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Schematic representations in arithmetical problem solving: Analysis of their impact on grade 4 students

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Abstract While the value of 'schematic representations' in problem solving requires no further demonstration, the way in which students should be taught how to construct these representations invariably gives rise to various debates. This study, conducted on 146 grade 4 students in Luxembourg, analyzes the effect of two types of 'schematic representation' (diagrams vs. schematic drawings) on the solving of arithmetical problems. The results show that the presence of schematic representations has a clear positive effect on overall student performance and that a non negligible proportion of students manage to reuse the representations encountered in order to solve new problems. While showing an effect slightly in favor of diagrams as opposed to schematic drawings, our results do not really permit us to draw any conclusions about the form that these representations should take, in particular since a differential effect was observed depending on the type of problem.

Keywords Arithmetical problem solving · Schematic representations · Diagrams · Schematic drawings · Non-routine problems

1 Introduction

The role of representations in the problem solving process is now widely recognized by the scientific community in the field of mathematics teaching and learning. In recent decades, the value of representations, generally defined as a configuration of symbols, images or concrete objects standing for some other entity (Dewindt-King & Goldin, 2003; Monoyiou,

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Papageorgiou, & Gagatsis, 2007), has been highlighted by different strands of research in mathematics education. In cognitive psychology, several authors (Julo, 2002; Schoenfeld, 1992; Thevenot, Barrouillet, & Fayol, 2010) have shown that an expert problem solving approach is defined as a multistep process in which the phase of representing the problem is a central element. In this context, effective representations consist of 'schematic representations' (Hegarty & Kozhenikov, 1999) which bring out the problem's main data and the relationships between them. In the field of arithmetical problems, Julo (2002) explains that two series of complementary work led to the emergence in the 1980s of a research focus on the role of these 'schematic representations' in problem solving: firstly, the work of Vergnaud (1983) on the conceptual field of addition and multiplication problems, and secondly, the work of Schoenfeld (1992), showing that a key feature of expertise was the ability to categorize the problems presented. Thus, according to Julo (2002), numerous studies have sought to analyze the possibilities of using representational tools to help students to distinguish certain classes of problems. The studies conducted in the tradition of research in cognitive psychology have thus mainly emphasized the teaching of predefined representations corresponding to categories of problems as an 'external aid' in problem solving.

More recent studies, conducted from a sociocultural perspective, have brought out the reciprocal link between meaning in mathematics and the construction of 'models' (Cobb, Yackle, & McClain, 2000; Gravemeijer, Lehrer, van Oers, & Verschaffel, 2002). The authors emphasize that the value of models lies mainly in the modeling activity itself, in that it is the construction of models that enables a person to make sense of mathematical situations and vice versa. From this perspective, models, like representations, are symbolic configurations, but in this case there is the idea that models represent not so much entities as the actions undertaken in the context of the teaching/learning situations in question (Gravemeijer et al., 2002; van Dijk, van Oers, & Terwel, 2003). Rather than provide students with 'predefined' models or representations as problem-solving aids, it is thus up to the students themselves to construct the symbols which will help them to create meaning in communication situations. The production of models is seen not as an external aid to problem solving, but as an internal process developed by students and forming part of the problem solving process itself.

These debates on the issue of representations or models reflect the very nature of all symbolization which, as Stylianou (2010) recently pointed out, refers both to a final product and to a process, i.e. both to the act of capturing a mathematical concept or relationship in a given form and to the form of this symbolization itself.

The overall objective of this article is to analyze the impact of 'schematic representations' on the solving of 'non-routine' problems. These are defined as problems where the solution does not appear immediately and the resolution of which does not involve applying a procedure which has just been presented in class (Diezmann, 2002; Pantziara, Gagatsis, & Elia, 2009). More specifically, the empirical study that is the subject of this paper seeks to assess firstly the effect of the presence of 'schematic representations' on the solving of arithmetical problems by grade 4 students (effect of the product), and secondly the manner in which these same students reuse these 'schematic representations' later on (effect of the product on the process).

2 Theoretical framework

While the value of 'schematic representations' or 'models' in problem solving requires no further demonstration, the way in which students should be taught to represent a situation or a problem invariably gives rise to various debates. Several studies (Hegarty & Kozhenikov, 1999; Uesaka, Manalo, & Ichikawa, 2007; van Garderen & Montague, 2003) show quite clearly that the construction of an appropriate representation (mentally or externally on paper) is a factor related to performance in problem solving. 'Schematic' representations (which bring out the problem's main data and the relationships between them) are effective whereas 'pictorial' representations (which represent the overall situation and emphasize the visual appearance of objects rather than the mathematical relationships) are relatively ineffective (Hegarty & Kozhenikov, 1999). Several questions remain unanswered, in particular concerning the precise form that these 'schematic representations' should take, and how students can be effectively led to construct them.

From the early 1980s, research in cognitive psychology showed the effectiveness of teaching predefined schemas corresponding to categories of problem in order to improve students' performance in problem solving. The objective of this theory of 'problem schemas' was to teach students schemas that they were supposed to select and reproduce in order to solve a problem. More recently, especially in France, the diagrams proposed by Vergnaud have been seen as significant in making explicit the relational structure characterizing each class of problem (Julo, 2002). In this type of approach, these schemas are regarded as more or less directly linked to problem solving procedures, in that they reflect the different categories of problem, and their use (mentally or externally on paper) enabling problems to be solved (Thevenot et al., 2010). Several studies have shown the effectiveness of this 'problem schema' approach (see in particular Levain, Le Borgne, & Simar, 2006 for multiplication problems inspired by the typology of Vergnaud, 1983; Willis & Fuson, 1988 for problems inspired by the typology of Riley, Greeno, & Heller, 1983).

While adhering to this concept of 'problem schemas' and the importance of categorizing problems, Julo (2002) criticizes this type of teaching approach by raising questions about the complex status of the 'schematic representations' proposed. He believes that this approach is risky, in that 'problem schemas' could be created by students using very different logic to that proposed. In other words, Julo (2002) calls into question the didactic approach which aims to teach predefined schematic representations, but does not really appear to question the notion itself of 'problem schemas' in the sense of a mental representation used to categorize types of problem and which is a sign of expertise.

We may note that other recent research, also in the context of cognitive approaches, now questions this theory of 'problem schemas'. Thus a study by Coquin-Viennot and Moreau (2003) in particular showed the impact of context on the use of problem solving procedures, thus suggesting the use of representations that are less formal than 'schemas' corresponding to different categories of problem, involving instead a 'mental model' (or 'episodic situation model') which takes into account the context in which the mathematical situation is set (Thevenot et al., 2010). Furthermore, the predefined schemas only seem to work for a limited number of problems which correspond to the categories they represent. After carrying out a literature review focusing especially on arithmetical problems, Verschaffel, Greer and De Corte (2007) also point very clearly to the limits of these schematizations (and hence of the experiments based on teaching them): although they seem effective and well suited to routine problems which can be symbolized by an arithmetical operation using the numbers drawn from the problem, it is questionable whether they are able to represent all the arithmetical problems that might be encountered in the real world. In other words, rather than being aids to solving varied and non-routine problems, Verschaffel et al. (2007) hold that "most of this research has instead used a very restricted set of scholastic word problem versions of the real world, without much reflection on the complexities from a sociocultural perspective" (p. 584). Moreover, from our point of view, there are also reasons for fearing that the systematic teaching of this type of schematization can lead to the development of superficial strategies in students (trying to 'guess' the appropriate schema rather than actually constructing a representation of the situation). On the basis of the results mentioned above, Thevenot et al. (2010) think it would be preferable to have children work on a wide variety of problems (with the emphasis on analysis and interpretation of problem situations) rather than having them mechanically learn procedures relating to particular types of problems. Thus they propose in particular that children should be encouraged to construct schematic representations themselves to represent the relationships between the various elements of the problem.

Another body of research has also looked at 'schematic representations', but from a different perspective than that described above. Rather than relying on categories of problems, Novick and colleagues (Novick, 2006; Novick, Hurley, & Francis, 1999) propose an analysis of various schematizations that they call 'spatial diagrams', which would seem to cover a wide range of problems and adapt well to complex and non-routine problems. Figure 1 illustrates the four types of 'spatial diagram' identified by these researchers.

The original aspect of this approach is the idea that these 'spatial diagrams' are not tied to a specific type of problem, firstly because the same schema can be used to represent two different problem structures (e.g. the matrix for a logic problem and for a multiplicative combination problem) and, secondly, the same problem structure can be represented by two different types of diagram (e.g. a matrix or a hierarchical schema for a multiplicative combination problem). In addition, work which has focused on teaching students to use this type of diagram (see in particular Diezmann, 2002; van Garderen, 2007) shows that they seem to allow flexibility which takes into account the situational context, as evidenced in particular by the illustrations in Fig. 2. In this sense, this kind of 'schematic representation' seems to us to meet some of the criticisms made of the theory of 'problem schemas', and to allow representations to be constructed that are more akin to the concept of the 'episodic situation model' which has emerged in recent work in cognitive psychology (Thevenot et al., 2010). They also seem to us to enable a broader set of problems to be handled than the schematizations directly based on problem typologies (Verchaffel et al., 2007).

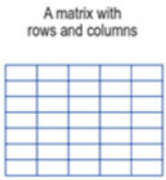
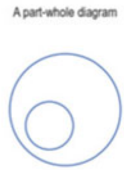
Network type diagrams	Matrix type diagrams	Hierarchy type diagrams	Part-whole type diagrams
			
Networks allow information to be arranged in a chronological or geographic sequence, such as timelines or train route maps respectively. These diagrams consist of sets of nodes (points) with one or more lines linking the nodes together.	**Matrices** accommodate categories of information. They are useful in combinatorial or deductive problems. Matrices use the two dimensions to convey the relationships between two sets of problem data, thereby making implicit relationships within the information more explicit.	**Hierarchies** represent levels of information that either increase or decrease. Tree diagrams, family trees, genetics and probability are some useful applications of hierarchies.	**Part-whole diagrams** show the relationship between a referent and at least one of its parts. For example, a diagram of a chocolate bar and a piece of chocolate is a part-whole diagram. Venn Diagrams are also part-whole diagrams

Fig. 1 Spatial diagrams proposed to represent non-routine problems (from Diezmann, 2002, pp. 4–5)

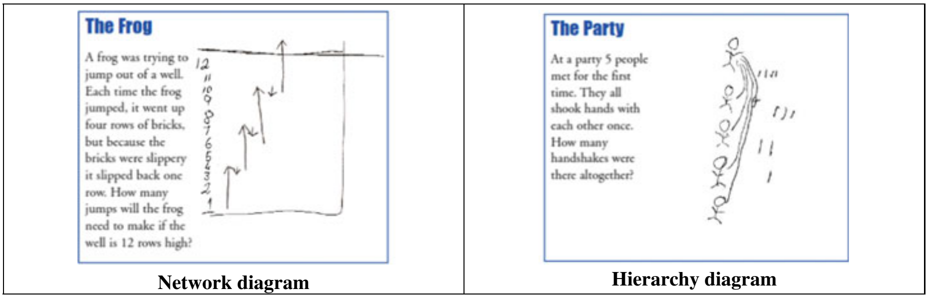


Fig. 2 Schematizations produced by ten-year-old students who had followed a training program based on spatial diagrams (from Diezmann, 2002, p. 8)

Several studies have shown the effectiveness of educational programs based on learning this type of schematization (Diezmann, 2002; van Garderen, 2007), including for students with learning disabilities (Jitendra, DiPipi, & Perron-Jones, 2002; Montague, Wagner, & Morgan, 2000). However, these results are qualified by the work of Pantziara et al. (2009). These authors examined the effect of the presence of these diagrams on solving non-routine problems by contrasting the results obtained by students when solving these problems freely with those obtained when the same students were confronted with diagrams to complete. The results show in particular that while the presence of these schemas helps some students, it seems to confuse others. Moreover, it also appears that the presence of schemas affects the hierarchy between problems, which we believe shows the importance of taking into account the structure of the problems presented. Finally, Pantziara et al. (2009) suggest that the ability to interpret and choose a 'diagram' which relates to a given problem can be an obstacle for some students, as the diagrams may be incompatible with the mental representations that they have constructed. The criticisms often directed at the teaching of predefined representations relate to students' difficulty in making sense of the conventions used, which belong to an adult way of thinking which is not necessarily meaningful to them (van Dijk et al., 2003).

From this perspective, the research carried out in the tradition of sociocultural approaches provides more support for the idea of having students produce their own models (Bednarz, Dufour-Janvier, Poirier, & Bacon, 1993; Gravemeijer, 1997; van Dijk et al., 2003). The production of models (or schematic representations) aims, as in the other approaches, to enable students to make sense of mathematical problems and to assist them in the process of solving them, but it occurs more specifically in communication situations involving argument and negotiation about 'taken-as-shared' meanings (Gravemeijer, 1997) which will enable students' mathematical reasoning to develop from the starting point of their informal symbolizations. The emphasis is more on the 'process' (the construction of the schematic representation and communication relating to this construction) than on the 'product' (the type of schematic representation produced) (van Dijk et al., 2003), the idea being that the meaning and the activity of modeling co-emerge (Cobb et al., 2000).

The debate on the different approaches that may be promoted in order to teach schematization could be further highlighted by the field of research which examines cognitive flexibility in problem solving strategies and, more recently, in representations (Heinze, Star, & Verschaffel, 2009). From this perspective, it is contended in particular that one characteristic of mathematical competence is the ability to "choose flexibly and adaptively between available representations" (Heinze et al., 2009, p. 537). In this sense, both schematizations produced by the students themselves ('self-generated drawings') and schematizations derived from the typology of Novick et al. (1999) should permit greater flexibility than those derived from research on schemas which are tied to specific categories of problems (schemas from the typologies of Riley et al., 1983 or Vergnaud, 1983).

Furthermore, although recent studies (van Dijk et al., 2003; Diezmann, 2002; van Garderen, 2007) broadly favor the construction of representations by students and hence the importance of ‘process’, the question remains as to the starting point of this construction process. Is it a good idea to give students ‘predefined representations’ so that they can subsequently reappropriate them in their own way (Diezmann, 2002; van Garderen, 2007), or is it better to take their own models as the starting point (Gravemeijer, 2002; van Dijk et al., 2003)? Starting from students ‘self-generated drawings’ could be an interesting alternative to the teaching of predefined schematizations. The few studies on the construction of models by students in problem solving seem to demonstrate a positive effect (Mousoulides, Christou, & Sriraman, 2008; van Essen & Hammaker, 1990; van Dijk et al., 2003; Van Meter & Garner, 2005).

According to van Dijk et al. (2003), having students produce models implies that they must invent something new: they must produce models which emerge from their own work. Two questions thus arise: Can students create such models from scratch (van Dijk et al., 2003) and what should one do if they do not invent the mathematical models that need to be taught (Gravemeijer, 1997)?

By means of an empirical study in the fourth grade of primary school in Luxembourg (grade 4, students aged around 10 years), our study aims to provide data to clarify some of the gray areas left by previous research. It aims to analyze the effect of two types of ‘schematic representation’ on students: ‘diagrams’ belonging to the typology of Novick (2006) and highly contextualized ‘schematic drawings’ which are close to the informal models that students might construct themselves. While the ‘diagrams’ can represent an obstacle for some students because of the conventions they use (Gravemeijer, 1997) and/or incompatibility with their mental models (Pantziara et al., 2009), the ‘schematic drawings’ which are closer to students’ self-generated drawings use fewer conventions and could prove more accessible to students. These ‘schematic drawings’ are more contextualized and could in this sense be consistent with the theory of ‘episodic situation models’ (Thevenot et al., 2010) and allow greater flexibility (Verschaffel et al., 2007); on the other hand, they could also complicate the emergence of categories of problems and hinder (or at least not help) the construction of a ‘problem schema’ (Julo, 2002).

This study aims to measure the effect of two types of ‘schematic representation’ (diagrams vs. schematic drawings) on the solving of arithmetical problems. More specifically, three research questions were defined: (1) What is the effect of the presence of each type of ‘schematic representation’ (diagrams vs. schematic drawings) on the solving of arithmetical problems (effect of the product)? (2) How do students reuse each of these schematic representations in problem solving (effect of the product on the process)? (3) To what extent, if at all, does the presence and/or production of these schematic representations have a differential effect on the hierarchization of problems?

3 Method

A paper and pencil test was given to 11 grade four classes in six Luxembourg primary schools. A total of 146 students participated in the three parts of the test. The students in the sample were not very familiar with creating representations in order to solve problems. A recent analysis of mathematics textbooks widely used by teachers in Luxembourg reflects the lack of emphasis on this skill in problem solving (Fagnant & Burton, 2009). The tests were conducted between 18 January and 6 February 2010. Testing took place in three parts, each consisting of four problems. The first two parts of the test were performed one directly after the other, with the third part being given after a slight delay (in the afternoon of the same day or on another day of the same week).

The first part was the same for all students. It contained four complex problems which students were asked to solve, putting down their workings on the sheet. No specific instructions were given on how to solve the problems. On the basis of the analysis of textbooks mentioned previously (Fagnant & Burton, 2009), these four problems can be described as non-routine for the students being tested due to the type of the problems (their mathematical structure), the position of the unknown quantity, and lack of familiarity with the wording. Table 1 presents the four problems and describes their characteristics.

In the second part, four problems with the same structure were given to the students. These consisted of parallel versions of the problems in Part 1, similar in terms of mathematical structure and the size of the numbers involved, but with few surface similarities. This time, each problem was accompanied by one of the two 'schematic representations' (diagram or schematic drawing). Two different versions of the test were set depending on the type of schematization associated with the problem, and were randomly distributed in the classes (in each class, half the students received Version A and half received Version B). Version A had diagrams based on the classification of Novick (2006), while Version B had schematic drawings inspired by free drawings produced by students in the fourth grade of primary school during another research project (Demonty, Fagnant, & Lejong, 2004). Table 2 shows the types of problem, the types of diagram or drawing proposed and the codes used to identify the problems in the rest of this paper. Examples of schematic representations are given in Appendix 1.

The *third part* of the test presented four new problems parallel to those set previously. This time, students were invited to solve them by producing a representation of the same type as those they encountered in the second part of the test. Once again, two versions of the test were set, differing in the specific instructions given to the students on how to construct representations. The goal was to see to what extent students are able to make effective reuse of the types of representation previously encountered.

The first two steps of the test procedure (Parts 1 and 2) will enable us to analyze the first research question (effect of the product). These steps 'reproduce' the approach proposed by Pantziara et al. (2009), while adding the possibility of a comparison between two modes of schematization.

In addition, the use of a three-step procedure allows us to throw light on the second research question (effect of the product on the process) by bringing us closer to a quasi-experimental design in which Part 1 corresponds to a pretest phase, Part 2 to a (kind of) intervention phase in which we present students with a problem solving tool and Part 3 to a post-test phase. From this standpoint, it will be interesting to compare the results obtained in Part 1 with those obtained in Part 3. Obviously, the second phase is not really a 'teaching experiment': the students were given a problem solving tool, but no explanation was provided and no discussion/confrontation/correction phase was organized following the completion of this step.

Light will be thrown on the third research question (the differential impact of representations according to the type of problem) with the help of an item response model (the Rasch model) which will enable all the problems (the two versions and three parts) to be placed on a common scale.

Finally, as the chosen procedure does not preclude a possible 'testing effect', precautions were taken to ensure that students did not identify the parallelism between the problems in

Table 1 The four types of problems presented to the students

	Problems in part 1	Type of problem	Position of the unknown quantity	Familiarity F: familiar NF: non-familiar
P1	Antoine collects cards from the series 'The Simpsons'. He already has 25 different cards. His friend Hassan gives him eight new cards. Antoine sorts the cards and notices that he has some doubles. He gives these to Paula and Paula gives him four cards in return. After this, he counts all his cards and finds that he has 30. *How many cards did Antoine give to Paula?*	Change type addition problem	Intermediary	F
P2	Mr Lavetout has to clean a large window. He climbs nine rungs to reach the middle of the window. Suddenly he notices a spot that he has missed lower down and goes down five rungs. When he has cleaned up the spot, he checks the time on his watch. Finally he goes up to the twelfth rung where he can finish cleaning the window. *How many rungs did he climb after he checked the time?*	Change type addition problem	Intermediary	NF The starting point is not given explicitly
P3	At the start of the year, Mrs LEMAITRE orders supplies for her class. She buys 10 atlases for 15 euros each and 22 mathematics textbooks for 5 euros each. She also buys a large geographical map for 30 euros and a shelf to store the new books. All of these purchases together cost her 350 euros. *How much did the shelf cost?*	Problem involving products and a part-whole type relationship	Intermediary	F
P4	The Crusty bakery sells wholemeal bread, brown bread and white bread. Each type of bread is available as a baguette, a round loaf and a square loaf. *How many different loaves can you buy at this bakery?*	Multiplicative combination type problem	Final	NF

Table 2 Schematic representations corresponding to the four types of problems set in the two versions of the test (with the codes used subsequently)

	Type of problem	Familiarity	Diagram (version A)	Schematic drawing (version B)	Code used
P1	Change type addition problem (Change)	F	Network	Contextualized representation of states and transformations	Change-F
P2	Change type addition problem (Change)	NF	Network	Contextualized representation of states and transformations	Change-NF
P3	Problem involving products and a part-whole type relation (P-whole)	F	Part-whole diagram	Contextualized representation of parts and whole	P-whole-F
P4	Multiplicative combination type problem (Combine)	NF	Matrix	Contextualized representation of all possible combinations	Combine-NF

the three parts of the test too readily. Thus the problems were set in different contexts (i.e. they had few surface similarities), and the problem types were mixed up from one part of the test to another. For example, for problem 'P2' (see Table 1 and Figure 5, Appendix 1), there is a window cleaner in Part 1, a snail climbing up a wall in Part 2 and a flea trainer in Part 3. To achieve parallelism, all three cases involve an addition problem with several steps, with the unknown quantity relating to the last step and number sizes of less than 20.

4 Results

Before analyzing the impact of the schematic representations, two questions are worth asking: firstly, whether the problems set were sufficiently complex to require the construction of an externalized representation and, secondly, whether the students spontaneously use this type of representation. When problems are too simple, schematic representations seem to make them harder to solve by increasing the cognitive load without providing any genuine help (Berends & Van Lieshout, 2009; Elia, Gagatsis, & Demetriou, 2007). Moreover, if students spontaneously use effective representations, one might ask whether it is appropriate to seek to suggest other representations to them.

The analysis of the students' results for the four problems in Part 1 reveals significant difficulties in solving these non-routine problems. Success rates varied, depending on the problem, between 19 % and 34 %. It should be remembered that no special instructions were given regarding the problems and that the students were not explicitly asked to produce schematic representations. The 146 students were each presented with four problems, giving 584 opportunities to construct representations. However, only nine schematic representations were produced spontaneously. These nine schematic representations were drawings which mainly related to the 'multiplicative combination' type problem. This involved combining types of bread (white, brown, and wholemeal) with loaf shapes (square loaf, round loaf or baguette). The students generally drew the different possibilities or, in incorrect cases, only some of them. Out of the nine schematic representations, six led to a correct answer.

The vast majority of students had difficulty solving these types of problems, but did not spontaneously resort to schematic representations externalized on paper, despite the fact that these should potentially help them deal with these complex tasks. We will therefore investigate this in the next part of this article, following the directions of analysis which correspond to our research questions. Section 4.1 looks at the overall results relating to the effect of schematic representations and their reuse (questions 1 and 2). Section 4.2 considers the potential impact of the type of problem set, and whether there is a differential impact according to the type of problem and the type of schematization proposed (question 3).

4.1 The effect of schematic representations and their reuse

To measure the impact of the presence of schematic representations given along with the problems, we compared the mean scores of the two groups between Part 1 of the test (problems only) and Part 2 (problems accompanied by schematic representations). Then, to measure the possible 'reuse' of representations, we compared the scores between Part 1 of the test (problems only) and Part 3 (explicit invitation to produce a schematic representation of the same type as that found in Part 2). In both cases, we calculated an 'effect size' in order to quantify the magnitude of the difference between the two parts of the test, that is to say, the 'gain' contributed by the representations. Generally, following Cohen (1992), we consider an effect size of 0.2 as small, of 0.5 as moderate and of 0.8 as high. The effect size provides information on the size of the difference between two observed means, but does not necessarily allow generalizations to be made. These results are presented in Table 3 below.

Comparison of the mean success rates between Parts 1 and 2 shows that the presence of schematic representations immediately and clearly increases the students' overall success rates for problems, although these were still relatively low. The figures for the effect size reflect the non negligible impact of the presence of these representations; this positive effect occurs for both types of schematic representation investigated, diagrams (0.69) and schematic drawings (0.65), with diagrams coming out slightly better.

Students' mean scores are lower in Part 3 than in Part 2, which is contrary to what one would expect from a simple 'testing effect'. This suggests that, in the context of the test, the presence of schematic representations is a more effective aid for solving problems than encouragement to produce a representation oneself, even if such representations have been presented previously. It should be recalled that in the third phase of the test, students were explicitly asked to solve problems by generating schematic representations similar to those they had encountered in Part 2. In Version A, students were given three examples of diagrams (without data) as a reminder, but no indication was given about the type of diagram to be used for any given problem. In Version B, it was stated that drawings needed to be made which could take any form but must contain certain information (the main numerical data, the unknown quantity that needs to be worked out and an indication of the connections between the information).

Our results show that after very limited input, consisting simply of exposing students to a possible tool to help with problem solving, they were partially able to produce schematic representations and improve their problem solving performance (see comparison between

Table 3 Mean results of students for the four problems in Parts 1, 2 and 3 and effect size

	Average % success (and standard deviation)			Effect size	
	Part 1	Part 2	Part 3	Comparison of parts 1 and 2	Comparison of parts 1 and 3
Version A (diagrams)	27 % (30 %)	49 % (34 %)	42 % (32 %)	0.69	0.48
Version B (schematic drawings)	23 % (27 %)	42 % (33 %)	37 % (34 %)	0.65	0.46

Table 4 Mean percentage of representations produced by the students for the four problems in Part 3 of the test

	Version A—diagrams N=74	Version B—schematic drawings N=72
Percentage of representations	91 %	84 %
Percentage of correct representations, either complete or partial	44 %	36 %
Percentage of correct and complete representations	30 %	15 %

Parts 1 and 3 of the test, with an effect size of 0.48 and 0.46 respectively). These results seem to indicate considerable potential for schematic representations as an aid to problem solving and a possibility for teaching their use in an effective manner.

In order to examine in greater depth the reuse of the schematic representations given, we need to look at whether the improvement in scores can be attributed to the construction of schematic representations by the students. Table 4 shows the percentage of representations produced by the students in the two groups, on average for the four problems in Part 3 of the test, as well as the proportion of correct representations, whether complete (containing all the data in the problem and the relationships between them) or partial[1].

In response to the explicit request made in Part 3, a large majority of students produced schematic representations to solve the problems. Hardly any diagrams were produced in Part 1, whereas in Part 3 an average of more than 90 % of the group receiving Version A produced diagrams and nearly 85 % of the group receiving Version B.

Approximately 40 % of the students on average produced correct representations (44 % for Version A and 36 % for Version B), but fewer of them produced complete representations (30 % on average for Version A and 15 % for Version B). In Version B, the low percentages are partly explained by the large proportion of 'pictorial' representations (Hegarty & Kozhenikov, 1999; van Garderen & Montague, 2003) illustrating the context described in the problem but neglecting the numerical data.

Finally, it should again be noted that only the representations put down on paper could be analyzed, and that it is also possible that the effect of the presence of schematic representations in Part 2 also played a role in the construction of mental representations in Part 3. This hypothesis could explain the fact that the success rates in Part 3 (42 % for diagrams and 37 % for schematized drawings: see Table 3) exceed by far the percentages for complete and correct representations (30 % for diagrams and 15 % for schematized drawings: see Table 4).

Finally, the results described here show an overall positive effect of schematic representations on students' success in solving the problems. So far, however, the results have been analyzed in overall terms, and the question arises of whether there is a differential effect according to the type of problems. This is discussed in the following section.

4.2 Do the different types of schematic representation have a differential impact according to the type of problem?

To analyze the impact of schematic representations according to the structure of problems (P1, P2, P3 and P4, see Table 1), we have used an item response model (the Rasch model)

[1] The procedure for assessing the representations was, on the one hand, to identify the presence of the numerical data (the presence of figures or drawings representing the elements to be counted) and, on the other hand, to identify the presence of indicators of relationships between the data (eg, a set to represent a part-whole relationship, an arrow to represent a change, a crossed out element to signify a withdrawal, …).

which will enable us to analyze all the problems in the test on the same hierarchical scale, although some of them were given to different samples of students. The model makes it possible to create an anchor on the basis of the four problems common to the 146 students (the four problems in Part 1), and then to position the other 16 problems on the same scale (the four problems in Parts 2 and 3, considering the problems in Versions A and B as different since they are not accompanied by the same representations and the same instructions). In this analysis, we will look at how the twenty problems relate to one another hierarchically, thus moving away from the previous analyses which focused primarily on comparisons between Parts 1 and 2 (effect of the product) and between Parts 1 and 3 (effect of the product on the process). In order to do this, we will refer to the three parts of the test as 'the problems on their own' (Part 1), 'the problems accompanied by a representation provided from the outset' (Part 2) and 'the problems accompanied by explicit encouragement to produce a representation' (i.e. 'after having been shown this same type of representation before, in Part 2') (Part 3).

The item hierarchy as revealed by the Rasch model is given in Appendix 2. The most complex problems are found at the top of the graph and the simplest at the bottom. The crosses represent the students, positioned in relation to the problems for which they have a 50 % probability of success. The problems are named as explained in Table 2, and in addition, the numbers 1, 2 and 3 specify the parts of the test concerned and the letters a and b specify the version of the test (a for the diagrams and b for the schematic drawings). For example, the problem *Combine_NF_2a*, relates to the 'multiplicative combination' type problem with 'Non-familiar' presentation, from Part 2, Version A—diagrams (see Table 2 above).

Figure 3 distinguishes the three parts of the tests and the two kinds of schematic representation (Versions A and B). Part 1 is clearly distinct from the two other parts (the four problems in this part are the most complex in the test), which testifies to the help provided by the two types of schematic representations, under both conditions (schematizations given at the outset or explicit encouragement to produce a schematization). The diagram also shows extensive overlap in degree of difficulty between Parts 2 and 3, both for Version A (diagram) and Version B (schematic drawing). Finally, it can also be seen that the hierarchy of the four problems varies from one version to the other (A or B) and from one part of the test to another (Parts 1, 2 or 3), indicating a differential effect of the impact of representations on different types of problem.

Figure 4 arranges the items so as to illustrate the observed hierarchies for each type of problem.

For problems of the type **'part-whole familiar' (P-whole_F)**: in both versions (A and B), representation provided at the outset was more effective than encouragement to generate a representation (2a was slightly easier than 3a and 2b was slightly easier than 3b). The diagrams were more effective than the schematic drawings (2a was easier than 2b, and 3a easier than 3b).

For problems of the type '**change familiar' (change_F)**: in both versions (A and B), encouragement to generate a representation was more effective than representation provided at the outset (3a was easier than 2a and 3b easier than 2b). The diagrams were more effective than the schematic drawings (2a was easier than 2b) or were equally effective (3a and 3b were of equal difficulty).

For problems of the type **'change—non-familiar' (change_NF)**: in both versions (A and B), representation provided at the outset was more effective than encouragement to generate a representation (2a was very clearly easier than 3a and 2b was clearly easier than 3b). The diagrams were more effective than the schematic drawings (2a was clearly easier than 2b and 3a slightly easier than 3b).

For problems of the type '**multiplicative combination—non-familiar' (combine_NF)**: in Version B, representation provided at the outset was more effective than encouragement to

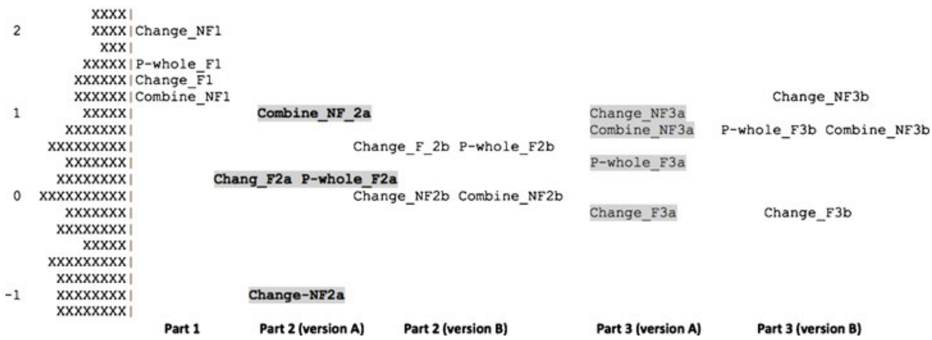


Fig. 3 Hierarchy of problems established by the Rasch model, showing the ranking of parts of the test as a function of the types of schematic representation

generate a representation (2b was clearly easier than 3b), while in Version A the two alternatives were slightly similar in effectiveness (2a was slightly similar in difficulty to 3a). The schematic drawings were more effective than the diagrams when provided at the outset (2b was clearly easier than 2a), while they were of a comparable level of effectiveness when students had to produce them themselves (3a and 3b were of equal difficulty).

Ultimately, in most cases there is a hierarchy across the three versions of the test: a problem on its own is more complex than a problem where the student is encouraged to produce a representation, which is more complex than a problem accompanied by a representation provided at the outset. The problems of the 'change-familiar' type were the only ones that really deviated from this hierarchy, since in both versions the students progressed in Part 3 of the test by seeming to make effective reuse of the representations that had been suggested, whether these were diagrams or schematic drawings. For this type of specific problem, the results are therefore very encouraging with regard to the possible effectiveness of teaching students to use schematic representations effectively in problem solving.

Finally, looking again at the comparison between the two forms of schematic representations, we find confirmation, in line with the results presented earlier, that diagrams appear to be slightly more effective than schematic drawings, with the exception of the matrix diagram for the problem involving a multiplicative combination (combine_NF_2a).

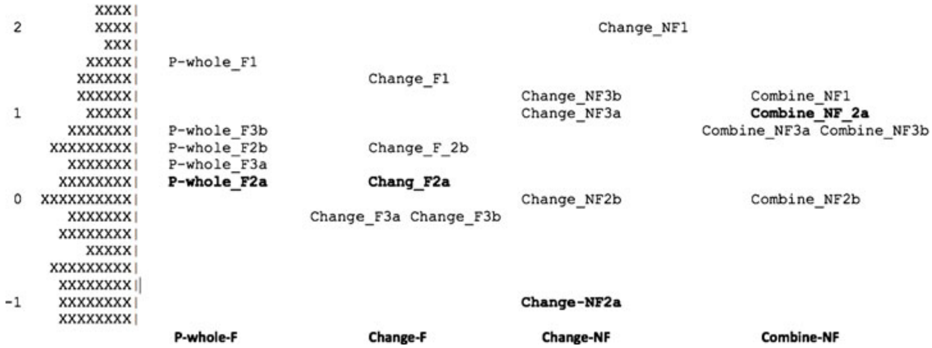


Fig. 4 Hierarchy of problems established by the Rasch model, showing the ranking of types of problems

While the general trends observed across the whole sample (results in Parts 2 and 3 better than those observed in Part 1) are found in the vast majority of classes (an increase in mean scores is observed between Parts 1 and 2 of the test in ten out of eleven classes for Version A and in eight out of eleven classes for Version B, and an increase in scores between Parts 1 and 3 of the test in nine out of eleven classes for both kinds of representation), the same is not true at the individual level. Additional analyses show fluctuations in students' results in the three parts of the test: although the schematic representations seem to have helped some students, they seem to have confused others. Table 5 shows these fluctuations in results for the four types of problem. The three-digit code corresponds to the three parts of the test and the digits 0 and 1 respectively indicate failure or success for each problem (for example, 0-1-1 indicates failure in Part 1 and success in Parts 2 and 3).

On average for all problems, the effect of schematic representations was positive for 39 % of students: 14 % failed at the first problem, then succeeded with the next two and 25 % failed in Part 1 but succeeded in either Part 2 (15 %) or Part 3 (10 %). Conversely, there was on average a negative effect for 5 % of students (success in Part 1 followed by failure in Parts 2 and 3) and a fairly negative effect for 7 %. For the 10 % of students on average who managed to solve only the problems in Part 3 correctly, while a possible 'testing effect' cannot be totally excluded (although it should be recalled that the problems were mixed up, that their superficial characteristics differed from one part to another, and that Part 3 was set at a different time from the other two parts), it can be hypothesized that the presence of representations in Part 2 somehow 'equipped' students to construct effective representations themselves (either mentally or externally on paper) in Part 3. With regard to the effect of the two forms of schematic representation, there was again a more positive overall effect for Version A (diagrams) than for Version B (schematic drawings), except for the multiplicative combination problem.

5 Discussion and conclusion

This study examined the impact of two types of schematic representation on the performance of grade 4 students in solving non-routine arithmetic problems. The

Table 5 Fluctuations in results between the three parts of the test

	No effect (0-0-0)	Fairly negative effect		Clear negative effect (1-0-0)	Fairly positive effect		Clear positive effect (0-1-1)	Help not needed (1-1-1)
		(1-1-0)	(1-0-1)		(0-1-0)	(0-0-1)		
P-whole-F A	38 %	4 %	3 %	4 %	11 %	11 %	16 %	14 %
P-whole F B	49 %	3 %	3 %	4 %	10 %	6 %	14 %	13 %
Change-F A	27 %	3 %	3 %	7 %	9 %	16 %	18 %	18 %
Change-F B	38 %	4 %	7 %	4 %	4 %	14 %	19 %	10 %
Change-NF A	24 %	3 %	1 %	3 %	36 %	3 %	18 %	12 %
Change-NF B	39 %	8 %	3 %	3 %	21 %	8 %	13 %	6 %
Combine-NF A	43 %	4 %	1 %	9 %	8 %	12 %	3 %	19 %
Combine-NF B	33 %	3 %	7 %	7 %	22 %	7 %	13 %	8 %
Mean	36 %	4 %	3 %	5 %	15 %	10 %	14 %	12 %

initial results showed that without specific instruction, the students in this study did not spontaneously use any schematic representations to solve these problems, complex though they were, as the low success rates show. After these initial findings which confirmed the interest of the study, questions focused on the influence of the presence of schematic representations, their possible reuse, and whether there was a differential effect according to the type of problem.

With regard to the *effect of the presence of schematic representations*, we found a non negligible positive effect (effect size greater than 0.65) on overall student performance, for both diagrams and schematic drawings. These results contradict those of the study of Pantziara et al. (2009), who found no overall positive effect when problems were accompanied by predefined diagrams to be completed (schematic drawings were not used by these authors). This could be explained by the classroom practices of Luxembourg students unaccustomed to these problems and to schematic representations, whereas some of the types of problems used by Pantziara et al. (2009) were found in the students' textbooks. Their greater familiarity with the contents could therefore have counteracted the effect of the schematizations in some way. Another explanatory hypothesis is that the schemas suggested in our study were fairly complete and explicit, whereas in the other study they were rather abstract and incomplete (barely outlined in some cases).

With regard to *the reuse of these schematic representations*, we found that a large majority of students produced or attempted to produce representations in Part 3 of the test. The problems in this last part were generally answered more successfully than those in the first part of the test. If one regards Part 2 to be a (very brief) educational intervention, the comparison of results between Parts 1 and 3 reveals the potential for a genuine intervention with the aim of teaching students to produce schematic representations. Even if, despite all the precautions that were taken, we cannot completely exclude a testing effect due to the three versions, our results suggest that this improvement in scores would seem to be related to the more effective use of the representations, whether drawn on paper or internalized. We may also note that the results were generally lower in Part 3 than in Part 2, which is not in accordance with an interpretation in terms of a testing effect. Finally, as the three versions of the test were relatively close, the reuse of representations in Part 3 appears at best to be a 'near transfer', both in terms of time and in terms of the type of task. To evaluate a possible 'far transfer' effect, it would have been interesting to set one or two problems with another mathematical structure in Part 3 and possibly also to set a delayed post-test a few weeks later.

With regard to *types of problem*, we found, like Pantziara et al. (2009), differences in impact depending on the type of problem and the category of predefined schema, as well as a change in the hierarchy of problem types. Regarding the diagrams, we observed a clear positive effect for three types of problem (especially the 'change' type problems), but no effect for the matrix diagram provided along with the 'multiplicative combination' problem. This result can be explained by the relatively abstract nature of a matrix diagram, where the implicit rules need to be properly decoded. By contrast, the 'network' and 'part-whole' type diagrams present very few implicit rules to be understood. Regarding the schematic drawings, the presence of representations had a positive effect for each type of problem, without any particularly marked distinction.

Schematic representations are an important aid for students in solving complex problems. The results reveal a clear overall improvement in student performance, despite the fact that they had encountered such representations only once and without any explicit teaching or discussion among the children having taken place. Overall, the results are slightly in favor of *diagrams* derived from the typology of Novick et al. (1999) rather than schematic drawings similar to *self-generated drawings*. At first sight, one might think that this difference is explained by the fact that these representations are closer to the production of an arithmetical operation. However, this explanation seems rather narrow, since it is fairly clear that the transition from schematic representation to solving the problem seems to go beyond mere 'translation'. On average for all problems combined, 36 % of students did not derive any benefit from the schematic representations, since they failed all three parts of the test. The results are relatively similar for both forms of representation and vary more depending on the type of problem. They demonstrate the inadequacy of the method consisting of presenting a predefined schematic representation (diagram or schematic drawing) and reinforce the idea that these should be constructed with the students (van Dijk et al., 2003).

A key challenge is to get students to realize the importance of this potentially powerful tool for problem solving. Explicit learning of different forms of representation could be integrated into the teaching of problem solving, whether spontaneous representations to be used and discussed in class or more conventional schemas to be constructed gradually with the students. While our results show an effect slightly in favor of diagrams rather than schematic drawings, they do not really allow us draw conclusions about the form which these representations should take, since variations are observed not just according to the type of problem, but also according to the particular student. Like the study of Pantziara et al. (2009), our results also show that, for each type of problem and for both forms of schematization, schematic representations seem to have helped some students (39 % on average), but they also appear to have confused others (7 % on average). Pantziara et al. (2009) argue for complementarity between the diagrams and other more 'inventive' constructions: 'teachers could give students opportunities not only to use presented diagrams but also to invent or search for their own solution strategy' (Pantziara et al., 2009, p. 56).

Along the same lines and with reference to research on cognitive flexibility in the use of representations (Heinze et al., 2009), it would also be worth looking at which forms of teaching/learning best allow this 'representational flexibility' to be developed. Our results show a differential effect for representations not only according to the type of problem but also according to the individual student. It is also possible that the results vary according to certain characteristics of classes and other sociocultural factors that we have not analyzed here. In this sense, further studies are needed to investigate the extent to which students should be taught to use 'the representation that best suits a particular type of problem', or whether it is better to aim for greater flexibility of practice, following the example of Nistal, Van Dooren, Clarebout, Elen, and Verschaffel (2009) for whom the choice of an appropriate representation depends not only on the type of problem, but also on student characteristics and the particular context.

Showing the value of not being too rigid and allowing an adaptability which takes into account the situational context (Diezmann, 2002), the diagrams drawn from the classification of Novick (2006) appear to be assimilable to the concept of the 'episodic situation model' highlighted in recent work in cognitive psychology (Thevenot et al., 2010). However, where these are imposed on students as the only models of the situations that they represent, these diagrams will always be open to the criticisms made by Julo (2002) of research aiming to teach students to construct schematic representations directly inspired by problem typologies: this type of approach consists of immediately associating a particular form of symbolism with the formation of a particular mental schema corresponding to one or more abstract categories of problem. Spontaneous symbolization, which is particularly in line with work done within a sociocultural approach (Bednarz et al., 1993; Gravemeijer, 1997; van Dijk et al., 2003), would meet such criticism but may not lead readily to the formation of problem categories if it remains too contextualized and hence does not allow the decontextualization which is necessary for the emergence of the invariants on which these categories are based. From this point of view, it would probably be useful to consider whether an approach to teaching and learning could be developed that aims to start from the symbolizations produced spontaneously by the students and to lead them gradually, with the teacher supporting the construction process, to more conventional symbolization which helps in the abstraction of these categorizations. From another point of view, questions might also be asked about the impact of these categories of problems on young students. Although categorizations are a sign of expertise, are young students really capable of such a level of abstraction and is it really appropriate to focus teaching on the emergence of these categorizations? In this sense, interventionist research should be carried out to compare the effect of the various methods of teaching/learning schematizations according to whether or not they take the students' spontaneous symbolizations as the starting point.

Appendix 1

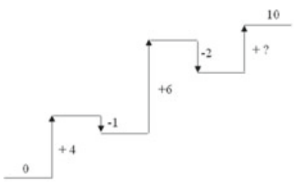
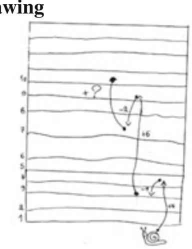
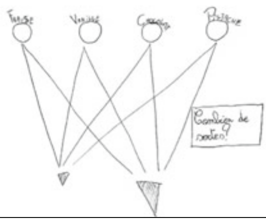
<table>
<tr><th>Non-familiar change type problem</th><th>Non-familiar multiplicative combination type problem</th></tr>
<tr><td>A snail tries to climb a brick wall. First it climbs up the first four bricks, but is then exhausted, stops and falls asleep. While it is asleep it slips down one brick. When it wakes up it climbs up six bricks, then goes to sleep again and slips down two bricks. On its last attempt it reaches the tenth brick. How many bricks did the snail climb on its last attempt?</td><td>The ice-cream seller where I live sells scoops of strawberry, vanilla, chocolate and pistachio ice-cream. He has two types of cone: small and large. How many different kinds of one-scoop ice-cream can the seller make?</td></tr>
<tr><td>Diagram (network type)
</td><td>Diagram (matrix type)
<table><tr><th></th><th>Fraise</th><th>Vanille</th><th>Chocolat</th><th>Pistache</th></tr><tr><td>Petits cornets</td><td></td><td></td><td></td><td></td></tr><tr><td>Grands cornets</td><td></td><td></td><td></td><td></td></tr></table></td></tr>
<tr><td>Schematic drawing
</td><td>Schematic drawing
</td></tr>
</table>

Fig. 5 Examples of problems and schematizations proposed in Part 2

Appendix 2

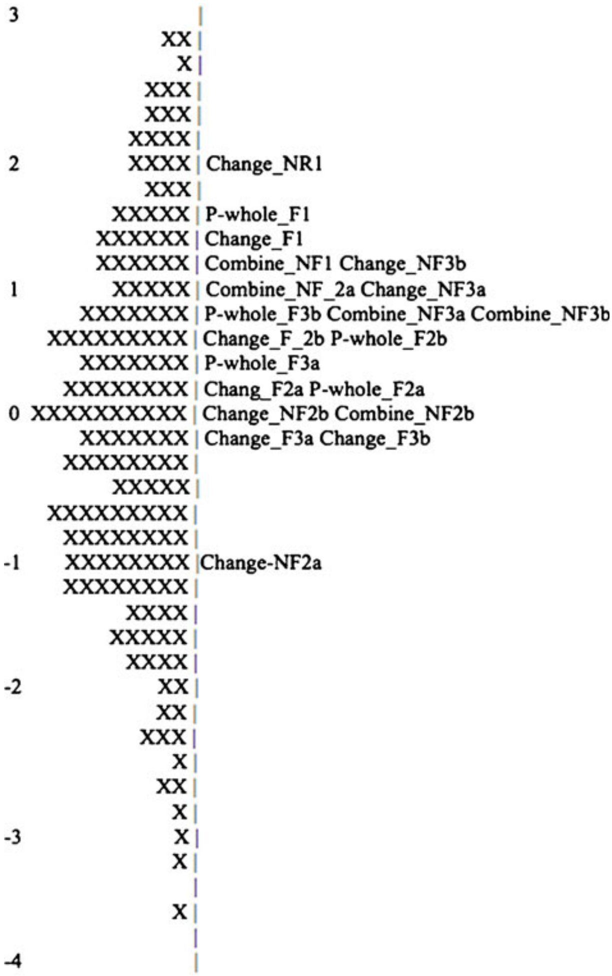


Fig. 6 Graph showing the hierarchy of items established by the Rasch analysis (The test fits the model fairly well: all the infits fall between 0.85 and 1.15, except for two problems for which the infits fall below the lower standards: combine-NR2a (0.66) and change-R-3b (0.69). The removal of these items does not change the ranking of the other items.)

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