Higher symmetries of the conformal Laplacian

J.-P. Michel, J. Silhan, R.

Corfu, May 2013

Introduction

- On $(\mathbb{R}^2, \mathrm{g}_0)$, we consider the Helmholtz equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

Introduction

- On $(\mathbb{R}^2, \mathrm{g}_0)$, we consider the Helmholtz equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

- Coordinates (u, v) separate this equation $\Longleftrightarrow$ $\exists$ solution of the form $f(u)g(v)$

Introduction

- On $(\mathbb{R}^2, \mathrm{g}_0)$, we consider the Helmholtz equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

- Coordinates (u, v) separate this equation $\Longleftrightarrow \exists$ solution of the form $f(u)g(v)$
- Coordinates (u, v) orthogonal $\Longleftrightarrow \mathrm{g}_0(\partial_u, \partial_v) = 0$

- There exist 4 families of orthogonal separating coordinates systems :

- There exist 4 families of orthogonal separating coordinates systems :
  1. Cartesian coordinates

- There exist 4 families of orthogonal separating coordinates systems :

  1. Cartesian coordinates
  2. Polar coordinates (r, θ) :

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{cases}$$

- There exist 4 families of orthogonal separating coordinates systems :
  1. Cartesian coordinates
  2. Polar coordinates (r, θ) :

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

  3. Parabolic coordinates (ξ, η) :

$$\begin{cases} x = \xi\eta \\ y = \frac{1}{2}(\xi^2 - \eta^2) \end{cases}$$

- There exist 4 families of orthogonal separating coordinates systems :

  1. Cartesian coordinates
  2. Polar coordinates (r, θ) :

  $$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

  3. Parabolic coordinates (ξ, η) :

  $$\begin{cases} x = \xi\eta \\ y = \frac{1}{2}(\xi^2 - \eta^2) \end{cases}$$

  4. Elliptic coordinates (α, β) :

  $$\begin{cases} x = \sqrt{d}\cos(\alpha)\cosh(\beta) \\ y = \sqrt{d}\sin(\alpha)\sinh(\beta) \end{cases}$$

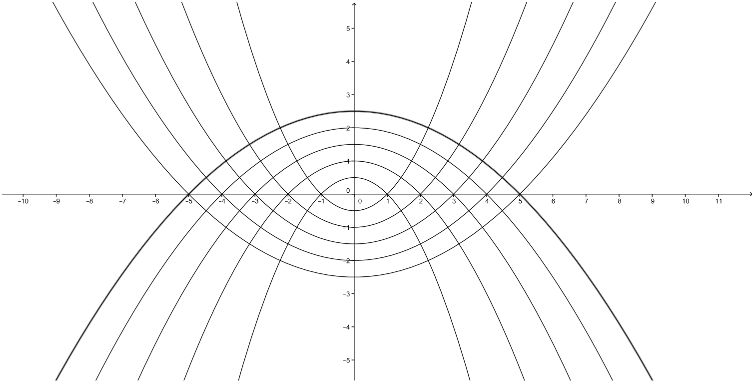

Figure: Coordinates lines corresponding to the parabolic coordinates system

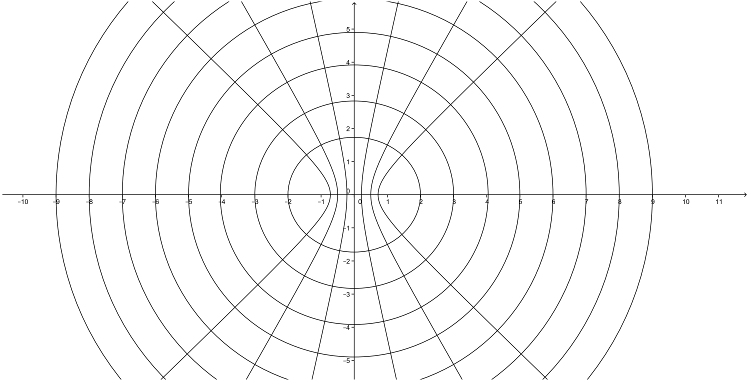

Figure: Coordinates lines corresponding to the elliptic coordinates system

- Bijective correspondence

$$\{\text{Separating coordinates systems}\}$$

$$\longleftrightarrow$$

$$\{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\}$$

- Bijective correspondence

$$\{\text{Separating coordinates systems}\}$$

$$\longleftrightarrow$$

$$\{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\}$$

Coordinates system	Symmetry
(x, y)	∂_x^2
(r, θ)	L_θ^2
(ξ, η)	$\frac{1}{2}(\partial_x L_\theta + L_\theta \partial_x)$
(α, β)	$L_\theta^2 + d\partial_x^2$

with $L_\theta = x\partial_y - y\partial_x$

- Link between the symmetry and the coordinates system : if the second-order part of D reads as

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} A \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

the eigenvectors of A are tangent to the coordinates lines.

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{R},$$

where R is the scalar curvature of g.

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{R},$$

  where R is the scalar curvature of g.
- $\lambda_0 = \frac{n-2}{2n}$, $\mu_0 = \frac{n+2}{2n}$
- If Δ_Y acts between λ_0- and μ_0-densities, Δ_Y conformally invariant : $\Delta_Y(\tilde{\mathrm{g}}) = \Delta_Y(\mathrm{g})$ if $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$.

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{R},$$

  where R is the scalar curvature of g.
- $\lambda_0 = \frac{n-2}{2n}$, $\mu_0 = \frac{n+2}{2n}$
- If Δ_Y acts between λ_0- and μ_0-densities, Δ_Y conformally invariant : $\Delta_Y(\tilde{\mathrm{g}}) = \Delta_Y(\mathrm{g})$ if $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$.
- Symmetry of Δ_Y : $D \in \mathcal{D}_{\lambda_0,\lambda_0}(M)$ such that $[\Delta_Y, D] = 0$

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{R},$$

where R is the scalar curvature of g.

- $\lambda_0 = \frac{n-2}{2n}$, $\mu_0 = \frac{n+2}{2n}$
- If Δ_Y acts between λ_0- and μ_0-densities, Δ_Y conformally invariant : $\Delta_Y(\tilde{\mathrm{g}}) = \Delta_Y(\mathrm{g})$ if $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$.
- Symmetry of Δ_Y : $D \in \mathcal{D}_{\lambda_0,\lambda_0}(M)$ such that $[\Delta_Y, D] = 0$
- Conformal symmetry of Δ_Y : $D_1 \in \mathcal{D}_{\lambda_0,\lambda_0}(M)$ such that $\exists D_2 \in \mathcal{D}_{\mu_0,\mu_0}(M)$ such that $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$

- (M, g) conformally flat : in a coordinates system, g is conformally equivalent to the canonical metric in $\mathbb{R}^n$

  Conformal symmetries of Δ_Y known (M. Eastwood, J.-P. Michel)

- (M, g) conformally flat : in a coordinates system, g is conformally equivalent to the canonical metric in $\mathbb{R}^n$

  Conformal symmetries of Δ_Y known (M. Eastwood, J.-P. Michel)
- (M, g) Einstein : $\mathrm{Ric} = f\mathrm{g}$
  Existence of a second order symmetry (B. Carter)

- If $D \in \mathcal{D}^k_{\lambda_0,\lambda_0}(M)$ reads

$$\sum_{|\alpha|\leqslant k} D^\alpha \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

- If $D \in \mathcal{D}^k_{\lambda_0,\lambda_0}(M)$ reads

$$\sum_{|\alpha| \leqslant k} D^\alpha \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

$$\sigma(D) = \sum_{|\alpha|=k} D^\alpha p_1^{\alpha_1} \dots p_n^{\alpha_n},$$

where (x^i, p_i) are the canonical coordinates on T^*M

- If $D \in \mathcal{D}^k_{\lambda_0,\lambda_0}(M)$ reads

$$\sum_{|\alpha|\leqslant k} D^\alpha \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

$$\sigma(D) = \sum_{|\alpha|=k} D^\alpha p_1^{\alpha_1} \dots p_n^{\alpha_n},$$

where (x^i, p_i) are the canonical coordinates on T^*M

- $\sigma(D)$ can be viewed as a contravariant symmetric tensor of degree k :

$$\sigma(D) = \sum_{|\alpha|=k} D^\alpha \partial_1^{\alpha_1} \vee \dots \vee \partial_n^{\alpha_n}$$

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = \mathrm{g}^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = \mathrm{g}^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
- Conformal Killing tensor K : trace-free part of $\nabla_{(i_0} K_{i_1 \cdots i_k)}$ vanishes

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = \mathrm{g}^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
- Conformal Killing tensor K : trace-free part of $\nabla_{(i_0} K_{i_1 \cdots i_k)}$ vanishes
- If D is a symmetry of Δ_Y, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = \mathrm{g}^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
- Conformal Killing tensor K : trace-free part of $\nabla_{(i_0} K_{i_1 \cdots i_k)}$ vanishes
- If D is a symmetry of Δ_Y, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor
- Killing tensor K : $\nabla_{(i_0} K_{i_1 \cdots i_k)} = 0$

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = g^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
- Conformal Killing tensor K : trace-free part of $\nabla_{(i_0} K_{i_1 \cdots i_k)}$ vanishes
- If D is a symmetry of Δ_Y, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor
- Killing tensor K : $\nabla_{(i_0} K_{i_1 \cdots i_k)} = 0$
- The existence of a (conformal) Killing tensor is necessary to have the existence of a (conformal) symmetry of Δ_Y

- If D is a conformal symmetry of Δ_Y, there exists an operator D' such that $\Delta_Y \circ D = D' \circ \Delta_Y$
- $\sigma(\Delta_Y) = H = g^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
- Conformal Killing tensor K : trace-free part of $\nabla_{(i_0} K_{i_1 \cdots i_k)}$ vanishes
- If D is a symmetry of Δ_Y, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor
- Killing tensor K : $\nabla_{(i_0} K_{i_1 \cdots i_k)} = 0$
- The existence of a (conformal) Killing tensor is necessary to have the existence of a (conformal) symmetry of Δ_Y
- Is this condition sufficient ?

Definition

A quantization on M is a linear bijection $\mathcal{Q}^M_{\lambda,\lambda}$ from the space of symbols $\mathrm{Pol}(T^*M)$ to the space of differential operators $\mathcal{D}_{\lambda,\lambda}(M)$ such that

$$\sigma(\mathcal{Q}^M_{\lambda,\lambda}(S)) = S, \quad \forall S \in \mathrm{Pol}(T^*M)$$

Definition

A natural and conformally invariant quantization is the data for every manifold M of a quantization $\mathcal{Q}^M_{\lambda,\lambda}$ depending on a pseudo-Riemannian metric defined on M such that

- *If Φ is a local diffeomorphism from M to a manifold N, then one has*

$$\mathcal{Q}^M_{\lambda,\lambda}(\Phi^*\mathrm{g})(\Phi^* S) = \Phi^*(\mathcal{Q}^N_{\lambda,\lambda}(\mathrm{g})(S)),$$

  for all pseudo-Riemannian metric g on N and all $S \in \mathrm{Pol}(T^*N)$
- *$\mathcal{Q}^M_{\lambda,\lambda}(\mathrm{g}) = \mathcal{Q}^M_{\lambda,\lambda}(\tilde{\mathrm{g}})$ whenever g and $\tilde{\mathrm{g}}$ are conformally equivalent, i.e. whenever there exists a function Υ such that $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$.*

- Proof of the existence of $\mathcal{Q}^M_{\lambda,\lambda}$:
  1. Work by A. Cap, J. Silhan
  2. Work by P. Mathonet, R.

- If K is a conformal Killing tensor of degree 2, there exists a conformal symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3 A_{jl}{}^i \right)$$

- If K is a conformal Killing tensor of degree 2, there exists a conformal symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3 A_{jl}{}^i \right)$$

- C : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}(\mathrm{g}_{a[c}\mathrm{Ric}_{d]b} - \mathrm{g}_{b[c}\mathrm{Ric}_{d]a}) + \frac{2}{(n-1)(n-2)}\mathrm{R}\mathrm{g}_{a[c}\mathrm{g}_{d]b}$$

- If K is a conformal Killing tensor of degree 2, there exists a conformal symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3A_{jl}{}^i \right)$$

- C : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}(\mathrm{g}_{a[c}\mathrm{Ric}_{d]b} - \mathrm{g}_{b[c}\mathrm{Ric}_{d]a}) + \frac{2}{(n-1)(n-2)}\mathrm{R}\mathrm{g}_{a[c}\mathrm{g}_{d]b}$$

- A : Cotton-York tensor :

$$A_{ijk} = \nabla_k \mathrm{Ric}_{ij} - \nabla_j \mathrm{Ric}_{ik} + \frac{1}{2(n-1)} (\nabla_j \mathrm{R} g_{ik} - \nabla_k \mathrm{R} g_{ij})$$

- If $\mathrm{Obs}(K)^\flat = 2df$, the conformal symmetries of Δ_Y whose the principal symbol is given by K are of the form

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) - f + L_X^{\lambda_0} + c,$$

where X is a conformal Killing vector field and where $c \in \mathbb{R}$

- If K is a Killing tensor of degree 2, there exists a symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form

- If K is a Killing tensor of degree 2, there exists a symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form
- If $\mathrm{Obs}(K)^\flat = 2df$, the symmetries of Δ_Y whose the principal symbol is given by K are of the form

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) - f + L_X^{\lambda_0} + c,$$

where X is a Killing vector field and where $c \in \mathbb{R}$

- Remarks :
  1. If (M, g) is conformally flat, no condition on the (conformal) Killing tensor K

- Remarks :
  1. If (M, g) is conformally flat, no condition on the (conformal) Killing tensor K
  2. If $\mathrm{Ric} = \frac{1}{n}\mathrm{Rg}$ and if K is a Killing tensor of degree 2, then

$$\mathrm{Obs}(K)^\flat = d\left(\frac{2-n}{2(n+1)}(\nabla_i\nabla_j K^{ij}) + \frac{2-n}{2n(n-1)}\mathrm{Rg}_{ij}K^{ij}\right)$$

and $\nabla_i K^{ij}\nabla_j$ is a symmetry of Δ_Y

- On $\mathbb{R}^3$, pairs of (local) diagonal metrics and (local) Killing tensors are classified :

- On $\mathbb{R}^3$, pairs of (local) diagonal metrics and (local) Killing tensors are classified :
  Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$:

$$\frac{1}{2(\gamma(x_1, x_2) + c(x_3))} \left(a(x_1, x_2) p_1^2 + b(x_1, x_2) p_2^2 + p_3^2 \right) ,$$

- On $\mathbb{R}^3$, pairs of (local) diagonal metrics and (local) Killing tensors are classified :
  Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$:

  $$\frac{1}{2(\gamma(x_1, x_2) + c(x_3))} \left(a(x_1, x_2) p_1^2 + b(x_1, x_2) p_2^2 + p_3^2 \right),$$

  Killing tensor K :

  $$\frac{c(x_3) a(x_1, x_2) p_1^2 + c(x_3) b(x_1, x_2) p_2^2 - \gamma(x_1, x_2) p_3^2}{\gamma(x_1, x_2) + c(x_3)},$$

  $a, b, \gamma \in C^\infty(\mathbb{R}^2)$, $c \in C^\infty(\mathbb{R})$.

- If $\tilde{g} = \frac{1}{2(\gamma(x_1,x_2)+c(x_3))} g$, then

$$\mathrm{Obs}(K)^\flat = d(-\frac{1}{8}(3\mathrm{Ric}_{ij} - \mathrm{R}\tilde{g}_{ij})K^{ij})$$

- If $\tilde{\mathrm{g}} = \frac{1}{2(\gamma(x_1,x_2)+c(x_3))}\mathrm{g}$, then

$$\mathrm{Obs}(K)^\flat = d(-\frac{1}{8}(3\mathrm{Ric}_{ij} - \mathrm{R}\tilde{\mathrm{g}}_{ij})K^{ij})$$

- Symmetry of Δ_Y :

$$\nabla_i K^{ij}\nabla_j - \frac{1}{16}(\nabla_i\nabla_j K^{ij}) - \frac{1}{8}\mathrm{Ric}_{ij}K^{ij}$$

- Four-dimensional fiber bundle M over S^2 with coordinates (ψ, r, θ, ϕ)
- Taub-NUT metric g :

$$\left(1+\frac{2m}{r}\right)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right) + \frac{4m^2}{1+\frac{2m}{r}}\left(d\psi+\cos\theta d\phi\right)^2$$

- Four-dimensional fiber bundle M over S^2 with coordinates (ψ, r, θ, ϕ)
- Taub-NUT metric g :

$$\left(1+\frac{2m}{r}\right)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right) + \frac{4m^2}{1+\frac{2m}{r}}\left(d\psi+\cos\theta d\phi\right)^2$$

- g hyperkähler : there exist three complex structures J_i which are covariantly constant and which satisfy the quaternion relations

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathrm{Id}.$$

- The skewsymmetric tensor Y of degree 2 is Killing-Yano iff $\nabla_{(\lambda}Y_{\mu)\nu} = 0$

- The skewsymmetric tensor Y of degree 2 is Killing-Yano iff $\nabla_{(\lambda}Y_{\mu)\nu} = 0$
- Killing-Yano tensor Y :

$$2m^2\,(d\psi + \cos\theta d\phi)\wedge dr + r(r+m)(r+2m)\sin\theta d\theta\wedge d\phi$$

- The skewsymmetric tensor Y of degree 2 is Killing-Yano iff $\nabla_{(\lambda} Y_{\mu)\nu} = 0$
- Killing-Yano tensor Y :

  $2m^2\,(d\psi + \cos\theta d\phi)\wedge dr + r(r+m)(r+2m)\sin\theta d\theta\wedge d\phi$

- $*Y$ conformal Killing-Yano tensor :

$$\nabla_{(\lambda} * Y_{\mu)\nu} = \frac{2}{3}(\mathrm{g}_{\lambda\mu}\nabla_\kappa(*Y^\kappa_\nu) + \nabla_\kappa(*Y^\kappa_{(\lambda})\mathrm{g}_{\mu)\nu})$$

- The skewsymmetric tensor Y of degree 2 is Killing-Yano iff $\nabla_{(\lambda} Y_{\mu)\nu} = 0$
- Killing-Yano tensor Y :

  $2m^2\,(d\psi + \cos\theta d\phi)\wedge dr + r(r+m)(r+2m)\sin\theta d\theta\wedge d\phi$

- $*Y$ conformal Killing-Yano tensor :

$$\nabla_{(\lambda} * Y_{\mu)\nu} = \frac{2}{3}(\mathrm{g}_{\lambda\mu}\nabla_\kappa(*Y^\kappa_\nu) + \nabla_\kappa(*Y^\kappa_{(\lambda})\mathrm{g}_{\mu)\nu})$$

- J_i Killing-Yano tensors, hence

$$K_i = p_\mu p_\nu \left(*Y^{(\mu}_\lambda J^{\nu)\lambda}_i\right)$$

  conformal Killing tensors

- The skewsymmetric tensor Y of degree 2 is Killing-Yano iff $\nabla_{(\lambda}Y_{\mu)\nu} = 0$
- Killing-Yano tensor Y :

  $2m^2\,(d\psi + \cos\theta d\phi)\wedge dr + r(r+m)(r+2m)\sin\theta d\theta\wedge d\phi$

- $*Y$ conformal Killing-Yano tensor :

$$\nabla_{(\lambda} * Y_{\mu)\nu} = \frac{2}{3}(\mathrm{g}_{\lambda\mu}\nabla_\kappa(*Y^\kappa_\nu) + \nabla_\kappa(*Y^\kappa_{(\lambda})\mathrm{g}_{\mu)\nu})$$

- J_i Killing-Yano tensors, hence

$$K_i = p_\mu p_\nu \left(*Y_\lambda^{(\mu} J_i^{\nu)\lambda}\right)$$

  conformal Killing tensors
- $\mathrm{Obs}(K_i)^\flat$ not exact, then there are no conformal symmetries whose principal symbols are the K_i

- Laplace equation : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential

- Laplace equation : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential
- Helmholtz equation : $(\Delta_Y + V)\psi = E\psi$, $V \in C^\infty(M)$ is a fixed potential and $E \in \mathbb{R}$ a free parameter

- Laplace equation : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential
- Helmholtz equation : $(\Delta_Y + V)\psi = E\psi$, $V \in C^\infty(M)$ is a fixed potential and $E \in \mathbb{R}$ a free parameter
- Solving Helmholtz equation : finding a solution for all E

- Laplace equation R-separable in an orthogonal coordinates system (x^i) ($\mathrm{g}_{ij} = 0$ if $i \neq j$)

$$\Longleftrightarrow$$

$\exists$ $n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R = \sum_{i=1}^{n} h_i L_i .$$

- Helmholtz equation R-separable in an orthogonal coordinates system (x^i)

$$\Longleftrightarrow$$

$\forall\ E \in \mathbb{R},\ \exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^{n} h_i L_i .$$

- Helmholtz equation R-separable in an orthogonal coordinates system (x^i)

$$\Longleftrightarrow$$

$\forall\ E \in \mathbb{R}$, $\exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^{n} h_i L_i.$$

- $R\prod_{i=1}^n \phi_i(x^i)$ solution of the Laplace or Helmholtz equation

$$\Longleftrightarrow$$

$$L_i\phi_i = 0 \quad \forall i$$

- Laplace equation R-separates in an orthogonal coordinate system if and only if :

- Laplace equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ conformal Killing tensors (K_i), where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,

- Laplace equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ conformal Killing tensors (K_i), where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \ldots, K_n\}$ is linearly independent over $C^\infty(M)$,

- Laplace equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ conformal Killing tensors (K_i), where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \ldots, K_n\}$ is linearly independent over $C^\infty(M)$,
    - as endomorphisms of TM, $K_1, \ldots, K_n$ admit a basis of common eigenvectors.

- Laplace equation R-separates in an orthogonal coordinate system if and only if :
  - (a) $\exists$ a family of $n-1$ conformal Killing tensors (K_i), where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \ldots, K_n\}$ is linearly independent over $C^\infty(M)$,
    - as endomorphisms of TM, $K_1, \ldots, K_n$ admit a basis of common eigenvectors.
  - (b) $\exists$ second order conformal symmetries D_i, i.e. $[\Delta_Y + V, D_i] \in (\Delta_Y + V)$, with principal symbols $\sigma_2(D_i) = K_i$, for all $i = 2, \ldots, n$.

- Helmholtz equation R-separates in an orthogonal coordinate system if and only if :

- Helmholtz equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ Killing tensors (K_i) where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,

- Helmholtz equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ Killing tensors (K_i) where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \ldots, K_n\}$ is linearly independent over $C^\infty(M)$,

- Helmholtz equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ Killing tensors (K_i) where $i = 2, \dots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \dots, K_n\}$ is linearly independent over $C^\infty(M)$,
    - as endomorphisms of TM, $K_1, \dots, K_n$ admit a basis of common eigenvectors.

- Helmholtz equation R-separates in an orthogonal coordinate system if and only if :
  (a) $\exists$ a family of $n-1$ Killing tensors (K_i) where $i = 2, \ldots, n$, such that, together with $K_1 = \mathrm{g}^{ij} p_i p_j$,
    - they Poisson commute : $\{K_i, K_j\} = 0$ for all i, j,
    - the set $\{K_1, \ldots, K_n\}$ is linearly independent over $C^\infty(M)$,
    - as endomorphisms of TM, $K_1, \ldots, K_n$ admit a basis of common eigenvectors.

  (b) $\exists$ second order symmetries D_i, i.e. $[\Delta_Y + V, D_i] = 0$, with principal symbols $\sigma_2(D_i) = K_i$, for all $i = 2, \ldots, n$.

- Link between the (conformal) symmetries and the R-separating coordinate systems :

- Link between the (conformal) symmetries and the R-separating coordinate systems :
- Eigenvectors of the $K_i \longleftrightarrow$ integrable distributions

- Link between the (conformal) symmetries and the R-separating coordinate systems :
- Eigenvectors of the $K_i \longleftrightarrow$ integrable distributions
- Leaves of the corresponding foliations $\longleftrightarrow$ Coordinates lines of the R-separating coordinate systems