more precise analysis. A rigorous approach like the one applied in § 5.1 but now with various damping coefficients. Figure 5 shows again that modal coupling affects much more the out-of-plane bending moment than the in-plane ones. The expected behaviour can be checked: the discrepancies are small for slightly damped structures. It must be noted that the difference can be as important as 30%!

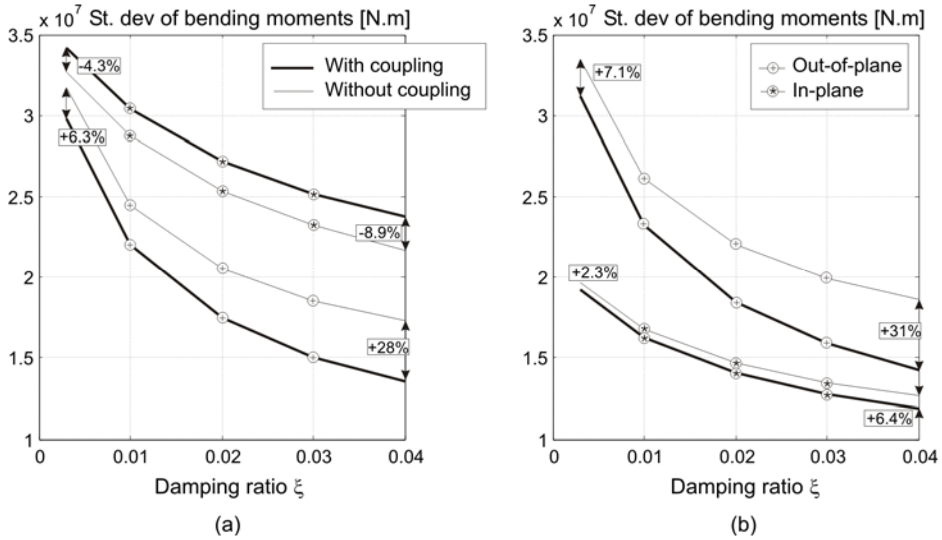

Figure 5: Parametric study - Comparison of bending moments in the bridge deck: (a) on the pile and (b) under the pylon

7 CONCLUSIONS

It is sometimes thought that cross-correlation terms must be accounted for in case of closely spaced natural frequencies. In this paper we have used to classical decomposition of the modal response into a background (quasi-static) and a resonant contribution to show that modal cross-correlation terms can have quasi-static origin.

This kind of effect has been illustrated on the Viaduct of Millau. It has been shown that the discrepancies between results obtained with and without cross-correlation could go up to 30% if the structural damping was high. For structures exhibiting an important quasi-static behavior (like “highly” damped structures) the correlation coefficient of the generalized forces can be considered as estimations of the correlation coefficients of the generalized coordinates.

In the most general case, the lack of modal coupling shouldn’t be concluded without having inspected (i) the proximity of the natural frequencies and (ii) the correlation coefficients of the generalized forces.

8 ACKNOWLEDGMENTS

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9 REFERENCES

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