Extensions of surperalgebras of Krichever-Novikov type

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1. Algebras of Krichever-Novikov type

2. Cocycles on $\mathscr{L}_{g,N}$ and on $\mathscr{J}_{g,N}$

First introduced in 1987 by Krichever and Novikov

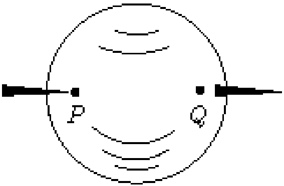


Fig. 1: Virasoro situation
$(g = 0, \; N = 2)$

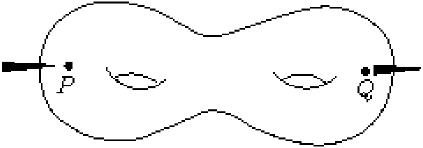


Fig. 2: An example of a Krichever - Novikov situation

- Let M, be a compact Riemann surface of genus g;
- Consider a set of two points : $A = \{P\} \cup \{Q\}$

Further studied by Schlichenmaier around 1990

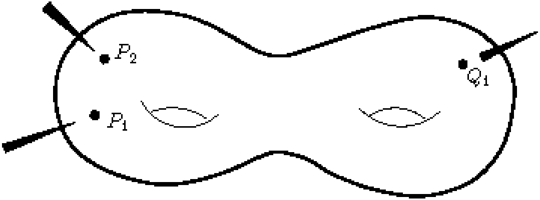
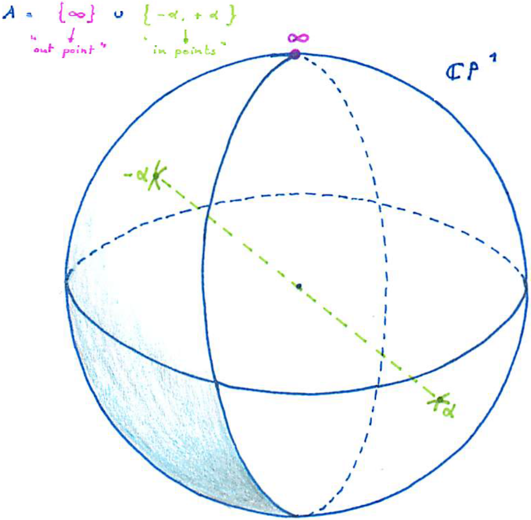


Fig. 3: An example of a generalized situation ($N = 3$, 2 in-points, 1 out-point)

- Let M, be a compact Riemann surface of genus g ;
- Consider the union of two sets of ordered points and fix the splitting :

$$A = \underbrace{\{P_1, \ldots, P_K\}}_{:=I} \cup \underbrace{\{Q_1, \ldots, Q_{N-K}\}}_{:=O}$$

Special case also considered by Schlichenmaier



Generators of $\mathscr{G}_{0,3} \cong \mathscr{F}_{-1}$:

$$V_{2k}(z) = z(z-\alpha)^k(z+\alpha)^k \frac{d}{dz},$$
$$V_{2k+1}(z) = (z-\alpha)^{k+1}(z+\alpha)^{k+1} \frac{d}{dz}$$

with relations :

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & n, m \text{ odd,} \\ (m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2} & n \text{ odd , } m \text{ even,} \\ (m-n)(V_{n+m} + \alpha^2 V_{n+m-2}) & n, m \text{ even .} \end{cases}$$

Generators of $\mathscr{A}_{0,3} \cong \mathscr{F}_0$:

$$G_{2k}(z) = (z-\alpha)^k(z+\alpha)^k,$$
$$G_{2k+1}(z) = z(z-\alpha)^k(z+\alpha)^k$$

with relations :

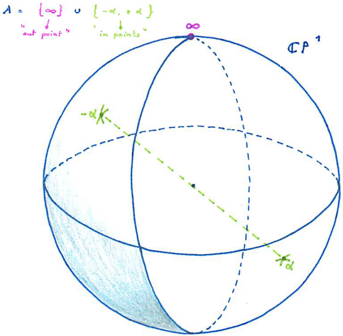
$$G_n.G_m = \begin{cases} G_{n+m} + \alpha^2 G_{n+m-2} & n, m \text{ odd,} \\ G_{n+m} & \text{otherwise .} \end{cases}$$

Supercase studied by Morier-Genoud & Leidwanger (2011)

$$\mathscr{L}_{g,N} = \mathscr{G}_{g,N} \oplus \mathscr{F}_{-1/2} \qquad \mathscr{J}_{g,N} = \mathscr{A}_{g,N} \oplus \mathscr{F}_{-1/2}$$

Special case :

$$\mathscr{L}_{0,3} = \mathscr{G}_{0,3} \oplus \mathscr{F}_{-1/2} \qquad \mathscr{J}_{0,3} = \mathscr{A}_{0,3} \oplus \mathscr{F}_{-1/2}$$



Generators of $\mathscr{L}_{0,3}$

$$V_{2k}(z) = z(z-\alpha)^k(z+\alpha)^k \tfrac{d}{dz}, \qquad \varphi_{2k+\frac{1}{2}}(z) = \sqrt{2}z(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2},$$

$$V_{2k+1}(z) = (z-\alpha)^{k+1}(z+\alpha)^{k+1} \tfrac{d}{dz}, \quad \varphi_{2k-\frac{1}{2}}(z) = \sqrt{2}(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2}.$$

with relations :

$$[V_n, \varphi_i] = \begin{cases} (i-\frac{n}{2})\varphi_{n+i} & n, i-\frac{1}{2} \text{ odd}, \\ (i-\frac{n}{2})\varphi_{n+i} + (i-\frac{n}{2}-1)\alpha^2\varphi_{n+i-2} & n \text{ odd }, i-\frac{1}{2} \text{ even}, \\ (i-\frac{n}{2})\varphi_{n+i} + (i-\frac{n}{2}+\frac{1}{2})\alpha^2\varphi_{n+i+2} & n \text{ even }, i-\frac{1}{2} \text{ odd}, \\ (i-\frac{n}{2})\varphi_{n+i} + (i-\frac{n}{2}-\frac{1}{2})\alpha^2\varphi_{n+i+2} & n, i-\frac{1}{2} \text{ even }. \end{cases}$$

$$[\varphi_i, \varphi_j] = \begin{cases} V_{n+i} + \alpha^2 V_{i+j-2} & i-\frac{1}{2}, j-\frac{1}{2} \text{ even}, \\ V_{i+j} & \text{otherwise }. \end{cases}$$

Generators of $\mathscr{J}_{0,3}$

$$G_{2k}(z)=(z-\alpha)^k(z+\alpha)^k, \qquad \varphi_{2k+\frac{1}{2}}(z)=\sqrt{2}z(z-\alpha)^k(z+\alpha)^k\,dz^{-1/2},$$

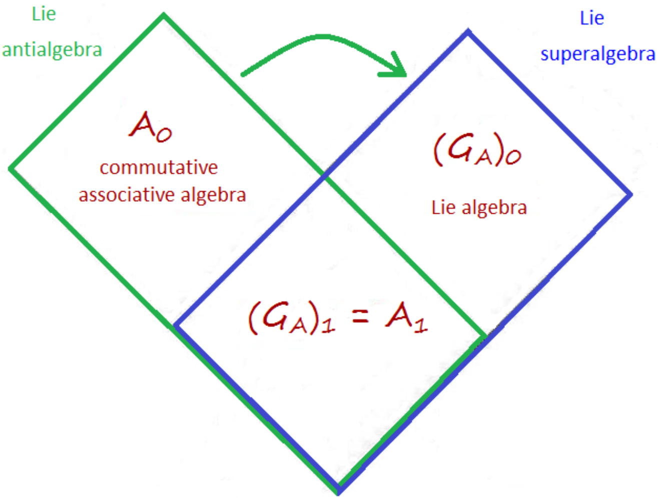
$$G_{2k+1}(z)=z(z-\alpha)^k(z+\alpha)^k, \quad \varphi_{2k-\frac{1}{2}}(z)=\sqrt{2}(z-\alpha)^k(z+\alpha)^k\,dz^{-1/2}.$$

with relations :

$$G_n.\varphi_i=\begin{cases}\frac{1}{2}\varphi_{n+i} & n \text{ even or } i-\frac{1}{2} \text{ odd,}\\ \frac{1}{2}(\varphi_{n+i}+\alpha^2\varphi_{n+i-2}) & n \text{ odd and } i-\frac{1}{2} \text{ even,}\end{cases}$$

$$\varphi_i.\varphi_j=\begin{cases}(j-i)G_{i+j} & i-\frac{1}{2} \text{ odd }, j-\frac{1}{2} \text{ even,}\\ (j-i)G_{i+j}+(j-i+1)\alpha^2G_{i+j-2} & i-\frac{1}{2} \text{ aven }, j-\frac{1}{2} \text{ odd,}\\ (j-i)(G_{i+j}+\alpha^2G_{i+j-2}) & i-\frac{1}{2} \text{ aven }, j-\frac{1}{2} \text{ even.}\end{cases}$$

Introduction by Ovsienko of a class of $\mathbb{Z}_2$-graded commutative but not associative algebras (2007)



Interesting examples of Lie superalgebras

1. Symplectic orthogonal $osp(1|2)$:

$$\begin{pmatrix} 0 & u & v \\ v & p & q \\ -u & r & -p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & q \\ 0 & r & -p \end{pmatrix} \oplus \begin{pmatrix} 0 & u & v \\ v & 0 & 0 \\ -u & 0 & 0 \end{pmatrix}$$

$$osp(1|2) \quad = \quad < Y, H, X > \quad \oplus \quad < A, B >$$

2. Conformal Neveu-Schwarz $\mathscr{K}(1)$ = $< \alpha_n > \oplus < b_i >$ with relations

$$[\alpha_n, \alpha_m] = (m-n)\alpha_{n+m}, \qquad [\alpha_n, b_i] = (i - \tfrac{n}{2})b_{i+n}, \qquad [b_i, b_j] = -\alpha_{i+j}.$$

3. Lie superalgebra of Krichever-Novikov type, $\mathscr{L}_{g,N}$.

Interresting examples of Lie Antialgebras

1. Tiny Kaplansky, $K_3 = <\varepsilon> \oplus <a,b>$ with relations

$$\varepsilon.\varepsilon = \varepsilon, \qquad \varepsilon.a = \frac{1}{2}a, \qquad \varepsilon.b = \frac{1}{2}b, \qquad a.b = \frac{1}{2}\varepsilon$$

2. Full derivation-Conformal Algebra, $\mathscr{AK}(1) = <\varepsilon_n> \oplus <a_i>$ with relations

$$\varepsilon_n.\varepsilon_m = \varepsilon_{m+n}, \qquad \varepsilon_n.a_i = \frac{1}{2}a_{i+n}, \qquad a_i.a_j = \frac{1}{2}(j-i)\varepsilon_{i+j}.$$

3. Jordan superalgebra of Krichever-Novikov type, $\mathscr{J}_{g,N}$.

1 Algebras of Krichever-Novikov type

2 Cocycles on $\mathscr{L}_{g,N}$ and on $\mathscr{J}_{g,N}$

Trivial central extension on a Lie superalgebra

- A 2-cocycle on a Lie superalgebra $\mathcal{L}$ is an even bilinear function $c : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ satisfying the following conditions :

  (C1) **skewsymmetry** : $c(u,v) = -(-1)^{\bar{u}\bar{v}} c(v,u)$

  (C2) **Jacobi identity** : $c(u,[v,w]) = c([u,v],w) + (-1)^{\bar{u}\bar{v}} c(v,[u,w])$

- As in the usual Lie case, a 2-cocycle defines a central extension of $\mathcal{L}$.

$$\mathcal{L} \xrightarrow{\quad \textit{extend} \quad} \mathcal{L} \oplus \mathcal{B}$$

- A 2-cocycle is called trivial , or a coboundary if it is of the form $c(u,v) = f([u,v])$, where f is a linear function on $\mathcal{L}$. Otherwise, c is called non-trivial .

- $Z^2(\mathcal{L})$: space of all 2-cocycles ; $B^2(\mathcal{L})$: space of 2-coboundaries
  $H^2(\mathcal{L}) = Z^2(\mathcal{L})/B^2(\mathcal{L})$: second cohomology space of $\mathcal{L}$.
  This space classifies non-trivial central extensions of $\mathcal{L}$.

Existence of a non-trivial almost graded 2-cocycle on $\mathscr{L}_{g,N}$

Theorem

The even bilinear map $c : \mathscr{L}_{g,N} \times \mathscr{L}_{g,N} \to \mathbb{C}$ given by

$$
\begin{aligned}
c\left(e(z)\tfrac{d}{dz}, f(z)\tfrac{d}{dz}\right) &= \frac{-1}{2i\pi}\int_{\mathscr{C}} \frac{1}{2}(e^{'''}f - ef^{'''}) - R(e'f - ef')dz,\\
c\left(\varphi(z)dz^{-1/2}, \psi(z)dz^{-1/2}\right) &= \frac{1}{2i\pi}\int_{\mathscr{C}} \frac{1}{2}\left(\varphi^{''}\psi + \varphi\psi^{''}\right) - \frac{1}{2}R\varphi\psi dz,\\
c\left(e(z)\tfrac{d}{dz}, \psi(z)dz^{-1/2}\right) &= 0
\end{aligned}
$$

is a well defined almost-graded non trivial local 2-cocyle, where $\mathscr{C}$ is a separating cycle and R is a projective connection.

Induced 1-cocycle on a Lie superalgebra

Given a 2-cocycle on a Lie (super) algebra $c : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$, one can define a 1-cocycle, C, defined as follow

$$\langle C(x), y\rangle := c(x,y) \quad , \quad C : \mathscr{L} \to \mathscr{L}^*.$$

The 1-cocycle condition is satisfied : $C([x,y]) = ad_x^*(C(y)) - (-1)^{\bar{x}\bar{y}} ad_y^*(C(x))$.

Corollary

An almost-graded local 1-cocycle on $\mathscr{L}_{g,N}$ is given by

$$\begin{aligned} C\left(e(z)\frac{d}{dz}\right) &= -\left(e''' - 2Re' - R'e\right) dz^2, \\ C\left(\varphi(z)dz^{-1/2}\right) &= \left(\varphi'' - \frac{1}{2}R\varphi\right) dz^{3/2} \end{aligned}$$

The converse construction does not work since c is not necessarily skewsymmetric.

1-cocycle on a Lie antialgebra

A 1-cocycle on a Lie antialgebra $\mathscr{A}$ with coefficients in an $\mathscr{A}$-module $\mathscr{B}$, is an even linear map $\mathscr{C} : \mathscr{A} \longrightarrow \mathscr{B}$ such that

$$\mathscr{C}\,(u\,\centerdot\,v) = \rho_u\,(\mathscr{C}(v)) + (-1)^{\bar{u}\bar{v}}\rho_v\,(\mathscr{C}(u))\,.$$

The dual space, $\mathscr{A}^*$, is an $\mathscr{A}$-module as well.

Theorem

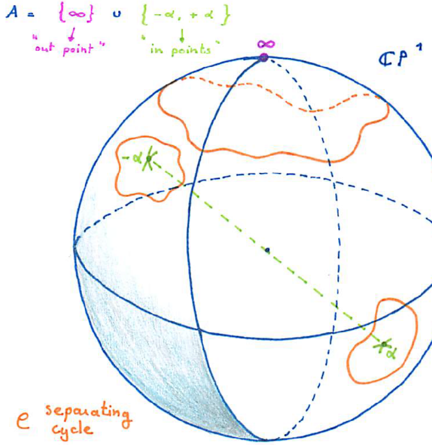
With respect to the splitting, the expression

$$\mathscr{C}\,(\varepsilon(z)) = -\varepsilon'(z)dz, \qquad \mathscr{C}\left(\psi(z)dz^{-1/2}\right) = \left(\psi''(z) - \frac{1}{2}R\,\psi(z)\right)dz^{3/2}$$

defines an almost-graded local 1-cocycle on $\mathscr{J}_{g,N}$ with coefficients in $\mathscr{J}^*_{g,N}$.

What does happen on $\mathscr{L}_{0,3}$ and $\mathscr{J}_{0,3}$?

We have a clear basis on these spaces, so that we can compute thanks to the residues theorem the different cocycles on the elements of the basis.



What does happen on $\mathscr{L}_{0,3}$ and $\mathscr{J}_{0,3}$?

Proposition

Up to isomorphy, the 2-cocycle on the Lie superalgebra $\mathscr{L}_{0,3}$, for all $k, l \in \mathbb{Z}$, is given by

$$\begin{aligned}
c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l+\frac{1}{2}}\right) &= c\left(\varphi_{2k-\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) = 0 \\
c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) &= 4k(2k+1)\delta_{k+l,0} + 8\alpha^2 k(k-1)\delta_{k+l,1} \\
c(V_{2k}, V_{2l}) &= -2k(4k^2-1)\delta_{k+l,0} - 8\alpha^2 k(k-1)(2k-1)\delta_{k+l,1} \\
&\quad -8\alpha^4 k(k-1)(k-2)\delta_{k+l,2} \\
c(V_{2k+1}, V_{2l+1}) &= -8\alpha^2(k+1)k(k-1)\delta_{k+l,0} - 4k(k+1)(2k+1)\delta_{k+l,-1} \\
c(V_{2k}, V_{2l+1}) &= 0.
\end{aligned}$$

What does happen on $\mathscr{L}_{0,3}$ and $\mathscr{J}_{0,3}$?

Proposition

Up to isomorphy, the 1-cocycle on the Lie superalgebra $\mathscr{L}_{0,3}$ is given by :

$$C(V_n) = -n(n-1)(n+1)V^*_{-n} - 2\alpha^2 n(n-2)(n-1)V^*_{-n+2} - \alpha^4 n(n-2)(n-4)V^*_{-n+4}$$

$$C(V_m) = -(m+1)m(m-1)V^*_{-m} - \alpha^2(m+1)(m-1)(m-3)V^*_{-m+2},$$

$$C(\varphi_i) = 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2},$$

$$C(\varphi_j) = 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2},$$

where $i-\frac{1}{2}$, n are even and $j-\frac{1}{2}$, m are odd.

What does happen on $\mathscr{L}_{0,3}$ and $\mathscr{J}_{0,3}$?

Proposition

Up to isomorphy, the 1-cocycle on the algebra $\mathscr{J}_{0,3}$ is given by

$$C(G_n) = \; -nG^*_{-n},$$

$$C(G_m) = \; -mG^*_{-m} - \alpha^2(m-1)G^*_{-m+2},$$

$$C(\varphi_i) = \; 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2},$$

$$C(\varphi_j) = \; 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2},$$

where $i-\frac{1}{2}$ and n are even and $j-\frac{1}{2}$, m are odd.

Thank you for your attention

Reference : arXiv :1204.4338