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ON THE LINK BETWEEN DESIGN AGAINST FATIGUE AND FRACTURE MECHANICS

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ABSTRACT

The gradient method, often used in design against fatigue, is analyzed from the point of view of its link with fracture mechanics. By a logical step, a law is deduced for the variation of the endurance limit in terms of the relative stress gradient. The proposed expression is found to be in good agreement with older ones and may easily be extended to a large class of materials.

INTRODUCTION

There are two fundamentally distinct approaches of fatigue strength analysis. The first one, which may be called the traditional approach, is based on the concept of endurance limit and makes use of stress concentration factors and fatigue notch factors [2,3,6,7,8,9,14,15,17,19,21]. The second one is based on fracture mechanics and crack propagation [5,10,11,18]. Surprisingly, even when both points of view are exposed in a same book, a possible link between them is almost never envisaged, in such a way that both approaches seem to be completely orthogonal.

Concerning the traditional methods, it has to be noted that the so-called gradient method tends to become a standard in Germany [19,21] and in France [2]. In this method, the local endurance limit is implicitly or explicitly supposed to increase with the relative stress gradient. This approach constitutes a considerable simplification from older methods using separate scale and notch effects which are difficult to evaluate precisely, due to the large amount of experiences which are necessary. The crucial point of the gradient method is the adoption of a suitable expression of the local endurance limit and the gradient, and it is where authors diverge. In fact, most of proposed expressions come from heuristic considerations and some experiments or from statistical data.

A more sound theoretical background seems to be highly desirable. In this way, if it is postulated that the endurance limit depends on the relative stress gradient, a necessary physical condition should be the compatibility of the resulting theory with the fracture mechanics whose results are now generally admitted. This constitutes a first limiting condition, corresponding to very high gradients. At the other side, where the gradient tends to zero, it is well known that a full stress concentration factor has to be applied. A very simple interpolation between these two extreme cases leads to a variation law of the endurance limit which seems very reasonable and which is moreover

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in good accordance, in the case of steels, with Petersen's law. Based on two measurable characteristics of the considered material, our law opens the way to computations on less known materials such as Titanium or other ones.

1. THEORETICAL BACKGROUND OF THE GRADIENT METHOD

Fatigue tests show that the endurance limit cannot be considered as a material constant, since it depends on a lot of factors, the most significant ones being the size, the shape and the surface finish of the part. This last effect, whose role is secondary, will be omitted in what follows.

1.1 Scale effect

When testing smooth parts, that is to say without stress concentrations, it is found that the endurance limit may vary with the size of the part. This is the case for bending and torsion, where a higher endurance limit is obtained when the size of the part is decreased. No such effect is obtained in axial loading. The first conclusion of these results is *that the true intrinsic characteristic of the material is its endurance limit in axial loading.* It will be noted σ_{Do}.

Turning now to the fact that different limits are obtained in bending, it is clear that the fundamental difference between bending and axial loading is the existence of a stress gradient. In fact, high stresses only occur in a zone whose depth may be measured by the quantity

$$\frac{\sigma_{max}}{\left(\frac{d\sigma}{dx}\right) at\ the\ \max imum}$$

or equivalently, by the *relative stress gradient*

$$\chi_o = \frac{\left(\frac{d\sigma}{dx}\right) at\ the\ \max}{\sigma_{max}} = \frac{2}{d} \qquad (1)$$

Experimental results may then be explained by admitting that the endurance limit is an increasing function of the relative stress gradient, that is,

$$\sigma_D = \sum(\chi_o), \qquad (2)$$

$\sum(\chi_o)$ being an increasing function of the relative stress gradient, verifying the condition

$$\sum(o) = \sigma_{D_o} \qquad (3)$$

in order to obtain the correct value in axial loading. A consequence of this assumption is that for very great parts submitted to bending, the endurance limit tends to be equal to the axial loading limit,

$$\lim_{d\to\infty} \sigma_{fD} = \lim_{\chi_o \to o} \sum(\chi_o) = \sum(o) = \sigma_{D_o} \qquad (4)$$

1.2 Notch effect

In the case of notched parts, engineering practice is to compute the maximum stress σ_{max} indirectly from some easily computable nominal stress σ_n, through the so-called stress concentration factor K_t,

$$\sigma_{max} = K_t \sigma_n \qquad (5)$$

At a first glance, the endurance relation should be

$$\sigma_{max\,D} = \sum(\chi_o)$$

that is

$$\sigma_{nD} = \frac{\sum(\chi_o)}{K_t} \qquad (6)$$

This however is not true and a lot of experiences showed that the true relation is of the form

$$\sigma_{nD} = \frac{\sum(\chi_o)}{K_f} \qquad (7)$$

with a so-called *notch effect factor.* Generally noted K_f, this factor also varies in an intricate manner with the scale, the shape of the part and the material, a fact which may be symbolically written

$$K_f = K_f(\chi_o, K_t, material) \qquad (8)$$

This renders design computations highly delicate, since in most cases, sufficiently accurate values of K_f are not available.

However, some tendencies are known,

a) When increasing the size of the part, all other factors being equal, K_f tends to K_t.

b) For a given size, it is clear that the stress concentration factor increases with a decrease of the notch radius. Now, for great radii, K_f is of the same order as K_t, but for very low radii, K_f may be much lower than K_t and in practice, sharp notches lead to a non-zero, although low endurance limit.

c) Higher strength steels exhibit higher values of K_f.

If it is noted that at the vicinity of the notch, the relative stress gradient is of the form

$$\chi = \chi_o + \frac{C}{r} \qquad (9)$$

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results (a) and (b) may be explained by extending the arguments concerning the scale effect, that is by assuming that

$$\sigma_{\max D} = K_t \, \sigma_{nD} = \sum(\chi) \qquad (10)$$

with the value (9) of the gradient. Equivalently,

$$\sigma_{nD} = \frac{\sum(\chi)}{K_t} \qquad (11)$$

This is the basis of the *gradient method,* which is largely used in Germany [19,21] and also recommended in France by the CETIM [2]. It remains to determine a suitable function $\Sigma(\chi)$ and this is the point where different approaches diverge. Our purpose is to show that a rational endurance law may be obtained as an interpolation between the two limiting cases χ=0 and $\chi=\infty$.

2. PHYSICAL EXIGENCIES ON THE LAW $\Sigma(\chi)$

Assuming that the function $\Sigma(\chi)$ is an intrinsic characteristic of the material, it may be theoretically determined from model problems. In order to discard any secondary scale effect, axial loading problems will be considered.

First of all, a zero stress gradient is obtained in the case of the axial loading of an unnotched part. The result is

$$\sum(o) = \sigma_{Do} \qquad (12)$$

from which one may write in the general case

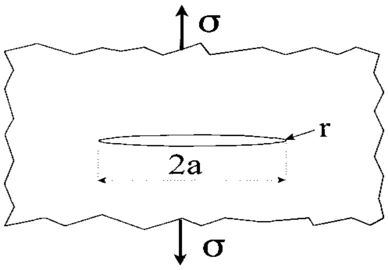
$$\sum(\chi) = \sigma_{Do} + f(\chi) \text{ with } f(0) = 0 \qquad (13)$$

the function $f(\chi)$ eventually depending on the material.

Let us now consider an elliptic hole of half length a and notch radius r (fig. 1). As is well known from elasticity theory, the relative stress gradient is then

$$\chi = \frac{2.33}{r} \qquad (14)$$

Figure 1 : elliptic hole



When the notch radius tends to zero, the hole becomes a crack, and the problem may be treated by using fracture mechanics [5,10,11,13]. As proved by IRWIN, the stress intensity factor K = KI is equal to

$$K = \lim_{r \to o} \frac{1}{2} \sigma_{\max} \sqrt{\pi r} \qquad (15)$$

and, using the value (14) of the relative stress gradient, this reduces to

$$K = \lim_{\chi \to \infty} \frac{1}{2} \sigma_{\max} \sqrt{\frac{2.33\pi}{\chi}} \qquad (16)$$

In the case of a fatigue loading, the stress intensity factors varies from a maximal value K_m to a minimal value K_m, and the classical notations are

$$\Delta K = K_M - K_m$$
$$R = K_m/K_M \qquad (17)$$

Considering an *alternate* load, R= -1, and $(\sigma_{max})_m = -(\sigma_{max})_M$ so that, from (16),

$$\Delta K\big|_{R=-1} = \lim_{\chi \to \infty} (\sigma_{\max})_M \sqrt{\frac{2.33\pi}{\chi}} \qquad (18)$$

Now, it is well known that crack propagation depends on ΔK and R, following a law of the form

$$\frac{da}{dN} = f(\Delta K, R)$$

and that there exists a *threshold* ΔK_{th} under which no crack propagation occurs. This threshold clearly defines the endurance limit $(\sigma_{\max})_M = \sum(\chi)$ so that

$$\Delta K_{th}\big|_{R=-1} = \lim_{\chi \to \infty} \sum(\chi) \sqrt{\frac{2.33\pi}{\chi}} \qquad (19)$$

This leads to the second exigency on the lax $\sum(\chi)$: *it has to verify the relation*

$$\lim_{\chi \to \infty} \frac{\sum(\chi)}{\sqrt{\chi}} = \frac{\Delta K_{th}\big|_{R=-1}}{\sqrt{2.33\pi}} \qquad (20)$$

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3. AN ELEMENTARY EXPRESSION OF $\Sigma(\chi)$

Expressing $\sum(\chi)$ in the form (14) and applying condition (20) leads to the condition

$$\lim_{\chi\to\infty}\frac{\sigma_{Do}+f(\chi)}{\sqrt{\chi}}=\lim_{\chi\to\infty}\frac{f(\chi)}{\sqrt{\chi}}=\frac{\Delta K_{th}\big|_{R=-1}}{\sqrt{2.33\pi}}$$

The most elementary function that verifies this condition is

$$f(\chi)=A\sqrt{\chi} \qquad (21)$$

with

$$A=\frac{\Delta K_{th}\big|_{R=-1}}{\sqrt{2.33\pi}} \qquad (22)$$

The result is thus

$$\sum(\chi)=\sigma_{Do}+A\sqrt{\chi} \qquad (23)$$

This simple law relates the endurance limit to two measurable characteristics of the material, namely, its axial loading endurance limit and its crack propagation threshold for R = -1.

4. PRACTICAL VALUES OF ΔK_{th} AND χ

Figure 2 : crack propagation threshold [12]

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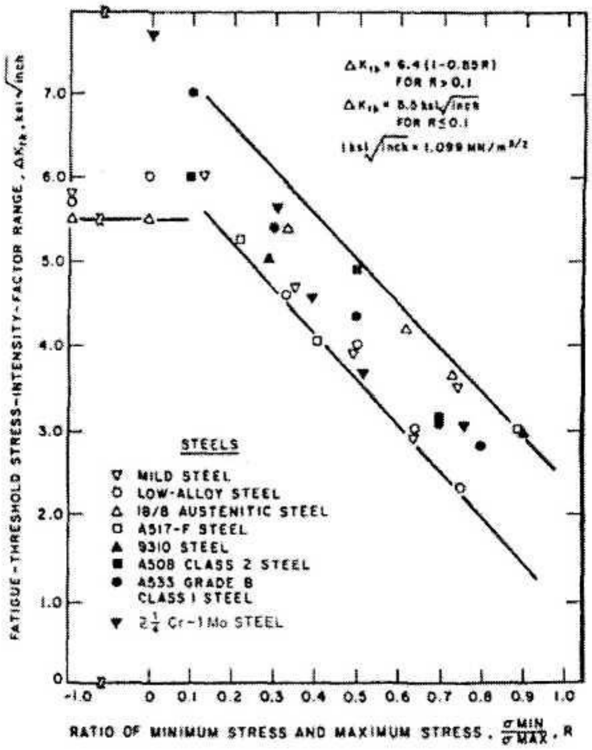

In order to perform a concrete discussion of formula (23), actual values of ΔK_{th} and χ are needed. Concerning the crack propagation threshold, results from BARSOM and WOLFE, as referred in [12], indicate that for a large number of steels, the following relation may be admitted (see fig. 2).

$$\Delta K_{th}\big|_{R=-1} \approx 6{,}045\ MPa\sqrt{m} = 191.2\ MPa\sqrt{mm} \qquad (24)$$

This leads to

$$A = 70.67\ MPa\sqrt{mm} \qquad (25)$$

Concerning the relative stress gradient, values are given from long in german litterature. Figure 3 is extracted from [19]. It has

to be noted that it is current practice to replace $\frac{2{,}33}{r}$ by the

simpler value $\frac{2}{r}$, leading to a maximum error of 8 % on the square root of the gradient, on the safe side.

Figure 3 : current values of the gradient [19]

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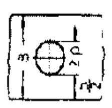
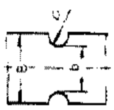
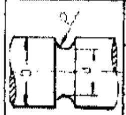
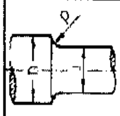
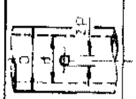
notch	loading	χ_0	χ
	axial	0	$2/\rho$
	bending	$4/b$	$4/b + 2/\rho$
	axial	0	$2/\rho$
	bending	$2/b$	$2/b + 2/\rho$
	axial	0	$2/\rho$
	bending	$2/d$	$2/d + 2/\rho$
	torsion	$2/d$	$2/d + 1/\rho$
	axial	0	$2/\rho$
	bending	$4/(D+d)$	$4/(D+d)+2/\rho$
	torsion	$4/(D+d)$	$4/(D+d)+1/\rho$
	torsion	$2/D$	$2/D + 1/\rho$
	bending	$2/D$	$2/D + 4/\rho$
	torsion	$2/D$	$2/D + 3/\rho$

5. A COMPARISON WITH OTHER PROPOSED FORMULAE

Analogous formulae to (23) have been previously proposed, but on different theoretical bases.

a) Siebel's formula, which is widely referred in the german literature [19,21], is equivalent to

$$\sum(\chi) = \sigma_{Do}(1+\sqrt{\rho^{*}\chi}) \qquad (26)$$

that is, the same expression as (23), with

$$A = \sigma_{Do}\sqrt{\rho^{*}} \qquad (27)$$

and a value of p* depending on the ultimate stress Rm of the steel. Admitting the approximate relation, $\sigma_{Do} \approx 0{,}45\ R_m$

Siebel's formula leads to the results of table 1

Table 1 : Siebel's formula

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Rm/Mpa	300	400	500	600	700
A/MPa √*mm*	31.37	38.61	43.86	48.29	50.81
Rm/Mpa	800	900	1000	1100	
A/MPa √*mm*	50.90	49.59	45	38.34	

The mean value of A is, following Siebel, equal to 44.09, that is 38 % lower than ours. It has to be noted that Siebel also gave values for other materials than steel.

b) Petersen's formula [19,21] may be written as

$$\sum(\chi) = \sigma_{Do}\left(1 + \frac{B}{Rm}\sqrt{\chi}\right) \quad (28)$$

with

$$B = 140\ MPa\sqrt{mm} \quad (29)$$

Assuming $\sigma_{Do} \approx 0.45\ Rm$, this leads to

$$A_{Peterson} = 140\chi 0.45 = 63.00\ MPa\sqrt{mm} \quad (30)$$

a value which differs from ours by 10 % only.

c) Heywood [17], who was probably the first to clearly assert the fundamental character of the endurance limit in axial loading, which does not depend on the scale, proposed for this type of loading the formula

$$K_f = \frac{K_t}{1 + 2\sqrt{\frac{a}{r}}} \quad (31)$$

where r is the notch radius. This is equivalent to

$$\sum(\chi) = \sigma_{Do}\left(1 + 2\sqrt{\frac{a}{r}}\right) = \sigma_{Do}\left(1 + 2\sqrt{2.33a}\ \sqrt{\chi}\right) \quad (32)$$

From tests on geometrically identical parts made from different steels, he found that endurance limits were correctly represented by the straight line

$$\sigma = \sigma_{Do} / K_t ,$$

and concluded that

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$$\sqrt{a} = \frac{C}{\sigma_{Do}} = \frac{C_1}{Rm} \qquad C, C_1 = cst \tag{33}$$

where a fixed ratio between σ_{Do} and R_m is assumed. This is in perfect accordance with Petersen. Later, he modified his formula as

$$K_f = \frac{K_t}{1 + 2\frac{K_t - 1}{K_t}\sqrt{\frac{a}{r}}}$$

in order to obtain $K_f = 1$ when $K_t = 1$. But as he had not made the link between $\frac{1}{r}$ and the relative stress gradient, he was not able to take the scale effect in account.

d) A gradient theory was also proposed by Brand and Sutterlin [2], with the following law

$$\sum(\chi) = a \log \chi^* + b \tag{34}$$

with

$$\chi^* = \max(\chi; 0.02) \tag{35}$$

and valid, following these authors, for $\chi < 10\ mm^{-1}$. This formula does not allow a limiting process for $\chi \to \infty$, that is to say, parts with a zero notch radius are considered to have a vanishing endurance limit, a fact that contradicts experience.

The salient point of the above comparison is the remarkable agreement between Petersen's formula and our results, however obtained by a totally different way. Siebel's values of A seem to be too conservative. Finally, the Brand- Sutterlin formula is not correct at the limit.

6. RELATION BETWEEN THE CRACK PROPAGATION THRESHOLD AND THE ENDURANCE LIMITS IN BENDING AND AXIAL LOADING

From formula (23), the conventional endurance limit for bending σ_D^*, which is obtained with smooth parts of 10 mm diameter, has to be

$$\sigma_D^* = \sum(0.2) = \sigma_{Do} + A\sqrt{0,2}$$

So, if σ_D^* and σ_{Do} are known, it is possible to deduce the value of the crack propagation threshold by

$$A = 2.236\,(\sigma_D^* - \sigma_{Do}) \tag{36}$$

and, from (22),

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$$\Delta K_{th}\big|_{R=-1} = 6.05\,(\sigma^*_D - \sigma_{Do}) \qquad (37)$$

It is interesting to compute these values from the numerous Smith diagrams contained in the german litterature [19]. The results are given in table 2. As can be seen, cemented steels and spheroidal graphite cast irons lead to the highest values. Concerning carbon steels and classical allied steels, the mean of the obtained values is

$$\left(\Delta K_{th}\big|_{R=-1}\right)_{mean} = 6.175\ MPa\,\sqrt{m}\ ,$$

in good accordance with Barsom-Rolfe's value (6.045 $\mathrm{MPa}\sqrt{m}$).

Table 2 : ΔK_{th} for steels and cast irons

Type	Name	σ_{Do} MPa	σ^*_D MPa	A MPa√mm	ΔK_{th} MPa√mm	ΔK_{th} MPa√m
Carbon steels	St37	175	200	55,90	151,3	4,785
	St42	190	220	67,08	181,5	5,740
	St50	230	260	67,08	181,5	5,740
	St60	270	300	67,08	181,5	5,740
	St70	300	340	89,44	242	7,653
Cemented steels	Ckl5	270	300	67,08	181,5	5,740
	15Cr3	320	350	67,08	181,5	5,740
	16MnCr5	400	450	111,8	302,5	9,566
	15CrNi6	500	550	111,8	302,5	9,566
	20MnCr5	540	600	134,2	363	11,48
	18CrNi8	580	650	156,5	423,5	13,39
Allied steels	Ck22	250	280	67,08	181,5	5,740
	Ck45	340	370	67,08	181,5	5,740
	40Mn4	400	440	89,44	242	7,653
	41Cr4	450	480	67,08	181,5	5,740
	50CrMo4	500	540	89,44	242	7,653
	30CrNiMo8	570	600	67,08	181,5	5,740
Spheroidal Graphite cast irons	GGG38	110	150	89,44	242	7,653
	GGG42	130	180	111,8	302,5	9,566
	GGG50	150	210	134,2	363	11,48
	GGG60	180	250	156,5	423,5	13,39
	GGG70	210	300	201,2	544,5	17,22

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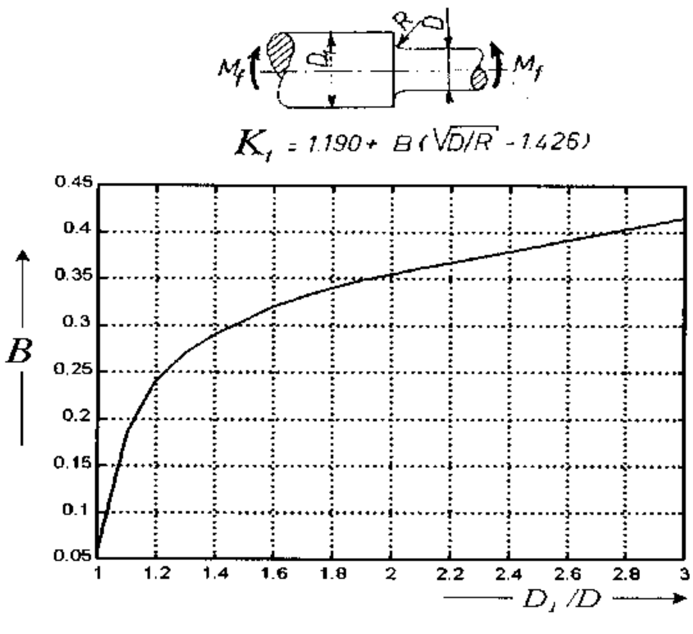
7. THE CASE OF SHARP NOTCHES

Being compatible with fracture mechanics, the proposed law renders possible a treatment of sharp notches, inasmuch an analytical expression of the stress concentration factor is known. As an example, let us consider a stepped shaft submitted to bending (fig. 4). From Peterson's curves [15] established between the limits D/R = 3.3 and D/R = 100, the following analytical expression may be adjusted,

$$K_t = 1.190 + B\left(\sqrt{\frac{D}{R}} - 1.426\right) \qquad (38)$$

with B depending on the ration D_1/D. The relative stress gradient is given by $\chi = \frac{4}{D + D_1} + \frac{2.33}{R}$

Figure 4 : stepped shaft



So, for a radius R tending to zero, all other dimensions being unchanged,

$$\sigma_{nD} = \lim_{R \to 0} \frac{\sum(\chi)}{K_t} = \frac{\sqrt{2.33}\ A}{B\sqrt{D}} \qquad (39)$$

From this result,

a) The right member depends on the material through the constant A, which is related to ΔK_{th}. Since this last quantity seems not to depend on the steel, the following conclusion is reached : *with a sharp notch, a stronger steel is not better.*

b) Sharp notches also lead to a *very strong scale effect,* namely of the form $1/\sqrt{D}$.

8. CONCLUSIONS

It appears that a design against fatigue by the gradient method is perfectly compatible with the results of fracture mechanics, if a suitable variation law of the endurance limit with the gradient is adopted. The proposed approach leads to a simple formula depending on two material parameters, its endurance limit in axial loading and its crack propagation threshold. From available data, this threshold seems to be very constant from one steel to another, a fact which is confirmed by classical data on endurance limits in bending and axial loading. A remarkable fact is the good agreement between the proposed formula and Petersen's one.

An essential feature of the proposed approach is that it can be easily extended to a wide class of metals without necessitating too large an experimentation, since only two material constants are necessary.

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