Lyapunov-based sufficient conditions for exponential stability in hybrid systems

A.R. Teel, F. Forni, L. Zaccarian

***Abstract*—Lyapunov-based sufficient conditions for exponential stability in hybrid systems are presented. The focus is on converting non-strict Lyapunov conditions, having certain observability properties, into strict Lyapunov conditions for exponential stability. Both local and global results are considered. The utility of the results is illustrated through an example.**

I. Introduction

This technical note is motivated by analysis problems that arise in tracking and estimation problems for mechanical systems with impacts (see, e.g., [19], [17] and references therein). For such problems, in [7], [8], [6], it has been shown how to construct a radially unbounded Lyapunov function that does not increase during flows, does not increase at impacts, and whose zero level set corresponds to zero position tracking error. Moreover, the derivative of the Lyapunov function is negative semidefinite with an observability property: constraining the derivative equal to zero on any time interval of nonzero length implies that the Lyapunov function's value must be identically zero. This fact, coupled with provable absence of Zeno solutions and the hybrid invariance principle [21], leads to the conclusion that the zero level set of the Lyapunov function is globally asymptotically stable. However, we wish to establish the stronger result that the zero level set of the Lyapunov function is globally exponentially stable. For such a conclusion, we rely on the results in this note.

Our contribution has connections to the hybrid invariance principle [15], [3], [21], [10], although in the absence of some homogeneity properties [12], the invariance principle is typically not strong enough to conclude exponential stability. There are also connections to Matrosov conditions for asymptotic stability in hybrid systems [16], [22]. In addition, there is a connection to a result from linear systems theory, often called the "Squashing" Lemma [20], [4].

The paper is organized as follows. In Section II we characterize exponential stability and its Lyapunov formulation. In Section III we formulate relaxed Lyapunov conditions. Later we show how homogeneity (Section IV) and observability with partially linear flow maps (Section V) are situations where those relaxed conditions are sufficient for exponential stability.

A.R. Teel is with the ECE Department, University of California, Santa Barbara, USA and is supported by AFOSR grant FA9550-12-1-0127 and NSF grants ECCS-0925637 and ECCS-1232035.

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II. Definitions and initial observation

We consider a hybrid system with state $x \in \mathbb{R}^n$ of the form

$$\mathcal{H}: \quad \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x)\ . \end{cases} \tag{1}$$

A solution to a hybrid system is any hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$, where the domain is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, that satisfies the constraints imposed in (1). See [9, pp. 39-42] for a precise definition of a hybrid arc and a solution to a hybrid system. A solution is said to be *complete* if its domain is unbounded.

Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the quantity $|x|_{\mathcal{A}}$ denotes the distance of x to $\mathcal{A}$, that is, $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$ where $|\cdot|$ is a vector (e.g., Euclidean) norm on $\mathbb{R}^n$. Given $\mu > 0$, $\mathcal{A} + \mu\mathbb{B}^\circ := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} < \mu\}$. For a set $C \subset \mathbb{R}^n$, $\overline{C}$ denotes the closure of C.

For the system (1), a closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *(locally) exponentially stable* if there exist strictly positive real numbers μ, k, λ such that each solution x satisfying $|x(0,0)|_{\mathcal{A}} < \mu$ also satisfies, for all $(t,j) \in \operatorname{dom} x$,

$$|x(t,j)|_{\mathcal{A}} \leq k \exp(-\lambda(t+j))|x(0,0)|_{\mathcal{A}}\ . \tag{2}$$

It is said to be *globally exponentially stable* if there exist strictly positive real numbers μ, k such that each solution x, regardless of the size of the initial condition, satisfies (2). Nothing in the definition of exponential stability guarantees that solutions exist or have unbounded time domains. Rather, the definition simply imposes a bound on the distance of the solution to the set $\mathcal{A}$ for each time in each solution's domain. A simple illustration of this fact is discussed at the end of Example 4 in Section V-B of this note.

A simple sufficient condition for exponential stability is given in the following theorem, the proof of which follows standard arguments extended to hybrid systems.

Theorem 1: For the system (1), the closed set $\mathcal{A} \subset \mathbb{R}^n$ is locally exponentially stable if there exist positive real numbers $\underline{\alpha}$, $\overline{\alpha}$, λ, μ, p, and a function $V : \operatorname{dom} V \to \mathbb{R}$, where $C \cup D \cup G(D) \subset \operatorname{dom} V$, that is continuously differentiable on an open set containing $\overline{C}$ and satisfies

$$\underline{\alpha} \cdot |x|^p_{\mathcal{A}} \leq V(x) \leq \overline{\alpha} \cdot |x|^p_{\mathcal{A}} \quad \forall x \in (C \cup D \cup G(D)) \cap (\mathcal{A} + \mu\mathbb{B}^\circ) \tag{3}$$

$$\langle \nabla V(x), f \rangle \leq -\lambda V(x) \quad \forall x \in C \cap (\mathcal{A} + \mu\mathbb{B}^\circ),\ f \in F(x) \tag{4}$$

$$V(g) \leq \exp(-\lambda) V(x) \quad \forall x \in D \cap (\mathcal{A} + \mu\mathbb{B}^\circ),\ g \in G(x). \tag{5}$$

If these bounds hold with $\mu = \infty$ then the set $\mathcal{A}$ is globally exponentially stable.

Proof: Consider the evolution of a solution x satisfying $|x(0,0)|_{\mathcal{A}} < (\underline{\alpha}/\overline{\alpha})^{\frac{1}{p}} \mu$. Since $x(0,0) \in \overline{C} \cup D$ by definition and V is continuously differentiable on $\overline{C}$, it follows from (3) that $V(x(0,0)) < \underline{\alpha} \cdot \mu^p$. Also note, according to (3), that $V(x) < \underline{\alpha} \cdot \mu^p$ implies that $x \in \mathcal{A} + \mu\mathbb{B}^\circ$. Thus, as long as V does not increase along the solution, we will have that $x(t,j) \in C \cap (\mathcal{A} + \mu\mathbb{B}^\circ)$ for each j and almost all t such that $(t,j) \in \operatorname{dom} x$ and we will have that $x(t,j) \in D \cap (\mathcal{A} + \mu\mathbb{B}^\circ)$ whenever $(t,j),(t,j+1) \in \operatorname{dom} x$. A flowing solution would take some positive amount of time to reach the condition $V(x(t,j)) \geq \underline{\alpha} \cdot \mu^p$. Due to (4), for almost all time in that interval, the derivative of $t \mapsto V(x(t,j))$ is not positive. Hence, it is impossible for x to leave the set $\mathcal{A}+\mu\mathbb{B}^\circ$ by flowing. Condition (5) rules out x leaving the set $\mathcal{A}+\mu\mathbb{B}^\circ$ by jumping. Thus, x never leaves the set $\mathcal{A}+\mu\mathbb{B}^\circ$ and we can apply (4) and (5) and a comparison principle [1, Lemma C.1] to conclude that, for all $(t,j) \in \operatorname{dom} x$,

$$\begin{aligned}\underline{\alpha} \cdot |x(t,j)|_{\mathcal{A}}^p \leq V(x(t,j)) &\leq \exp(-\lambda(t+j))V(x(0,0)) \\ &\leq \overline{\alpha}\exp(-\lambda(t+j))|x(0,0)|_{\mathcal{A}}^p\end{aligned}$$

so that if $|x(0,0)|_{\mathcal{A}} < (\underline{\alpha}/\overline{\alpha})^{\frac{1}{p}}\mu$ then, for all $(t,j) \in \operatorname{dom} x$,

$$|x(t,j)|_{\mathcal{A}} \leq (\overline{\alpha}/\underline{\alpha})^{\frac{1}{p}}\exp\left(-\lambda p^{-1}(t+j)\right)|x(0,0)|_{\mathcal{A}}$$

This bound establishes the result. ∎

III. Exponential stability under relaxed Lyapunov conditions

The purpose of this note is to provide relaxed Lyapunov-based conditions for exponential stability. In particular, motivated by results in [7], [8], [6], we consider the situation where a non-strict Lyapunov function has been found, in other words, a function satisfying the conditions of Theorem 1 with the caveat that the conditions (4) and (5) are weakened to

$$\langle \nabla V(x), f\rangle \leq -\rho(x) \quad \forall x \in C \cap (\mathcal{A}+\mu\mathbb{B}^\circ),\ f \in F(x) \quad (6)$$

where $\rho : \overline{C} \to \mathbb{R}_{\geq 0}$ is a continuous function, and

$$V(g) \leq V(x) \qquad \forall x \in D \cap (\mathcal{A}+\mu\mathbb{B}^\circ),\ g \in G(x)\ . \quad (7)$$

The conditions (6)-(7) guarantee that $\mathcal{A}$ is stable. In particular,

$$|x(0,0)|_{\mathcal{A}} < (\underline{\alpha}/\overline{\alpha})^{\frac{1}{p}}\mu \implies |x(t,j)|_{\mathcal{A}} \leq (\overline{\alpha}/\underline{\alpha})^{\frac{1}{p}}|x(0,0)|_{\mathcal{A}}\ .$$

On the other hand, in general the conditions are not sufficient for exponential, or even asymptotic, stability. Several things can go wrong. For example, without extra structure imposed on the hybrid system, there may exist a solution that only jumps and never flows. Since the only information we have about jumps is that V does not increase, there is no hope of concluding that $V(x(t,j))$ converges to zero as $t+j \to \infty$. A simple example where (3), (6)-(7) hold yet convergence of V to zero fails is $n=1$, $\mathcal{A}=\{0\}$, $C=\varnothing$, $D=\mathbb{R}$, $G(x)=x$ and $V(x)=x^2$. Another possibility is that we may have convergence of $V(x(t,j))$ to zero as $t+j \to \infty$ at a rate that is slower than any exponential function, even if ρ is positive definite with respect to $\mathcal{A}$. A simple example where (3), (6)-(7) hold with ρ positive definite with respect to $\mathcal{A}$ yet convergence of V to zero is slower than exponential is $x \in \mathbb{R}$ (namely $n=1$), $\mathcal{A}=\{0\}$, $C=\mathbb{R}$, $F(x)=-x^3$, $D=\varnothing$ and $V(x)=x^2$. In light of these examples, we look to impose extra structure on the hybrid system to ensure exponential stability.

IV. Using asymptotic stability plus homogeneity

In this section we illustrate how (global) exponential stability can be established using the properties of homogeneous hybrid systems. In particular, for hybrid systems that have an appropriate type of homogeneity property [12], one path that is available is to prove a weaker asymptotic stability property and then rely on homogeneity to establish exponential stability. Proving asymptotic stability can then be done by relying on the invariance principle [21], as illustrated in the examples discussed next. Note that using homogeneity, one does not have, in general, an explicit Lyapunov function (as in Theorem 1). An alternative route is taken later in Section V, where we rely on different assumptions. For those results, an explicit Lyapunov function is derived.

A. *Homogeneity with respect to the partially standard dilation*

Using [12, Lemma 4.2], it is possible to pass from asymptotic to exponential stability for hybrid systems with an appropriate type of homogeneity. For example, consider the system (1) and let the integers $n_1 \in \mathbb{Z}_{\geq 1}$ and $n_2 \in \mathbb{Z}_{\geq 0}$ satisfy $n_1 + n_2 = n$. Given $\lambda > 0$, define the matrix $M(\lambda) := \operatorname{diag}(\lambda I_{n_1}, I_{n_2})$, which characterizes a partially standard (non-proper) dilation. The hybrid system (1) is said to be *homogeneous of degree zero with respect to the partially standard dilation* M if, for each $\lambda > 0$,

$$\begin{aligned} M(\lambda)C &= C, \qquad M(\lambda)D = D, \\ F(M(\lambda)x) &= M(\lambda)F(x) \quad \forall x \in C, \\ G(M(\lambda)x) &= M(\lambda)G(x) \quad \forall x \in D. \end{aligned}$$

The necessity of the next result is a straightforward consequence of the bound in (2), while the sufficiency can be proven as in the proof of [12, Theorem 7.1], which pertains to systems with logic modes, given in [12, §7.2]. Indeed, solutions to the homogeneous hybrid system can be scaled as suggested in [12, Lemma 4.2] and, following the proof technique of [12, Proposition 4.3], the scaled solutions allow establishing a uniform contractivity property over a sequence of hybrid time intervals of size $t+j=T$. Note that while [12, Proposition 4.3] refers only to the origin, generalizing the results to the set $\mathcal{A}$, which allows for the extra n_2 states, is straightforward as illustrated in [12, §7.2] where those extra n_2 states correspond to suitable logic modes indexes.

Proposition 1: Suppose the system (1) is homogeneous of degree zero with respect to the partially standard dilation $M(\lambda) := \operatorname{diag}(\lambda I_{n_1}, I_{n_2})$. Then the set $\mathcal{A} = \{0\} \times \mathbb{R}^{n_2}$ is globally exponentially stable if and only if there exist $0 < r < R$ and $T > 0$ such that $|x(0,0)|_{\mathcal{A}} \leq R$, $(t,j) \in \operatorname{dom} x$, and $t+j \geq T$ imply $|x(t,j)|_{\mathcal{A}} \leq r$.

The condition for global exponential stability in Proposition 1 is guaranteed if the set $\mathcal{A}$ is uniformly locally asymptotically stable. Proposition 1 can also be applied to a "conical approximation" of a hybrid system, like in [12], to derive local exponential stability results.

B. *Uniform asymptotic stability from non-strict Lyapunov conditions*

Several tools can be invoked to characterize extra structure sufficient to get asymptotic stability from non-strict Lyapunov conditions. A popular tool from the hybrid systems literature, often found under the title "multiple Lyapunov functions" [5], involves establishing conditions under which the Lyapunov function strictly decreases eventually, although not necessarily monotonically; see also [18] and the references therein. An alternative tool, which has been developed recently for hybrid systems, is based on the construction of a family of Matrosov functions [22], [16], where in the latter reference strict Lyapunov functions are constructed from non-strict ones. Finally, in the case where the set $\mathcal{A}$ is compact and the data of the hybrid system satisfies appropriate regularity properties [11], [9], the hybrid invariance principle [21] (cf. [15], [3]) can be invoked to establish asymptotic stability. To illustrate the latter approach, in concert with Proposition 1, we consider two examples based on material in [18].

Example 1: (Cf. [18, Example 3]) Let $M, H, K \in \mathbb{R}^{n\times n}$ be symmetric with K positive semidefinite and M, H positive definite. Suppose the pair $(K, M^{-1}H)$ is observable. Let $\lambda_0, \lambda_1 \in \mathbb{R}_{>0}$ satisfy $\lambda_0\lambda_1 \leq 1$. Let $0 < T_1 \leq T_2$. Consider the system with $x = (q, p, \alpha, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ where

$$\begin{aligned} C &= \mathbb{R}^n \times \mathbb{R}^n \times \{0,1\} \times [0, T_2] \\ D &= \mathbb{R}^n \times \mathbb{R}^n \times \{0,1\} \times [T_1, T_2] \\ F(x) &= \begin{bmatrix} M^{-1}p \\ -Hq - \alpha K M^{-1} p \\ 0 \\ 1 \end{bmatrix} \quad G(x) = \begin{bmatrix} \lambda_a q : |\lambda_a| \leq \lambda_\alpha \\ \lambda_b p : |\lambda_b| \leq \lambda_\alpha \\ 1-\alpha \\ 0 \end{bmatrix} . \end{aligned} \tag{8}$$

The constraints on the evolution of the state τ dictate that the jumps, which toggle α in the set $\{0, 1\}$, are spaced by at least T_1 time units and no more than T_2 time units. We first study global asymptotic stability of the compact set $\mathcal{A}_c := \{0\} \times \{0\} \times \{0, 1\} \times [0, T_2]$. Consider the Lyapunov function candidate $V(x) = \lambda_\alpha \left(q^T H q + p^T M^{-1} p\right)$. Since $\lambda_\alpha > 0$ for each $\alpha \in \{0, 1\}$ and H and M are positive definite, V is radially unbounded and positive definite with respect to $\mathcal{A}_c$ when restricted to the set $C \cup D \cup G(D)$. Like in [18], it is easy to verify that $\langle \nabla V(x), F(x)\rangle = -2\lambda_\alpha \alpha p^T M^{-1} K M^{-1} p \leq 0$ for all $x \in C$. Moreover, using that $\lambda_{1-\alpha}\lambda_\alpha \leq 1$, for all $x \in D$ and all $g \in G(x)$, we have

$$\begin{aligned} V(g) &\leq \lambda_{1-\alpha}\lambda_\alpha^2 \left(q^T H q + p^T M^{-1} p\right) \\ &\leq \lambda_\alpha \left(q^T H q + p^T M^{-1} p\right) \; = \; V(x) \; . \end{aligned}$$

Let x be a complete solution so that, for some $c \geq 0$, $V(x(t, j)) = c$ for all $(t, j) \in \operatorname{dom} x$. Due to the nature of the flow set and the jump map for α, there exists $i \in \mathbb{Z}_{\geq 0}$ and a $t_i \geq 0$ such that $\alpha(s, i) = 1$ for all $s \in [t_i, t_i + T_1]$ and thus, from the expression for $\langle \nabla V(x), F(x)\rangle$, $KM^{-1}p(s, i) = 0$ for all $s \in [t_i, t_i + T_1]$. Following [18] and the references therein, by repeatedly differentiating this equality and using the flow dynamics for (q, p), we get $K(M^{-1}H)^j q(s, i) = 0$ for all $s \in [t_i, t_i + T_1]$ and all positive integers j. By the assumed observability of $(K, M^{-1}H)$, it follows that $M^{-1}Hq(s, i) = 0$ for all $s \in [t_i, t_i + T_1]$. Since M and H are positive definite, $q(s, i) = 0$ for all $s \in [t_i, t_i + T_1]$. Differentiating this equality and using the flow dynamics for q yields that $p(s, i) = 0$ for all $s \in [t_i, t_i + T_1]$. Then, by the definition of V, we must have $c = 0$. Consequently, according to the invariance principle in [21], the compact set $\mathcal{A}_c$ is globally asymptotically stable. According to [11, Theorem 6.5], the set $\mathcal{A}_c$ is uniformly globally asymptotically stable. Then, it follows that the closed set $\mathcal{A} = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$ is uniformly globally asymptotically stable, since $|x|_{\mathcal{A}_c} = |x|_{\mathcal{A}}$ for all $x \in C \cup D \cup G(D)$. Finally, we note that the system (8) is homogeneous with respect to the partially standard dilation M with $n_1 = 2n$ and $n_2 = 2$. We conclude from Proposition 1 that the set $\mathcal{A}$ is globally exponentially stable. ⌟

Example 2: (Based on [18, Examples 1 and 4]) Let $\lambda_0, \lambda_1 \in \mathbb{R}_{>0}$ satisfy $\lambda_0\lambda_1 \leq 1$. Let $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ be continuous functions that satisfy $s\psi_i(s) > 0$ for all $s \neq 0$ and $i \in \{1, 2\}$. In addition, suppose that $\psi_1(z) = a_+ z$ for $z > 0$ and $\psi_1(z) = a_- z$ for $z < 0$ where $a_+, a_- \in \mathbb{R}_{>0}$. Consider the system with state $x = (z, v, \alpha) \in \mathbb{R}^3$ where

$$\begin{aligned} C &= (\mathbb{R} \times \mathbb{R}_{\geq 0} \times \{0\}) \cup (\mathbb{R} \times \mathbb{R}_{\leq 0} \times \{1\}) \\ D &= (\mathbb{R}_{\geq 0} \times \{0\} \times \{0\}) \cup (\mathbb{R}_{\leq 0} \times \{0\} \times \{1\}) \end{aligned} \tag{9}$$

$$F(x) = \begin{bmatrix} v \\ -\psi_1(z) - \alpha\psi_2(v) \\ 0 \end{bmatrix} \qquad G(x) = \begin{bmatrix} \lambda z : \lambda \in [0, \lambda_\alpha] \\ 0 \\ 1-\alpha \end{bmatrix} . \tag{10}$$

Initially, we study global asymptotic stability of the compact set $\mathcal{A}_c := \{0\} \times \{0\} \times \{0, 1\}$. Consider the Lyapunov function candidate $V(x) = \lambda_\alpha \left(\int_0^z \psi_1(s)ds + 0.5v^2\right)$ (see also [14, Example 4.8]). Due to the properties of ψ_1, the function V is radially unbounded and positive definite with respect to $\mathcal{A}_c$ when restricted to $C \cup D \cup G(D)$. Also $\langle \nabla V(x), F(x)\rangle = -\alpha v \psi_2(v)$ for all $x \in C$. Moreover, using $\lambda_\alpha \lambda_{1-\alpha} \leq 1$ and the properties of ψ_1, we get that, for all $x \in D$ and $g \in G(x)$,

$$\begin{aligned} V(g) &\leq \lambda_{1-\alpha}\lambda_\alpha^2 \int_0^z \psi_1(s)ds \\ &\leq \lambda_\alpha \left(\int_0^z \psi_1(s)ds + 0.5v^2\right) \; = \; V(x) \; . \end{aligned}$$

It can be verified that if there exists a complete solution x and $c \geq 0$ that satisfy $V(x(t, j)) = c$ for all $(t, j) \in \operatorname{dom} x$ then necessarily $c = 0$. Indeed, suppose $c > 0$. Then there exists $i \in \mathbb{Z}_{\geq 0}$, $t_i \in \mathbb{R}_{\geq 0}$ and $T \in \mathbb{R}_{>0}$ such that $\alpha(s, i) = 1$ for all $s \in [t_i, t_i + T]$. In turn, $v(s, i) = 0$ for all $s \in [t_i, t_i + T]$ and thus, from the properties of ψ_1, $z(s, i) = 0$ for all $s \in [t_i, t_i + T]$. This observation contradicts the assumption that $c > 0$. Global asymptotic stability of $\mathcal{A}_c$ then follows from the hybrid invariance principle [21], and thus uniform global asymptotic stability of the closed set $\mathcal{A} = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$ follows like in the previous example. When $\psi_2(\theta v) = \theta\psi_2(v)$ for all $v \in \mathbb{R}$ and $\theta > 0$, system (9)-(10) is homogeneous of degree zero with respect to the partially standard dilation M with $n_1 = 2$ and $n_2 = 1$. In that case, we conclude from Proposition 1 that the set $\mathcal{A}$ is globally exponentially stable. Alternatively, when ψ_2 is differentiable at the origin and $\psi_2'(0) > 0$, we conclude from Proposition 1 and a straightforward generalization of the results in [12], analogous to the generalization discussed in [12, Section 7], that the set $\mathcal{A}$ is locally exponentially stable, in addition to uniformly globally asymptotically stable. ⌟

V. Observability and partially linear flow maps

In this section, we consider hybrid systems with a special form that is motivated by systems that appear in [7], [8], [6]. (A self-contained motivational example is given in Section V-B below.) Differently from Section IV, we assume partially linear flow maps with a suitable observability condition involving the function $\rho(\cdot)$ in (6). Then, under a suitable average dwell-time condition, we show how to construct a Lyapunov function satisfying the conditions of Theorem 1.

A. Main result

Consider the following assumption.

Assumption 1: The following conditions hold:

1) The state $x \in \mathbb{R}^n$ can be decomposed as $x = (x_1^T, x_2^T)^T$ where $x_1 \in \mathbb{R}^{n_1}$, $n_1 \in \{1, \ldots, n\}$,
2) $\mathcal{A} = \{0\} \times \mathbb{R}^{n-n_1}$,
3) F has the form $F(x) = \begin{bmatrix} F_1 x_1 \\ F_2(x) \end{bmatrix}$ where $F_1 \in \mathbb{R}^{n_1 \times n_1}$,
4) Conditions (3), (6)-(7) hold with $p = 2$ and $\rho(x) = x_1^T H_1^T H_1 x_1$ for all $x \in C$ where the pair (H_1, F_1) is observable (in the classical linear sense),
5) The jumps of the hybrid system satisfy an average dwell-time constraint[1] with average dwell-time parameters (δ, N) where $\delta > 0$ and $N \geq 1$. In particular, by augmenting system (1) with the automaton

$$\begin{array}{rclcrcl} \tau & \in & [0, N] & \qquad & \dot{\tau} & \in & [0, \delta] \\ \tau & \in & [1, N] & & \tau^+ & = & \tau - 1 \,, \end{array} \tag{11}$$

in other words, considering the system

$$\begin{array}{rclcl} (\tau, x) & \in & [0, N] \times C & & \left\{ \begin{array}{rcl} \dot{\tau} & \in & [0, \delta] \\ \dot{x} & \in & F(x) \end{array} \right. \\ (\tau, x) & \in & [1, N] \times D & & \left\{ \begin{array}{rcl} \tau^+ & = & \tau - 1 \\ x^+ & \in & G(x), \end{array} \right. \end{array} \tag{12}$$

a hybrid arc x is a solution of (1) if and only if it is the "x"-component of a solution to (12) initialized with $\tau(0, 0) = N$ [2, Proposition 1.1]. ■

In the spirit of [16], we state our main result.

Theorem 2: Under Assumption 1, the conditions of Theorem 1 hold and thus the set $\mathcal{A}$ is locally exponentially stable for the system (1). If item 4 of Assumption 1 holds with $\mu = \infty$ then the conditions of Theorem 1 hold with $\mu = \infty$ and thus the set $\mathcal{A}$ is globally exponentially stable.

We emphasize that the usual basic conditions on the data of a hybrid system to guarantee well-posedness, the invariance principle, converse Lyapunov theorems, etc., are not needed for the conclusion of Theorem 2.

B. Illustrative examples

First we provide an example to emphasize that, without imposing some extra condition, the conclusion of Theorem 2 does not hold if the observability condition on (H_1, F_1) in the fourth item of Assumption 1 is relaxed to detectability.

Example 3: Consider the hybrid system where $C = \mathbb{R} \times [0, 1]$, $D = \mathbb{R} \times \{1\}$,

$$f(x) = \begin{bmatrix} -x_1 \\ 1 \end{bmatrix} \,, \quad g(x) = \begin{bmatrix} \exp(1) x_1 \\ 0 \end{bmatrix} \,,$$

and $\mathcal{A} = \{0\} \times \mathbb{R}$ so that the first three conditions of Assumption 1 hold. The last condition of Assumption 1 holds with $N = 1$ and $\delta = 1$. Indeed, we can set $\tau(0, 0) = 1$, then pick $\dot{\tau} = 0$ until $x_2(t, 0) = 1$ and then we can synchronize τ with x_2 thereafter. Consider $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ defined as $V(x) = \exp(2x_2) x_1^2$ for each $x \in \mathbb{R}^2$. We get that (3) holds with $\underline{\alpha} = 1$ and $\overline{\alpha} = \exp(2)$. Also, $V(g(x)) = \exp(2) x_1^2 = \exp(2x_2) x_1^2 = V(x)$ for all $(x_1, x_2) \in \mathbb{R} \times \{1\}$ and $\langle \nabla V(x), f(x) \rangle = -2x_1^2 \exp(2x_2) + 2\exp(2x_2) x_1^2 = 0$ for all $(x_1, x_2) \in \mathbb{R} \times [0, 1]$. Thus for $H_1 = 0$, which gives that (H_1, F_1) is detectable since $F_1 = -1$ is Hurwitz, we get that the fourth condition of Assumption 1 would hold if the word "observable" were replaced by "detectable". Conversely, it is clear that the set $\mathcal{A}$ is not exponentially stable since each solution satisfies $t - j \leq 1$ for all $(t, j) \in \operatorname{dom} x$ and

$$\begin{array}{rcl} |x(t,j)|_{\mathcal{A}} \;=\; |x_1(t,j)| & = & \exp(-t + j)|x_1(0,0)| \\ & \geq & \exp(-1)|x_1(0,0)| \\ & = & \exp(-1)|x(0,0)|_{\mathcal{A}} \,. \end{array}$$

This example demonstrates that Theorem 2 does not hold with "detectable" in place of "observable" in the fourth item of Assumption 1. The basic obstruction is that detectability in the linear sense, through the identically zero output, for the flow dynamics does not imply detectability, through the identically zero output, for the hybrid dynamics. Figure 1 illustrates the nonconverging behavior of x_1. ⌟

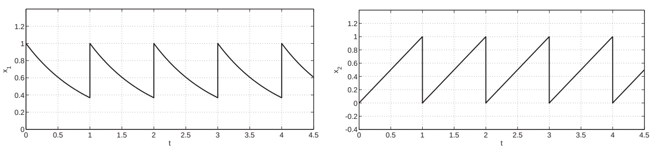

Figure 1. A simulation for Example 3 from the point $x_0 = [1 \;\; 0]^T$.

Next, we give an example, inspired by the results in [7], [8], [6], where the result of Theorem 2 is used to draw the global exponential stability conclusion.

Example 4: Consider the system with state $\xi = (\zeta_p, \zeta_v, z_p, z_v, q, \tau)^T \in \mathbb{R}^6$, where

$$\begin{array}{rcl} \widetilde{C} & = & \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{-1, 1\} \times [0, N] \\ \widetilde{D}_1 & = & \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{-1, 1\} \times [1, N] \\ \widetilde{D}_2 & = & \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{\leq 0} \times \{-1, 1\} \times [1, N] \\ \widetilde{D} & = & \widetilde{D}_1 \cup \widetilde{D}_2 \end{array}$$

$$\widetilde{F}(\xi) = \begin{bmatrix} \zeta_v \\ -\zeta_p - k(\zeta_v - q z_v) \\ z_v \\ -z_p \\ 0 \\ [0, \delta] \end{bmatrix} \quad \widetilde{G}_1(\xi) = \begin{bmatrix} -\zeta_p \\ -\zeta_v \\ z_p \\ z_v \\ -q \\ \tau - 1 \end{bmatrix} \quad \widetilde{G}_2(\xi) = \begin{bmatrix} \zeta_p \\ \zeta_v \\ -z_p \\ -z_v \\ -q \\ \tau - 1 \end{bmatrix}$$

and

$$\widetilde{G}(\xi) = \left\{ \begin{array}{ll} \widetilde{G}_1(\xi) & x \in \widetilde{D}_1 \backslash \widetilde{D}_2 \\ \widetilde{G}_2(\xi) & x \in \widetilde{D}_2 \backslash \widetilde{D}_1 \\ \widetilde{G}_1(\xi) \cup \widetilde{G}_2(\xi) & x \in \widetilde{D}_1 \cup \widetilde{D}_2 \,. \end{array} \right.$$

We let $k > 0$. The system can be viewed as a tracking system for mechanical variables that experience impacts. The (z_p, z_v) subsystem generates an oscillatory reference trajectory z_p that has a sign change in velocity when $z_p = 0$ and $z_v \leq 0$ (see

[1] A precise definition of the average dwell-time constraint with parameters (δ, N) is given in [2, Proposition 1.1]. Intuitively, the timer τ in (11), which evolves in $[0, N]$ and decreases by 1 at each jump, forbids more than N simultaneous jumps and restricts the number of jumps in each bounded ordinary time interval.

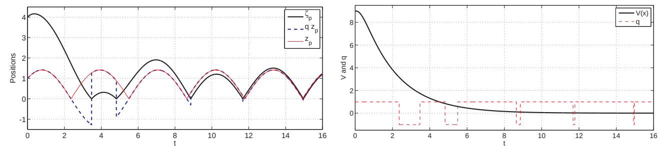

Figure 2. A simulation of the system in Example 4 from the initial condition $\zeta_0 = [4 \;\; 1]^T$, $z_0 = [1 \;\; 1]^T$, $q_0 = 1$, and $\tau_0 = 50$, for $k = 4$, $N = 100$, and $\delta = 2$.

the thin solid red curve in the upper plot of Figure 2). The (ζ_p, ζ_v) dynamics, which also experience a sign change in velocity when $\zeta_p = 0$ and $\zeta_v \le 0$, are acted on during flows through a control action that introduces the term $-k(\zeta_v - q z_p)$, which aims to make ζ_p track z_p (see the bold solid black curve in the upper plot of Figure 2). The relevant error is captured in the variables defined as $x_p := \zeta_p - q z_p$, $x_v := \zeta_v - q z_v$, $x_1 := (x_p, x_v)^T$. The bold dashed blue curve in the upper plot of Figure 2 clarifies graphically why ζ_p is compared to $q z_p$ rather than to z_p directly. We note that since the flow set and jump set yield the constraints $\zeta_p \ge 0$ and $z_p \ge 0$, the condition $x_p = 0$ necessarily implies $\zeta_p = z_p$, regardless of the value of $q \in \{-1, 1\}$. Thus, convergence of x_p to zero corresponds to convergence of ζ_p to z_p. We re-write the flow and jump dynamics in the coordinates $x := (x_p, x_v, z_p, z_v, q, \tau)$ to get a system with data (C, F, D, G) and, in these coordinates, establish global exponential stability of the closed set $\mathcal{A} := \{0\} \times \mathbb{R}^4$ using Theorem 2.

With the definitions $F_1 := \left[\begin{smallmatrix} 0 & 1 \\ -1 & -k \end{smallmatrix}\right]$, $H_1 := \left[\begin{smallmatrix} 0 & \sqrt{2k} \end{smallmatrix}\right]$ we have that $F_1^T + F_1 = -H_1^T H_1$, (H_1, F_1) is observable, and $\dot{x}_1 = F_1 x_1$. Note also that for jumps that correspond to $\xi \in \widetilde{D}_1$, we have that $x_1^+ = -x_1$ while for jumps that correspond to $\xi \in \widetilde{D}_2$, we have that $x_1^+ = x_1$. We define $V(x) := x_1^T x_1$ for all $x \in \mathbb{R}^6$ so that $|x|_{\mathcal{A}}^2 \le V(x) \le |x|_{\mathcal{A}}^2$ for all $x \in \mathbb{R}^6$. Implicitly, we are using the Euclidean norm as the underlying norm in the definition of the distance. We also have that $\langle \nabla V(x), f \rangle = x_1^T (F_1^T + F_1) x_1 = -x_1^T H_1^T H_1 x_1$ for all $x \in C$ and $f \in F(x)$, and $V(g) = |\pm x_1|^2 = |x_1|^2 = V(x)$ for all $x \in D$ and $g \in G(x)$. Since all of the conditions of Assumption 1 are satisfied, with $\mu = \infty$, the set $\mathcal{A}$ is globally exponentially stable according to Theorem 2. Figure 2 shows a typical response of the tracking system where the exponential convergence of V can be appreciated.

There are initial conditions from which complete solutions do not exist. For example, if $\tau(0,0) \in [0,1)$ and the rest of the state is initialized at a point in $C \cap D$ where it is not possible to flow, then the ensuing maximal solutions are trivial, that is, they have the time domain equal to the single point $(0,0)$. Picking $\delta > 0$ and $N \ge 1$ sufficiently large, initializing τ near N, and picking initial values for (z_p, z_v) with norm bounded away from zero typically is sufficient to guarantee that maximal solutions are complete. Because of the average dwell-time mechanism, if the time domain is unbounded then it is unbounded in the "ordinary" time direction. ⌟

C. Proof of main result

We start with a technical lemma that is closely related to the well-known "Squashing" lemma from linear systems theory [20], but is expressed in terms of a Lyapunov inequality. The proof of this lemma draws inspiration from the proof of [4, Proposition 2.1].

Lemma 1: Let the pair (H, F), with $F \in \mathbb{R}^{n \times n}$, be observable. Then there exist positive scalars $k \ge 1$, $\nu \ge 1$ and a matrix-valued function $P : [1, \infty) \to \mathbb{R}^{n \times n}$, such that, for all $h \in [1, \infty)$, $P(h)$ is symmetric and

$$\begin{array}{rcl} F^T P(h) + P(h) F & \le & -hP(h) + kh^\nu H^T H \\ I \le P(h) & \le & kh^\nu I \; . \end{array} \tag{13}$$

Proof: Consider the case where $H \in \mathbb{R}^{1 \times n}$ and (H, F) is in observer canonical form, that is,

$$H = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}^T, \; F = \bar{F}(c) := \begin{bmatrix} -c_1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & \ddots & 1 \\ -c_n & 0 & \dots & \dots & 0 \end{bmatrix} \tag{14}$$

for some $c = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^T \in \mathbb{R}^n$. Let $\beta \in \mathbb{R}^n$ be such that $\bar{F}(\beta)$ is Hurwitz with all eigenvalues having real part less than -1, and consider

$$K(h) := \begin{bmatrix} c_1 - h\beta_1 & c_2 - h^2\beta_2 & \dots & c_n - h^n \beta_n \end{bmatrix}^T. \tag{15}$$

Define $T(h) := \mathrm{diag}(h^{n-1}, h^{n-2}, \dots, 1)$ and note that

$$h\bar{F}(\beta) = T(h)[F + K(h)H]T(h)^{-1}. \tag{16}$$

Pick $\widehat{P} = \widehat{P}^T \ge I$ such that

$$(\bar{F}(\beta) + I)^T \widehat{P} + \widehat{P}(\bar{F}(\beta) + I) \le 0, \tag{17}$$

and take $P(h) := T(h)\widehat{P}T(h)$ so that $I \le P(h) \le \lambda_{max}(\hat{P}) h^{2(n-1)}$, where $\lambda_{max}(\hat{P})$ is clearly independent of h. Thus, using first (16) and then (17), we get

$$\begin{array}{l} [F + K(h)H]^T P(h) + P(h)[F + K(h)H] \; = \\ \; = \; hT(h)\Big(\bar{F}(\beta)^T \widehat{P} + \widehat{P}\bar{F}(\beta)\Big)T(h) \le \; -2hP(h). \end{array} \tag{18}$$

Using (15) and the fact that $h \ge 1$, we have $H^T K(h)^T K(h) H \le p_n h^{2n} H^T H$ for some $p_n > 0$ that depends on c, β, and n but is independent of h. Then from (18) and using in the second inequality below $A^T B + B^T A \le hB^T B + \frac{1}{h} A^T A$, with $A = P(h)^{1/2} K(h) H$, $B = -P(h)^{1/2}$, and $h > 0$, we get

$$\begin{array}{rl} F^T P(h) + P(h) F \; \le & \\ \le & -2hP(h) - H^T K(h)^T P(h) - P(h) K(h) H \\ \le & -2hP(h) + hP(h) + \frac{1}{h}[K(h)H]^T P(h)[K(h)H] \\ \le & -hP(h) + \lambda_{max}(\hat{P})\frac{h^{2(n-1)}}{h} H^T K(h)^T K(h) H \\ \le & -hP(h) + \lambda_{max}(\hat{P})\frac{h^{2(n-1)}}{h} p_n h^{2n} H^T H \\ = & -hP(h) + \lambda_{max}(\hat{P}) p_n h^{4n-3} H^T H \; . \end{array}$$

This calculation establishes the result with $\nu = 4n - 3$ when $H \in \mathbb{R}^{1 \times n}$ and (H, F) is in observer canonical form. For general (H, F) with $H \in \mathbb{R}^{1 \times n}$, the result of the lemma is obtained by invoking a coordinate transformation T such that the pair $(\widehat{H}, \widehat{F}) := (HT^{-1}, TFT^{-1})$ is in observer canonical form and then applying the argument above to $(\widehat{H}, \widehat{F})$. For general (H, F) with $H \in \mathbb{R}^{m \times n}$, we invoke Heymann's lemma, as done in [4], to reduce the problem to one with $H \in \mathbb{R}^{1\times n}$ and then apply the calculations just described for this case. In particular, Heymann's lemma [13] states that for any $v \in \mathbb{R}^m$ such that $v^T H \neq 0$, there exists $K_\circ$ such that the pair $(v^T H, F + K_\circ H)$ is observable. ■

We are now ready to prove Theorem 2. Throughout this proof we use the overloaded notation $|x|_P^2 := x^T P x$ when P is a symmetric positive semi-definite matrix.

Proof: (of Theorem 2) We first note that, according to the last item of Assumption 1, it is sufficient to establish exponential stability of the set $\mathcal{A}_\xi := [0, N] \times \mathcal{A}$ for system (12). We use $\xi := (\tau, x)$ for the full state of (12).

According to Assumption 1, we let the function V satisfy (3) as well as (6) and (7) with $p = 2$ and $\rho(x) = x_1^T H_1^T H_1 x_1$. Using Lemma 1, define $U(x) := |x_1|^2_{P(h)}$. From (13), (3), and the fact that $\mathcal{A} = \{0\} \times \mathbb{R}^{n_2}$ so that $|x|_{\mathcal{A}} = |x_1|$, we have that for each $x \in (C \cup D \cup G(D)) \cap (\mathcal{A} + \mu\mathbb{B}^\circ)$,

$$\frac{1}{\overline{\alpha}} V(x) \leq U(x) \leq \frac{kh^\nu}{\underline{\alpha}} V(x)\,. \tag{19}$$

Pick h large enough so that $1 > \gamma := \exp\left(-\frac{h}{2\delta}\right) kh^\nu \frac{\overline{\alpha}}{\underline{\alpha}}$ and define $\varphi(\tau) := \frac{1}{kh^\nu} \exp\left(\frac{h(\tau - N)}{2\delta}\right)$, so that $\frac{\partial\varphi}{\partial\tau} = \frac{h}{2\delta}\varphi(\tau)$, and $Y(\xi) = Y(\tau, x) := V(x) + \varphi(\tau)U(x)$. We establish the conditions (3)-(5) for the function Y, that is, replacing V by Y, $\mathcal{A}$ by $\mathcal{A}_\xi$ and x by ξ throughout (3)-(5). Condition (3) follows from (19) and the fact that φ is continuous and bounded away from zero on $[0, N]$, which is where the variable τ is constrained to evolve for the hybrid system (12).

For (4), considering (6), (13), (19), for all $\xi \in [0, N] \times C$ and $f \in [0, \delta] \times F(x)$, we get

$$\begin{array}{rcl}
\langle \nabla Y(\tau, x), f\rangle & \leq & \frac{\partial\varphi}{\partial\tau}\delta\, |x_1|^2_{P(h)} - |x_1|^2_{H_1{}^T H_1} \\
& & + \varphi(\tau)\left(-h\, |x_1|^2_{P(h)} + kh^\nu\, |x_1|^2_{H_1{}^T H_1}\right) \\
& = & -\, |x_1|^2_{H_1{}^T H_1} - \frac{h}{2}\varphi(\tau)\, |x_1|^2_{P(h)} \\
& & + \, kh^\nu \varphi(\tau)\, |x_1|^2_{H_1{}^T H_1} \\
& \leq & -\left[1 - \exp\left(\frac{h(\tau-N)}{2\delta}\right)\right] |x_1|^2_{H_1{}^T H_1} \\
& & - \, \frac{h}{2}\varphi(\tau)\, |x_1|^2_{P(h)} \\
& \leq & -\frac{h}{2}\varphi(\tau)\, |x_1|^2_{P(h)} \\
& \leq & -\frac{h}{4}\left(\frac{1}{kh^\nu}\exp\left(-\frac{hN}{2\delta}\right) + \varphi(\tau)\right) U(x) \\
& \leq & -\frac{h}{4}\left(\frac{1}{kh^\nu}\exp\left(-\frac{hN}{2\delta}\right)\frac{1}{\overline{\alpha}}V(x) + \varphi(\tau)U(x)\right) \\
& \leq & -\frac{h}{4}\min\left\{1, \frac{1}{kh^\nu}\exp\left(-\frac{hN}{2\delta}\right)\frac{1}{\overline{\alpha}}\right\} Y(\tau, x).
\end{array}$$

For (5), using (7), for $\tau \in [1, N]$, $x \in D$, $g \in G(x)$, we get

$$\begin{array}{rcl}
Y(\tau - 1, g) & = & V(g) + \varphi(\tau - 1)U(g) \\
& \leq & V(g) + \varphi(\tau - 1)\frac{kh^\nu}{\underline{\alpha}}V(g) \\
& \leq & V(x) + \varphi(\tau - 1)\frac{kh^\nu}{\underline{\alpha}}V(x) \\
& \leq & V(x) + \varphi(\tau - 1)\frac{kh^\nu \overline{\alpha}}{\underline{\alpha}}U(x) \\
& = & V(x) + \gamma\varphi(\tau)U(x) \\
& = & Y(\tau, x) + (\gamma - 1)\varphi(\tau)U(x) \\
& \leq & Y(\tau, x) + (\gamma - 1)c_1 \\
& \leq & Y(\tau, x) + (\gamma - 1)c_2
\end{array}$$

where

$$\begin{array}{rcl}
c_1 & := & \frac{1}{2}\left(\varphi(\tau)U(x) + \frac{1}{kh^\nu}\exp\left(-\frac{hN}{2\delta}\right)\frac{V(x)}{\overline{\alpha}}\right) \\
c_2 & := & \frac{1}{2}\min\left\{1, \frac{1}{kh^\nu}\exp\left(-\frac{hN}{2\delta}\right)\frac{1}{\overline{\alpha}}\right\} Y(\tau, x)\,.
\end{array}$$

Since $\gamma < 1$, there exists $\lambda > 0$ satisfying (4)-(5). ■

VI. Conclusion

In this note, Lyapunov-based sufficient conditions for exponential stability in hybrid systems are developed. The main contribution is the construction of a strict Lyapunov function that establishes exponential stability from a non-strict Lyapunov function when certain observability conditions apply. Our motivation is to provide a tool that can be used to establish global exponential tracking in a class of mechanical systems with impacts, as developed in [7], [8], [6].

References

[1] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[2] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems Part II:(pre) asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

[3] V.S. Chellaboina, S.P. Bhat, and W.M. Haddad. An invariance principle for nonlinear hybrid and impulsive dynamical systems. *Nonlinear Analysis*, 53(3-4):527–550, 2003.

[4] D. Cheng, L. Guo, Y. Lin, and Y. Wang. A note on overshoot estimation in pole placements. *Journal of Control Theory and Applications*, 2(2):161–164, 2004.

[5] R.A. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2000.

[6] F. Forni, A.R. Teel, and L. Zaccarian. Follow the bouncing ball: global results on tracking and state estimation with impacts. *IEEE Trans. Aut. Cont., submitted*, 2011.

[7] F. Forni, A.R. Teel, and L. Zaccarian. Tracking control in billiards using mirrors without smoke, part I: Lyapunov-based local tracking in polyhedral regions. In *Joint CDC-ECC*, pages 3283–3288, Orlando (FL), USA, December 2011.

[8] F. Forni, A.R. Teel, and L. Zaccarian. Tracking control in billiards using mirrors without smoke, part II: additional Lyapunov-based local and global results. In *Joint CDC-ECC*, pages 3289–3294, Orlando (FL), USA, December 2011.

[9] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *Control Systems Magazine, IEEE*, 29(2):28–93, April 2009.

[10] R. Goebel, R.G. Sanfelice, and A.R. Teel. Invariance principles for switching systems via hybrid systems techniques. *Systems & Control Letters*, 57(12):980–986, 2008.

[11] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573 – 587, 2006.

[12] R. Goebel and A.R. Teel. Preasymptotic stability and homogeneous approximations of hybrid dynamical systems. *SIAM Review*, 52(1):87–109, 2010.

[13] M. Heymann and W. Wonham. Comments on pole assignment in multi-input controllable linear systems. *IEEE Transactions on Automatic Control*, 13(6):748–749, 1968.

[14] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, USA, 3rd edition, 2002.

[15] J. Lygeros, K.H. Johansson, S.N. Simic, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48:2–17, 2003.

[16] M. Malisoff and F. Mazenc. Constructions of strict Lyapunov functions for discrete time and hybrid time-varying systems. *Nonlinear Analysis: Hybrid Systems*, 2(2):394–407, 2008.

[17] L. Menini and A. Tornambè. Asymptotic Tracking of Periodic Trajectories for a Simple Mechanical System Subject to Nonsmooth Impacts. *IEEE Trans. Aut. Control*, 46:1122–1126, 2001.

[18] A.N. Michel and L. Hou. Relaxation of hypotheses in LaSalle-Krasovskii type invariance results. *SIAM J. on Control and Optimization*, 49(4):1383–1403, 2011.

[19] I.C. Morarescu and B. Brogliato. Trajectory tracking control of multi-constraint complementarity Lagrangian systems. *IEEE Transactions on Automatic Control*, 55(6):1300–1313, 2010.

[20] F.M. Pait and A.S. Morse. A cyclic switching strategy for parameter-adaptive control. *IEEE Trans. Aut. Cont.*, 39(6):1172–1183, 1994.

[21] R.G. Sanfelice, R. Goebel, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12):2282–2297, 2007.

[22] R.G. Sanfelice and A.R. Teel. Asymptotic stability in hybrid systems via nested Matrosov functions. *IEEE Transactions on Automatic Control*, 54(7):1569–1574, 2009.