Optimal Fertility along the Lifecycle

Pierre Pestieau* and Gregory Ponthiere†

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Abstract

We explore the optimal fertility timing in a four-period OLG economy with physical capital, whose specificity is to include not one, but two reproduction periods. It is shown that, for a given total fertility rate, the economy exhibits quite different dynamics, depending on the timing of births. If all births take place in the late reproduction period, there exists no stable stationary equilibrium, and the econony exhibits cyclical dynamics due to labour growth fluctuations. We characterize the long-run social optimum, and show that optimal consumptions and capital depend on the optimal cohort growth factor, so that there is no one-to-one substitutability between early and late fertility. We also extend Samuelson's Serendipity Theorem to our economy, and study the robustness of our results to: (1) endogenizing fertility timing; (2) assuming rational anticipations about factor prices; (3) adding a third reproduction period.

Keywords: childbearing ages, early and late motherhoods, fertility, overlapping generations, social optimum.

JEL codes: E13, E21, J13.

*University of Liege, CORE, Paris School of Economics and CEPR.

†Paris School of Economics and Ecole Normale Superieure, Paris. [corresponding author] Address: Ecole Normale Supérieure, 48 Bd Jourdan, building B, second floor, office, B, 75014 Paris, France. Telephone: 0033-143136204. Fax: 0033-143136222. E-mail: gregory.ponthiere@ens.fr.

1 Introduction

The postponement of births is a key stylized fact of the last four decades. That change in the timing of births can be illustrated by the evolution of the average age of mothers at the first birth (Figure 1).[1] In Sweden, for instance, the average age of first-time mothers has grown from 25 to 29 years over 1970-2010. The postponement of parenthood is also illustrated by the observed rise in the average age of mothers for all births (Figure 2).[2]

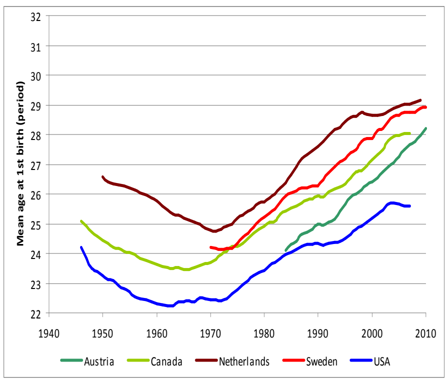

Figure 1: Average age of women at first birth (period)

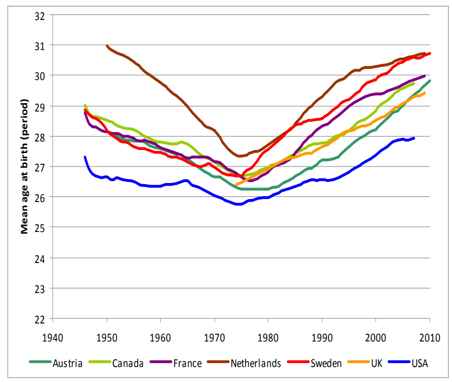

Figure 2: Average age of women at all births (period)

The observed postponement of births raises several questions. A first issue concerns the *causes* at work behind that phenomenon. In a pioneer article, Gustafsson (2001) reviewed major theoretical and empirical papers aimed at explaining that stylized fact. On the theoretical side, Happel *et al.* (1984) showed that consumption smoothing may imply delaying births, while Cigno and Ermisch (1989) argued that steeper earnings profiles induced by education lead to postponing births.[3] On the empirical side, the roles of the better earnings opportunities and better educational achievements for women were emphasized by Ermisch and Ogawa (1994) and Joshi (2002).[4]

Another important issue consists of the evaluation of the *effects* of that change in the timing of births on long-run economic dynamics. Momota (2009) examined, in a three-period overlapping generations (OLG) model with fixed total fertility, the impact of changes in the timing of births on the dynamics of the economy. D'Albis *et al* (2010) studied, in a continuous time OLG model, the joint dynamics of demography and economy under endogenous childbearing

[1] Data source: Human Fertility Database (2012).

[2] Data source: Human Fertility Database (2012).

[3] Cigno and Ermisch (1989) also found, on the basis of UK data, empirical support for that explanation of the observed heterogeneity in terms of fertility patterns.

[4] The timing of births influences in turn various outcomes. Ermisch and Pevalin (2005) showed that teen births worsens later outcomes on the marriage market.

ages, while assuming that the total number of children is decreasing in the timing of births. They proved that there exists a monetary steady-state if the average age of consumers is larger than the average age of producers.

A third question raised by the postponement of births consists of its social *optimality*. Is births delaying desirable from a social perspective? In a two-period OLG economy with a unique reproduction period, Samuelson (1975) showed that the (interior) optimal fertility rate balances, at the margin, two effects: on the one hand, the capital dilution effect due to a higher fertility; on the other hand, the intergenerational redistribution effect.[5] Samuelson proved also that fertility allows for the decentralization of the social optimum. This is the Serendipity Theorem: if there exists a unique stable stationary equilibrium, a perfectly competitive economy will converge towards the long-run social optimum when the optimal fertility rate is imposed.[6] Those results presuppose a unique reproduction period, and, hence, say little on the optimal fertility timing.

The goal of this paper is twofold. First, we propose to explore the consequences of fertility timing on long-run economic dynamics. That question can be formulated as follows: for a given total fertility rate (TFR), that is, a given total number of children along the lifecycle, is the timing of births neutral for economic dynamics? Then, in a second stage, we characterize the socially optimal timing of births. Here again, we examine the neutrality of fertility timing: is there, from the point of view of long-run social welfare, a one-to-one substitutability between early children and late children?

In order to answer those questions, we study a four-period perfectly competitive OLG economy with physical capital accumulation. The specificity of that economy is that there is here not one, but two reproduction periods: the second and third periods of life. To compare the long-run dynamics under different timing for births, we first study, as a baseline, an economy where parents take age-specific fertility rates as given, and analyze the impact of distinct fertility profiles on the long-run dynamics, under myopic anticipations about future factor prices. Then, we characterize the long-run social optimum, and study the optimal fertility timing. Finally, we explore the robustness of our results to the introduction of rational expectations about factor prices and endogenous fertility timing, as well as to the addition of a third reproduction period.

Anticipating on our results, we show that distinct timings for births lead to very different economic dynamics, even under a given total fertility rate. In particular, when all births take place in the late reproduction period, there exists no stable stationary equilibrium. Then, focusing on the long-run social optimum, we show that optimal consumption paths and optimal capital are defined

[5] Note that, as shown by Deardorff (1976), an interior optimal fertility rate does not always exist in a two-period OLG economy with Cobb-Douglas production and utility functions. In a more general model with CES production and utility functions, Michel and Pestieau (1993) emphasized that an interior optimal fertility rate requires a sufficiently low substitutability between capital and labour in the production process, and between first- and second-period consumptions in utility functions. Abio (2003) and Abio *et al* (2004) complemented those papers by studying optimal fertility under costly, endogenous fertility.

[6] Recently, Jaeger and Kuhle (2009) and de la Croix *et al* (2012) examined the robustness of the Serendipity Theorem to the introduction of debt and of risky lifetime.

in terms of the optimal long-run cohort growth factor, in which early and late fertility rates are no one-to-one substitutes (unlike in the TFR). We also derive an extended Serendipity Theorem: a perfectly competitive economy converges towards the social optimum, provided the government imposes the optimal long-run cohort growth factor. Finally, those results are shown to be robust to the introduction of rational expectations about factor prices, to the addition of a third reproduction period, as well as to endogenous fertility timing.[7]

The rest of the paper is organized as follows. Section 2 presents the baseline model. The long-run dynamics is studied in Section 3. Section 4 characterizes the long-run social optimum and studies its decentralization in line with Samuelson's Serendipity Theorem. Section 5 considers three extensions: (1) rational expectations about factor prices; (2) endogenous fertility; (3) addition of a third reproduction period. Section 6 concludes.

2 The model

We consider a four-period OLG model with physical capital accumulation. Its specificity lies in the existence of two - instead of one - reproduction periods. Period 1 consists of childhood. In periods 2 and 3, agents supply their labour inelastically, consume, save and make children. In period 4, they are retired.

2.1 Demography

Throughout this paper, we assume initial conditions insuring that the economy exhibits a strictly positive number of births at any period: $N_{-1} > 0$, $N_0 > 0$, where N_t denotes the number of individuals born at period t.

Individuals have n children in period 2, and m children in period 3. The total fertility rate (TFR) is, without loss of generality, assumed to be strictly positive, i.e. $n + m > 0$. The total number of individuals born at time t is:

$$N_t = nN_{t-1} + mN_{t-2} \tag{1}$$

The cohort size growth factor $g_t \equiv \frac{N_t}{N_{t-1}}$ is obtained by dividing (1) by N_{t-1}:

$$g_t = n + m\frac{N_{t-2}}{N_{t-1}} = n + \frac{m}{g_{t-1}} \tag{2}$$

If all children are born from young parents (i.e. $m = 0$), the cohort growth factor g_t is constant over time, and equal to n. If, on the contrary, all children are born from old adults (i.e. $n = 0$), g_t is no longer constant over time, but, rather, exhibits a two-period cycle.[8] In the general case where $n > 0$, $m > 0$, g_t

[7] Except the Extended Serendipity Theorem.

[8] To see this, note that

$$\begin{aligned} g_1 &= \frac{m}{g_0};\ g_2 = \frac{m}{g_1} = g_0;\ g_3 = \frac{m}{g_2} = \frac{m}{g_0} = g_1; \\ g_4 &= \frac{m}{g_3} = \frac{m}{g_1} = g_0;\ g_5 = \frac{m}{g_4} = \frac{m}{g_0} = g_1, \text{ etc...} \end{aligned}$$

converges, in the long-run, towards a unique level, equal to: $g = \frac{n+\sqrt[2]{n^2+4m}}{2}$.

To see this, denote $g_{t+1} \equiv f(g_t) = n + \frac{m}{g_t}$. We have that $\lim_{g_t \to 0} f(g_t) = +\infty$, and $\lim_{g_t \to +\infty} f(g_t) = n > 0$. In the (g_t, g_{t+1}) space, $f(g_t)$ lies above the $45°$ line for low g_t levels, but below the $45°$ line for high g_t levels. Thus, by continuity, $f(g_t)$ must cross the $45°$ line for some g_t, whose value is obtained by setting $g_{t+1} = g_t$ in $f(g_t)$, leading to $g = \frac{n+\sqrt[2]{n^2+4m}}{2}$. Given that $f'(g_t) \leq 0$, the intersection of $f(g_t)$ with the identity line is unique. Note also that, as $|f'(g)| = \frac{4m}{2n^2+4m+2n\sqrt[2]{n^2+4m}} < 1$ when $n > 0$, g_t converges, for any $g_0 > 0$, towards $\frac{n+\sqrt[2]{n^2+4m}}{2}$. This is not the case when $n = 0$, at which $|f'(g_t)| = 1$.

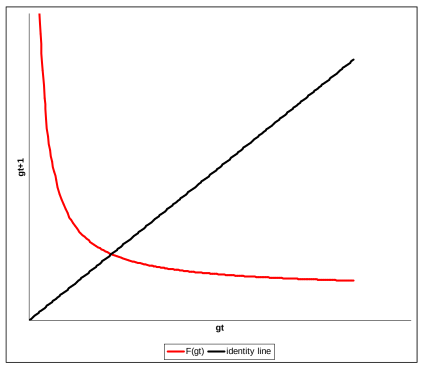

Figure 3: Dynamics of g_t

Therefore, under a *given* $TFR = n + m$, the population dynamics vary depending on the timing of births. Early and late births, although one-to-one substitutes in the TFR formula, are no one-to-one substitutes as far as population dynamics is concerned. That point is illustrated on Figure 4, which shows the dynamics of the number of births under $TFR = 1.15$, under two fertility profiles: $n = 1.15$, $m = 0$ and $n = 0$, $m = 1.15$.

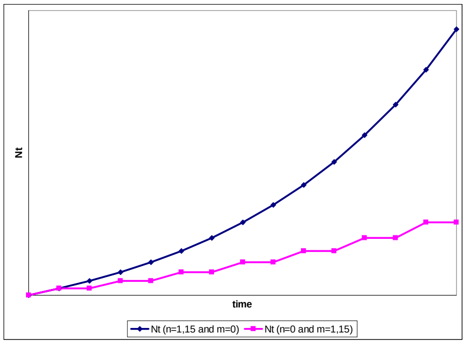

Figure 4: number of births under distinct fertility timing.

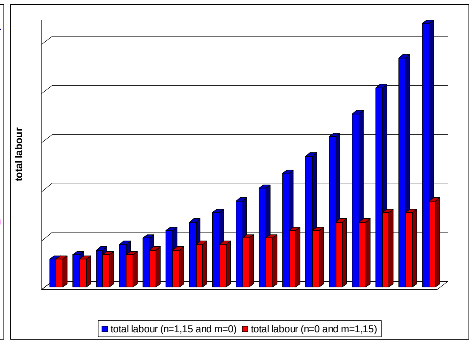

Figure 5: Total labour under distinct fertility timing.

The differences between the two patterns are twofold. First, as expected, the cohort size growth is larger when births take place earlier in the lifecycle. Second, whereas N_t grows constantly under the fertility profile $(1.15, 0)$, it exhibits a cyclical growth under the fertility profile $(0, 1.15)$.[9]

In order to examine the influence of birth timing on the economy, let us first describe the total labour force. Given that all agents supply their labour inelastically in their second and third periods of life, total labour force at t is:

$$L_t = N_{t-1} + N_{t-2} = g_{t-1}N_{t-2} + N_{t-2} \tag{3}$$

Dividing (3) by $L_{t-1} = N_{t-2} + N_{t-3}$ yields the labour growth factor:

$$\frac{L_t}{L_{t-1}} = \frac{g_{t-1}N_{t-2} + N_{t-2}}{N_{t-2} + N_{t-3}} = g_{t-2}\frac{1 + g_{t-1}}{1 + g_{t-2}} \tag{4}$$

If $n > 0$, g_t converges, in the long-run, towards $\frac{n+\sqrt[2]{n^2+4m}}{2}$, which is also the long-run labour growth factor. If $n = 0$, there is, in general, no convergence. Since $g_{t-2} \times g_{t-1} = m$, the labour force growth ratio is, in that case:

$$\frac{L_t}{L_{t-1}} = \frac{m(1 + g_{t-1})}{m + g_{t-1}} \tag{5}$$

thus varying over time, except in the special case of replacement fertility, i.e. $m = 1$, for which the labour supply is constant over time.

As shown on Figure 5, total labour can exhibit, under a given TFR, quite different patterns, depending on the timing of births. Labour grows at a constant rate when births are located in the early reproduction period, but grows at a fluctuating rate when births occur during the late reproduction period.

[9] Note that it is only in the special case where the TFR is at its replacement level (i.e. $n + m = 1$) that the two curves, then horizontal, coincide.

2.2 Production

The production of an output Y_t involves capital K_t and labour L_t, according to the function:[10]

$$Y_t = F(K_t, L_t) = \bar{F}(K_t, L_t) + (1-\delta)K_t \tag{6}$$

where δ is the depreciation rate of capital. The function $\bar{F}(K_t, L_t)$ is assumed to be homogeneous of degree one. Hence, the total production function $F(K_t, L_t)$ is also homogeneous of degree one, and the production can be written as:

$$y_t = F\left(k_t, 1 + \frac{N_{t-2}}{N_{t-1}}\right) \tag{7}$$

where $y_t = \frac{Y_t}{L_t^y} = \frac{Y_t}{N_{t-1}}$ denotes output per young worker, and $k_t = \frac{K_t}{L_t^y} = \frac{K_t}{N_{t-1}}$ denotes the capital per young worker.

The resource constraint of the economy, which states that what is produced is either consumed or invested, is:

$$F(K_t, L_t) = c_t N_{t-1} + d_t N_{t-2} + b_t N_{t-3} + K_{t+1}$$

where c_t, d_t and b_t are first-, second- and third-period consumptions.[11]

Dividing that constraint by the young labour force $L_t^y = N_{t-1}$, one gets:

$$F\left(k_t, 1 + \frac{1}{g_{t-1}}\right) = c_t + \frac{d_t}{g_{t-1}} + \frac{b_t}{g_{t-1}g_{t-2}} + k_{t+1}g_t \tag{8}$$

Finally, we assume that the economy is perfectly competitive, so that production factors are paid at their marginal productivity:

$$w_t = \left[F\left(k_t, 1 + \frac{1}{g_{t-1}}\right) - F_k\left(k_t, 1 + \frac{1}{g_{t-1}}\right)k_t\right]\frac{g_{t-1}}{1+g_{t-1}} \tag{9}$$

$$R_t = F_k\left(k_t, 1 + \frac{1}{g_{t-1}}\right) \tag{10}$$

where w_t denotes the wage rate, and R_t is the return on savings at period t.

2.3 Individual behavior

The problem of individuals can be written as:

$$\max_{c_t, d_{t+1}, b_{t+2}} u(c_t) + \beta u(d_{t+1}) + \beta^2 u(b_{t+2})$$

$$\text{s.t. } w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \frac{d_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1}R_{t+2}}$$

[10] It is assumed that the undepreciated units of capital are sold on the goods market.

[11] It is assumed that children live with their parents and share their consumption spending.

where the temporal utility function $u(\cdot)$ satisfies $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. The parameter β is a time preference factor ($0 < \beta < 1$).

Resolving that optimization problem allows us to derive savings in the second and third periods, denoted by s_t and z_{t+1}. Under perfect foresight, those optimal savings are functions of current and future factor prices:[12]

$$\begin{aligned} s_t &\equiv s\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right) \\ z_{t+1} &\equiv z\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right) \end{aligned}$$

3 Long-run dynamics

Backwarding the second savings equation by one period gives us z_t, i.e. the old worker's savings chosen at $t-1$. Then, substituting for s_t and z_t in $K_{t+1} = N_{t-1}s_t + N_{t-2}z_t$, and dividing by the number of young workers $L^y_{t+1} = N_t$, yields:

$$k_{t+1} = \frac{s\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right)}{g_t} + \frac{z\left(R_t, R_{t+1}, w_{t-1}, w_t\right)}{g_{t-1}g_t}$$

Given that $R_t = R(k_t)$ and $w_t = w(k_t)$, the dynamics of k_t is described by a difference equation of order 3.[13] As stressed by de la Croix and Michel (2002), the dynamics of capital under perfect foresight is quite complex when savings are made at several periods. There exist only a few ways to overcome that complexity. A first approach consists of imposing particular functional forms for utility and production functions. A second approach consists of keeping general functional forms, but of relaxing the perfect foresight assumption. If, for instance, one considers myopic anticipations, the number of time lags in the dynamic law of capital can be reduced.[14] In this section, we consider the myopic anticipations case. Then, in Section 5, we will consider rational expectations, under standard utility and production functions.

Under myopic anticipations on factor prices, the savings s_t and z_{t+1} can be rewritten by means of the following savings functions:

$$\begin{aligned} s_t &= s\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \sigma\left(k_t\right) \\ z_{t+1} &= z\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \zeta(k_t) \end{aligned}$$

Backwarding the second equation by one period and substituting for s_t and z_t in the capital accumulation equation yields: $k_{t+1} = \frac{\sigma(k_t)}{g_t} + \frac{\zeta(k_{t-1})}{g_{t-1}g_t}$. If one introduces the variable $\Omega_t \equiv \frac{\zeta(k_{t-1})}{g_{t-1}}$, the dynamics of the economy is summarized

[12] See de la Croix and Michel (2002, pp. 64-66).

[13] The highest-order term k_{t+2} comes from the interest factor at old adulthood for the young adult at t, i.e. R_{t+2}, whereas the lowest-order term k_{t-1} comes from the wage faced by old adults at t when being young workers at $t-1$, i.e. w_{t-1}.

[14] In our context, myopic anticipations mean that agents, when choosing their savings, take the *current* wages and interest rates as a proxy for future wages and interest rates.

by the following three-dimensional dynamic system:

$$
\begin{aligned}
k_{t+1} &\equiv G(k_t, \Omega_t, g_t) = \frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t} \\
\Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{g_t} \\
g_{t+1} &\equiv I(g_t) = n + \frac{m}{g_t}
\end{aligned}
\tag{A}
$$

The following proposition summarizes our results.

Proposition 1 *Assume myopic anticipations about factor prices. Assume that $\sigma(0) = 0$, $\sigma'(k_t) > 0$, $\zeta(0) = 0$ and $\zeta'(k_t) > 0$. Suppose $n + m > 0$. Denote $\sqrt[2]{n^2 + 4m}$ by Ψ.*

- *If $lim_{k\to 0}\ \frac{\Psi+n}{2}\left[1 - \frac{2\sigma'(k_t)}{n+\Psi}\right] < lim_{k\to 0}\ \frac{2\zeta'(k_t)}{n+\Psi}$ and if $lim_{k\to+\infty}\ \frac{\Psi+n}{2}\left[1 - \frac{2\sigma'(k_t)}{n+\Psi}\right] > lim_{k\to+\infty}\ \frac{2\zeta'(k_t)}{n+\Psi}$, there exists a stationary equilibrium with $k, \Omega, g > 0$.*
- *That stationary equilibrium is locally stable if and only if:*

  (i) $\frac{16m\zeta'(k)}{(n+\Psi)^4} < 1$

  (ii) $1 > \left[-\frac{4\zeta'(k)}{(n+\Psi)^2} - \frac{8m\sigma'(k)}{(n+\Psi)^3}\right] - \left[\frac{2\sigma'(k)}{n+\Psi} - \frac{4m}{(n+\Psi)^2}\right]\left[\frac{16m\zeta'(k)}{(n+\Psi)^4}\right] + \left[\frac{16m\zeta'(k)}{(n+\Psi)^4}\right]^2$

  (iii) $\left[\frac{4\zeta'(k)}{(n+\Psi)^2} + \frac{8m\sigma'(k)}{(n+\Psi)^3} - 1\right] < \frac{2\sigma'(k)}{n+\Psi} - \frac{4m}{(n+\Psi)^2} + \frac{16m\zeta'(k)}{(n+\Psi)^4} < \left[-\frac{4\zeta'(k)}{(n+\Psi)^2} - \frac{8m\sigma'(k)}{(n+\Psi)^3} + 1\right]$

Proof. See the Appendix. ■

The necessary and sufficient conditions for stability can be used to examine the impact of the timing of births on economic dynamics. For that purpose, Corollary 1 compares two economies differing on the timing of births.

Corollary 1 *Assume myopic anticipations about factor prices.*

- *Assume $n > 0$ and $m = 0$. Provided $\sigma(0) = 0$, $\zeta(0) = 0$, $lim_{k\to 0}\ n\left[1 - \frac{\sigma'(k_t)}{n}\right] < lim_{k\to 0}\ \frac{\zeta'(k_t)}{n}$ and $lim_{k\to+\infty}\ n\left[1 - \frac{\sigma'(k_t)}{n}\right] > lim_{k\to+\infty}\ \frac{\zeta'(k_t)}{n}$, there exists a stationary equilibrium with $k, \Omega, g > 0$. Provided $\frac{\zeta'(k)}{n^2} - 1 < \frac{\sigma'(k)}{n} < -\frac{\zeta'(k)}{n^2} + 1$, that equilibrium is locally stable.*
- *Assume $n = 0$ and $m > 0$. Provided $\sigma(0) = 0$, $\sigma'(k_t) > 0$, as well as $\zeta(0) = 0$, $\zeta'(k_t) > 0$, we have that, if $lim_{k\to 0}\ \sqrt[2]{m}\left[1 - \frac{\sigma'(k_t)}{\sqrt[2]{m}}\right] < lim_{k\to 0}\ \frac{\zeta'(k_t)}{\sqrt[2]{m}}$ and $lim_{k\to+\infty}\ \sqrt[2]{m}\left[1 - \frac{\sigma'(k_t)}{\sqrt[2]{m}}\right] > lim_{k\to+\infty}\ \frac{\zeta'(k_t)}{\sqrt[2]{m}}$, there exists a stationary equilibrium with $k, \Omega, g > 0$, which is unstable.*

Proof. See the Appendix. ■

Corollary 1 shows that the timing of births plays a crucial role in long-run economic dynamics. Whereas there exists, under mild conditions, a locally stable stationary equilibrium when all births take place early in life (i.e. $n > 0$ and $m = 0$), there exists no stable stationary equilibrium when all births are late births (i.e. $n = 0$ and $m > 0$).

The underlying intuition lies in the existence, under the latter fertility profile, of permanent fluctuations in labour growth caused by cycles in the cohort growth g_t (see Figure 5). Those permanent fluctuations in labour growth generate perpetual fluctuations in k_t, as well as in output and wages, and, hence, prevent the economy from converging towards a stationary equilibrium. As stated in Proposition 2, the economy with only late births is, under mild conditions, characterized by long-run cycles of period 2.

Proposition 2 *Assume myopic anticipations about factor prices. Suppose* $\sigma(0) = 0$, $\sigma'(k_t) > 0$, $\zeta(0) = 0$ *and* $\zeta'(k_t) > 0$.

Denote $\hat{D}(k_t) \equiv g_0 \left[\sigma^{-1}\left(\frac{m}{g_0}\left(k_t - \frac{\zeta(k_t)}{m}\right)\right) - \frac{\sigma(k_t)}{g_0}\right]$.

Denote $\check{D}(k_t) \equiv \frac{m}{g_0}\left[\sigma^{-1}\left(\left(k_t - \frac{\zeta(k_t)}{m}\right) g_0\right) - \frac{g_0\sigma(k_t)}{m}\right]$.

Assume that the equation $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0})}{g_0}$ *admits a non-negative solution and denote it by* $\Omega_t \equiv \hat{E}(k_t)$.

Assume that the equation $\Omega_t = \frac{g_0\zeta(\frac{g_0\sigma(k_t)}{m} + \frac{g_0\Omega_t}{m})}{m}$ *admits a non-negative solution and denote it by* $\Omega_t \equiv \check{E}(k_t)$.

- *If* $lim_{k\to\infty}\hat{D}(k_t) > lim_{k\to\infty}\hat{E}(k_t)$ *and* $lim_{k\to\infty}\check{D}(k_t) > lim_{k\to\infty}\check{E}(k_t)$, *the long-run dynamics is a two-period cycle* $(\hat{k}, \hat{\Omega}, g_0), (\check{k}, \check{\Omega}, \frac{m}{g_0})$.

- *Convergence to the cycle* $(\hat{k}, \hat{\Omega}, g_0), (\check{k}, \check{\Omega}, \frac{m}{g_0})$ *arises, iff:*

$$\left|\frac{\hat{Q}}{2} \pm \sqrt[2]{\frac{\hat{Q}^2 m g_0^2 - 4\zeta'(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0})\zeta'(\hat{k})}{4mg_0^2}}\right|, \left|\frac{\check{Q}}{2} \pm \sqrt[2]{\frac{\check{Q}^2 m^3 - 4\zeta'(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m})\zeta'(\check{k})}{4m^3}}\right| < 1,$$

$$\text{with } \hat{Q} \equiv \left[g_0^2\left(\sigma'\left(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0}\right)\sigma'(\hat{k}) + \zeta'(\hat{k})\right) + m\zeta'(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0})\right]/g_0^2 m$$

$$\check{Q} \equiv \left[m\sigma'\left(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m}\right)\sigma'\left(\check{k}\right) + m\zeta'(\check{k}) + \zeta'(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m})\right]/m^2.$$

Proof. See the Appendix. ■

In the light of this, the major role played by the timing of births could hardly be overemphasized. Whether births occur in the early or the late reproduction period makes a substantial difference. In the former case, the long-run dynamics is, in general, *stationary*, whereas, in the latter case, the dynamics is *cyclical*. Hence, even under an equal $TFR = n + m$, the two dynamics differ strongly, because of the distinct timings of births.

Finally, it is worth emphasizing a special case, where the economy will converge, in the long-run, towards a stationary equilibrium, despite all births being located during the late reproduction period.

Remark 1 *Assume myopic anticipations about factor prices. Assume* $N_{-1} = N_0 > 0$*,* $n = 0$ *and* $m = 1$*. If* $\sigma(0) = 0$*,* $\sigma'(k_t) > 0$*,* $\zeta(0) = 0$*,* $\zeta'(k_t) > 0$*,* $lim_{k \to 0}\, 1 - \sigma'(k_t) < lim_{k \to 0}\, \zeta'(k_t)$ *and* $lim_{k \to +\infty}\, 1 - \sigma'(k_t) > lim_{k \to +\infty}\, \zeta'(k_t)$*, there exists a locally stable stationary equilibrium with* $k, \Omega, g > 0$*.*

Proof. See the Appendix. ■

Thus, under particular initial conditions ($N_{-1} = N_0 > 0$), an economy with replacement fertility converges towards a stationary equilibrium despite all births being located in the second reproduction period. It should be stressed, however, that this convergence is achieved only because the initial level of the cohort growth factor g_0 takes, under those postulated initial conditions, its long-run equilibrium value (equal to unity). Therefore, in that special case, the question of the convergence of the cohort growth factor towards its long-run level is trivially solved. One can thus interpret the result presented in the above remark as a kind of "conditional" convergence result, where the influence of fertility timing is neutralized by initial conditions.

That special case, which involves specific initial conditions as well as a TFR equal to the replacement fertility level, does not question the general result obtained in this section (which does not presuppose specific initial conditions). The long-run dynamics of the economy - in particular the existence of a stable stationary equilibrium - is influenced by the timing of births, even for a given TFR. Therefore, if one only looks at the TFR, one misses a central aspect of the evolution of economies over time, since the nature of long-run dynamics - stationary or cyclical - depends on $n > 0$ or $n = 0$, whatever the TFR is.

4 Long-run social optimum

Let us now characterize the long-run social optimum in our economy. For that purpose, we will follow the approach pioneered by Samuelson (1975), who considers the problem faced by a social planner, who chooses the optimal levels of consumptions, capital and fertility, in such a way as to maximize the lifetime welfare of an agent living at the long-run equilibrium. The major difference with respect to Samuelson is that the social planning problem consists here of choosing not one, but two fertility rates: n and m.

4.1 The social planner's problem

Let us assume that there exists a unique stable stationary equilibrium in our economy.[15] Then, the social planner's problem can be written as follows:

$$
\begin{aligned}
& \max_{c,d,b,k,n,m} \; u(c) + \beta u(d) + \beta^2 u(b) \\
& \text{s.t. } F\left(k, \frac{n + \sqrt[2]{n^2+4m} + 2}{n + \sqrt[2]{n^2+4m}}\right) - k\frac{n + \sqrt[2]{n^2+4m}}{2} \\
= \; & c + d\frac{2}{n + \sqrt[2]{n^2+4m}} + b\left(\frac{2}{n + \sqrt[2]{n^2+4m}}\right)^2
\end{aligned}
$$

An interior optimum $(c^*, d^*, b^*, k^*, n^*, m^*)$ satisfies the following FOCs:

$$
\begin{aligned}
\frac{u'(c^*)}{\beta u'(d^*)} &= \frac{u'(d^*)}{\beta u'(b^*)} = \frac{n^* + \sqrt[2]{n^{*2}+4m^*}}{2} = g^* \\
F_k(k^*, \cdot) &= \frac{n^* + \sqrt[2]{n^{*2}+4m^*}}{2} = g^*
\end{aligned}
$$

The first expression implies that the MRS between consumptions at two successive periods is, at the optimum, equal to the optimal long-run cohort growth factor, $(n^* + \sqrt[2]{n^{*2}+4m^*})/2$. Thus, from the point of view of the optimal consumption profile, it is not the TFR $n^* + m^*$ that matters, but the cohort growth factor g^*, in which early and late births are no one-to-one substitutes.

The second expression is the Golden Rule: the optimal stock of capital per young worker k^* is such that the marginal productivity of capital is equal to the optimal cohort growth g^*. Here again, for a given total fertility rate $n^* + m^*$, the optimal capital will vary greatly with the optimal timing of births, since there is no one-to-one substitutability between early births n and late births m.

Regarding the FOCs for optimal n^* and m^*, we have:

$$
\begin{aligned}
& F_L(k^*, \cdot)\, 2\frac{1 + n^*\left(n^{*2}+4m^*\right)^{-1/2}}{\left(n^* + \sqrt[2]{n^{*2}+4m^*}\right)^2} + k^*\frac{1 + n^*\left(n^{*2}+4m^*\right)^{-1/2}}{2} \\
= \; & 2\frac{1 + n^*\left(n^{*2}+4m^*\right)^{-1/2}}{\left(n^* + \sqrt[2]{n^{*2}+4m^*}\right)^2}\left(d^* + \frac{4b^*}{n^* + \sqrt[2]{n^{*2}+4m^*}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& F_L(k^*, \cdot)\, 4\frac{\left(n^{*2}+4m^*\right)^{-1/2}}{\left(n^* + \sqrt[2]{n^{*2}+4m^*}\right)^2} + k^*\left(n^{*2}+4m^*\right)^{-1/2} \\
= \; & 4\frac{\left(n^{*2}+4m^*\right)^{-1/2}}{\left(n^* + \sqrt[2]{n^{*2}+4m^*}\right)^2}\left(d^* + \frac{4b^*}{n^* + \sqrt[2]{n^{*2}+4m^*}}\right)
\end{aligned}
$$

[15] We know from Section 3 that this assumption makes sense only if $n > 0$, as, under $n = 0$, the economy does not, in general, converge towards a stationary equilibrium. The assumption $n > 0$ is thus made throughout this section.

Those FOCs include the standard determinants of optimal fertility. On the LHS, we have the negative effects of fertility on the marginal productivity of labour (first term), as well as the capital widening effect (second term). That latter effect is known as the "Solow effect": a larger growth in the cohort size makes it more difficult to sustain a large capital per young worker level. On the RHS, we find the gains from intergenerational redistribution. This is the "Samuelson effect": a larger growth in the cohort size relaxes the resource constraint of the economy, by reducing the weight assigned to the old's consumption.

To interpret those FOCs, note that these can be written as:

$$\begin{aligned} g_n^* \left[\frac{-F_L(k^*, \cdot)}{g^{*2}} - k^* + \frac{d^*}{g^{*2}} + \frac{2b^*}{g^{*3}} \right] &= 0 \\ g_m^* \left[\frac{-F_L(k^*, \cdot)}{g^{*2}} - k^* + \frac{d^*}{g^{*2}} + \frac{2b^*}{g^{*3}} \right] &= 0 \end{aligned}$$

where $g_n^* = \frac{1}{2} + \frac{n^*}{2} \frac{1}{\sqrt[2]{n^{*2}+4m^*}} > 0$ and $g_m^* = \frac{1}{\sqrt[2]{n^{*2}+4m^*}} > 0$ denote the derivatives of the optimal cohort growth rate with respect to the optimal age-specific fertility rates n^* and m^* respectively. Focusing on the case of interior optimal age-specific fertility rates, i.e. $n^* > 0$, $m^* > 0$, it follows that g_n^* and g_m^* differ from 0. Therefore, the two above FOCs are satisfied if and only if:

$$k^* + \frac{F_L(k^*, \cdot)}{g^{*2}} = \frac{d^*}{g^{*2}} + \frac{2b^*}{g^{*3}} \tag{11}$$

The optimal cohort growth g^* is such that the marginal welfare loss from a higher cohort growth (the LHS) is equal to the marginal welfare gain from a higher cohort growth (the RHS). The negative welfare effects due to a higher cohort growth are the capital widening effect (1st term of the LHS) and the negative productivity effect (2nd term of the LHS), whereas the positive welfare effects are the intergenerational redistribution effects (1st and 2nd terms of the RHS). That condition for optimal cohort growth rate can be rewritten as:

$$g^{*3} + \frac{(F_L(k^*, \cdot) - d^*)}{k^*} g^* - \frac{2b^*}{k^*} = 0$$

In the Appendix, we solve that cubic equation, and derive the optimal cohort growth rate g^*. That variable determines both the optimal consumption paths and capital level k^*. Moreover, we know from above that, as long as g takes its optimum level g^*, the two FOCs characterizing the optimal age-specific fertility rates n^* and m^* are also satisfied. Hence the characterization of the social optimum requires, above all, a characterization of the optimal cohort growth rate g^*, rather than of the age-specific fertility rates, which affect optimal consumption paths and capital *only through* the optimal cohort growth rate.

Proposition 3 *Assume that there exists a unique stable stationary equilibrium.*

- *The long-run social optimum* $(c^*, d^*, b^*, k^*, n^*, m^*)$ *is such that:*

$$\frac{u'(c^*)}{\beta u'(d^*)} = \frac{u'(d^*)}{\beta u'(b^*)} = g^* = F_k\left(k^*, \cdot\right)$$
$$k^* + \frac{F_L\left(k^*, \cdot\right)}{g^{*2}} = \frac{d^*}{g^{*2}} + \frac{2b^*}{g^{*3}}$$

- *The optimal cohort growth* g^* *is characterized as follows:*

  - *if* $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} > 0,\ g^* = \sqrt[3]{\frac{\frac{2b^*}{k^*} + \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{2}} + \sqrt[3]{\frac{\frac{2b^*}{k^*} - \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{2}}$

  - *if* $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} = 0,\ g^* = \sqrt[3]{\frac{b^*}{k^*} - \sqrt[2]{\frac{b^{*2}}{k^{*2}} + \frac{\Phi^3}{27k^{*3}}}} - \frac{\Phi}{3k^* \sqrt[3]{\frac{b^*}{k^*} - \sqrt[2]{\frac{b^{*2}}{k^{*2}} + \frac{\Phi^3}{27k^{*3}}}}}$

  - *if* $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} < 0,\ g^* = \sqrt[3]{\frac{-\frac{2b^*}{k^*} - \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}} + \frac{2b^* i \sqrt[2]{3}}{k^*} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{4}} + \sqrt[3]{\frac{-\frac{2b^*}{k^*} + \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}} - \frac{2b^* i \sqrt[2]{3}}{k^*} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{4}}$

  where $\Phi \equiv \left(F_L\left(k^*, \cdot\right) - d^*\right)$.

Proof. See the Appendix. ■

The social optimum depends on age-specific fertility rates n^* and m^* *only insofar as* these yield the optimal cohort growth rate g^*. As long as n^* and m^* are such that $(n^* + \sqrt[2]{n^{*2} + 4m^*})/2$ is equal to g^*, the levels of n^* and m^* do not matter.[16] Hence there is not one, but *several* social optima $(c^*, d^*, b^*, k^*, n^*, m^*)$, since various pairs (n^*, m^*) yield the optimal cohort growth g^*.

Nevertheless, it should be stressed that there exists, from the perspective of long-run social welfare, no one-to-one substitutability between early and late births.[17] Hence, focusing only on the TFR is, here again, misleading. The importance of fertility timing is especially strong when the optimal cohort growth g^* is large, that is, when the intergenerational redistribution effect is large.

That point is illustrated on Figure 6, which shows iso-g lines, i.e. the set of (n, m) pairs such that $\frac{n + \sqrt[2]{n^2 + 4m}}{2} = g$. An economy whose g^* is equal to, for instance, 2, and which undergoes a fertility postponement from $n = 2$ to $n = 1$, can only sustain $g^* = 2$ provided m is raised from 0 to 2, implying a rise in TFR from 2 to 3. Thus the achievement of a high g^* imposes, in case of birth postponement, a strong rise in the total number of children.

In sum, an exclusive emphasis on the TFR $n+m$ is quite misleading from the point of view of long-run social welfare. There exists, in general, no one-to-one substitutability between early and late births, so that fertility timing matters for long-run social welfare.

[16] Remind, however, that the existence of a stable stationary equilibrium requires $n > 0$.

[17] The only exception is when $g = 1$. In that case, there is a one-to-one substiotuability between n and m, since the formula $(n + \sqrt[2]{n^2 + 4m})/2 = 1$ can be simplified to $n + m = 1$.

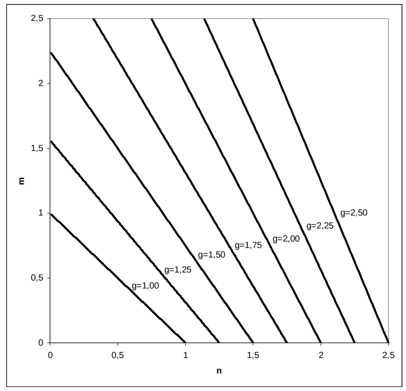

Figure 6: Iso-g lines.

4.2 The Serendipity Theorem

In his Serendipity Theorem, Samuelson (1975) showed that, if there exists a unique stable stationary equilibrium in a two-period OLG model with physical capital, then the perfectly competitive economy will converge towards the long-run social optimum provided the optimal fertility rate is imposed on individuals.

The purpose of this subsection is to examine whether the Serendipity Theorem remains valid in our environment. The question is the following. Assume that individuals behave like price-takers on competitive markets, and take fertility rates n and m as given. Is the economy going to converge towards the long-run social optimum when the optimal fertility is imposed?

The approach adopted by Samuelson is counterintuitive, since fertility is not, in the real world, taken "as given" by individuals, but is the outcome of parents' decisions.[18] However, we adopt Samuelson's approach, since it allows us to highlight the key role played by fertility - and fertility timing - in an intergenerational context, and, in particular, its capacity to allow for the decentralization of the social optimum in an otherwise decentralized economy.

To evaluate the robustness of Samuelson's result to the modelling of reproduction, let us first consider the problem faced by an agent living at the steady-state, who chooses second- and third-period savings, so as to maximize his lifetime welfare, while taking factor prices and fertility rates as given:

$$\max_{c,d,b} u(c) + \beta u(d) + \beta^2 u(b) \text{ s.t. } w + \frac{w}{R} = c + \frac{d}{R} + \frac{b}{R^2}$$

where $w = \left[F\left(k\left(,1+\frac{1}{g}\right)\right) - F_k\left(k,1+\frac{1}{g}\right)k\right]\frac{g}{1+g}$ and $R = F_k\left(k,1+\frac{1}{g}\right)$.

The FOCs are:

$$\frac{u'(c)}{\beta u'(d)} = \frac{u'(d)}{\beta u'(b)} = R$$

[18] See Barro and Becker (1989), and Ehrlich and Lui (1991).

Hence, if the planner fixes n and m such that $F_k(k^*, \cdot) = (n + \sqrt[2]{n^2 + 4m})/2 = g^*$, where k^* takes its socially optimal level, then individuals, being price-takers, will choose their savings optimally, since the above FOC then coincide with the FOC for optimal intergenerational allocations of resources.

Proposition 4 *Assume that there exists a unique stable stationary equilibrium. Then the perfectly competitive economy will converge towards the long-run social optimum provided the optimal cohort growth g^* is imposed. This amounts to impose fertility rates n and m such that $(n + \sqrt[2]{n^2 + 4m})/2 = g^*$.*

Proof. The proof follows from comparing the FOCs of the agent's problem and of the social planner's problem. ■

Proposition 4 shows that Samuelson's Serendipity Theorem is robust to the introduction of different ages of motherhood. Provided the social planner can impose optimal fertility, all other variables will, in a perfectly competitive economy, take their optimal values at the steady-state.

It is clear that fertility rates can hardly, in the real world, be "imposed" on individuals. That point, which could be formulated against Samuelson's initial framework, remains relevant here. However, in comparison to Samuelson's result, where the decentralization of the long-run social optimum was achieved by imposing the optimal fertility rate n (m being equal to 0), the government has here a larger degree of freedom, since it can play on *both* n and m. As shown above, the optimal cohort growth g^* can be achieved through various fertility profiles (n, m). Hence the government can decentralize the long-run social optimum through various fertility profiles. The only restriction is that there exists no one-to-one substitutability between early births (n) and late births (m), as shown on Figure 6. That limitation is especially strong when the optimal cohort growth factor g^* is large.

5 Extensions and robustness checks

5.1 Rational expectations

In Section 3, we showed that the long-run dynamics of the economy is, *ceteris paribus*, significantly affected by the timing of births when agents' anticipations about future production factor prices are myopic. In order to check the robustness of that result to the assumptions made regarding agent's anticipations, we solve here the dynamic system under rational expectations.[19] For that purpose, we impose particular functional forms for preferences and production. Assuming that $u(c) = log(c)$, the problem of the agent is:

$$\max_{c_t, d_{t+1}, b_{t+2}} log(c_t) + \beta log(d_{t+1}) + \beta^2 log(b_{t+2})$$
$$\text{s.t. } w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \frac{d_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1}R_{t+2}}$$

[19] That robustness check is necessary, since the precise distribution of market beliefs can have a significant impact on the form of economic dynamics (see Kurz 2011).

From the FOCs, we obtain the savings $s_t = \frac{(\beta+\beta^2)R_{t+1}w_t - w_{t+1}}{(1+\beta+\beta^2)R_{t+1}}$ and $z_{t+1} = \frac{\beta^2(w_{t+1}+R_{t+1}w_t)}{(1+\beta+\beta^2)}$. Substituting for s_t and z_t in the capital accumulation equation yields:

$$k_{t+1} = \frac{(\beta+\beta^2)R_{t+1}w_t - w_{t+1}}{g_t\left(1+\beta+\beta^2\right)R_{t+1}} + \frac{\beta^2 R_t w_{t-1} + \beta^2 w_t}{g_t g_{t-1}\left(1+\beta+\beta^2\right)} \tag{12}$$

Assuming that $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, substituting for $w_t = Ak_t^{\alpha}(1-\alpha)\left(\frac{g_{t-1}}{1+g_{t-1}}\right)^{\alpha}$ and $R_t = A\alpha k_t^{\alpha-1}\left(\frac{g_{t-1}}{1+g_{t-1}}\right)^{\alpha-1}$, and denoting $(1-\alpha)k_{t-1}^{\alpha}$ by X_t, we can, under $m > 0$, describe the dynamics of the economy by means of the following system:[20]

$$\begin{aligned}
k_{t+1} \quad &\equiv \quad G(k_t, X_t, g_t) = \frac{(\beta+\beta^2)Ak_t^{\alpha}\alpha(1-\alpha)\left(\frac{m}{g_t-n+m}\right)^{\alpha}(1+g_t)}{g_t\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+(1-\alpha)\right]} \\
&\quad + \frac{\beta^2A^2\alpha^2k_t^{\alpha-1}\left(\frac{m}{g_t-n+m}\right)^{\alpha-1}X_t\left(\frac{m(g_t-n)}{m-ng_t+n^2+m(g_t-n)}\right)^{\alpha}(1+g_t)}{\frac{g_t m}{g_t-n}\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+(1-\alpha)\right]} \\
&\quad + \frac{\beta^2A\alpha k_t^{\alpha}(1-\alpha)\left(\frac{m}{g_t-n+m}\right)^{\alpha}(1+g_t)}{\frac{g_t m}{g_t-n}\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+(1-\alpha)\right]} \\
X_{t+1} \quad &\equiv \quad H(k_t) = (1-\alpha)k_t^{\alpha} \\
g_{t+1} \quad &\equiv \quad I(g_t) = n + \frac{m}{g_t}
\end{aligned} \tag{B}$$

The following proposition summarizes our results.

Proposition 5 *Assume rational expectations about factor prices. Assume* $u(c) = log(c)$ *and* $F(K_t, L_t) = AK_t^{\alpha}L_t^{1-\alpha}$*, as well as* $m > 0$*.*

- *Provided* $lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)} > \frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)}$ *with* $g = \frac{n+\sqrt[2]{n^2+4m}}{2}$*, there exists a stationary equilibrium with* $k, X, g > 0$*.*
- *That stationary equilibrium is locally stable if and only if:*
  - *(i)* $\left|\frac{\Lambda\alpha m}{g^2}\right| < 1$
  - *(ii)* $1 > \frac{-\alpha m[1-A\Lambda]}{g^3} + \Lambda\left(\frac{Am(1-\alpha)}{g^2} - \alpha\right) - \left[\alpha - \Lambda A - \frac{m}{g^2}\right]\frac{\Lambda\alpha m}{g^2} + \left[\frac{\Lambda\alpha m}{g^2}\right]^2$
  - *(iii)* $\frac{m\alpha[1-\Lambda A]}{g^2} - \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) - 1 < \frac{\alpha g^2 - A\Lambda g^2 - m(1-\alpha\Lambda)}{g^2} < \frac{-m\alpha[1-\Lambda A]}{g^2} + \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) + 1$

[20] Under $m > 0$, we can rewrite $g_{t-1} = \frac{m}{g_t - n}$ and $g_{t-2} = \frac{m}{g_{t-1}-n}$, so that: $g_{t-2} = \frac{m(g_t-n)}{m-ng_t+n^2}$. Relaxing the assumption $m > 0$ to $m \geq 0$ and $n + m > 0$ would require a dynamic system whose dimension is strictly superior to 4, and, thus, hard to analyze using standard stability conditions on the Jacobian matrix.

$$\textit{where } \Lambda \equiv \frac{\beta^2 A^2 (1-\alpha)\alpha^2 k^{2\alpha-2}\left(\frac{m}{g-n+m}\right)^{\alpha-1}\left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha}(1+g)}{\frac{gm}{g-n}[(1+\beta+\beta^2)\alpha(1+g)+(1-\alpha)]}.$$

Proof. See the Appendix. ■

As in the myopic anticipations case, the conditions that are necessary and sufficient for the stability of a stationary equilibrium can be used to investigate the sensitivity of economic dynamics to the timing of births.

Corollary 2 *Assume rational expectations about factor prices. Assume* $u(c) = log(c)$ *and* $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$*. When* $n = 0$ *and* $m > 0$*, if* $\frac{(1-\alpha)\sqrt[2]{m}\left[\sqrt[2]{m}+\sqrt[2]{m}\beta-1\right]}{A\alpha\left(1+\sqrt[2]{m}\right)} < lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}m\left[\left(1+\beta+\beta^2\right)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+\sqrt[2]{m}}{\sqrt[2]{m}}\right)^{\alpha-1}\left(1+\sqrt[2]{m}\right)}$*, there exists a stationary equilibrium with* $k, X, g > 0$*. That equilibrium is not stable.*

Proof. See the Appendix. ■

In the light of Corollary 2, the robustness of our results to the assumptions made on expectations about factor prices can hardly be overemphasized. The cyclical dynamics prevailing under $n = 0$ is not due to particular assumptions about expectations, but is really due to the timing of births. Fluctuations in the cohort growth factor g_t lead to fluctuations in labour growth, and, also, in the capital per head and in the output per head. The sensitivity of economic dynamics to the timing of births is thus a general result, which is robust to assumptions made on anticipations about future factor prices.

5.2 Endogenous fertility

Let us now check the robustness of our results to another major assumption: the exogeneity of fertility and fertility timing. To answer that question, this section develops a simple model of lifecycle fertility choices, and examines the associated dynamics of the economy, as well as its long-run social optimum.[21]

Various motives were proposed to explain fertility choices, such as dynastic altruism (Barro and Becker 1989), or children as consumption and/or investment goods (Ehrlich and Lui 1991). For the sake of simplicity, we assume here that children are pure consumption goods.[22] Each early child has a cost $\theta > 0$, while each late child has a cost $\vartheta > 0$.[23] Thus the agent's problem becomes:

$$\max_{c_t, d_{t+1}, b_{t+2}, n_t, m_{t+1}} u(c_t) + \beta u(d_{t+1}) + \beta^2 u(b_{t+2}) + v(n_t) + \beta v(m_{t+1})$$

$$\text{s.t. } w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \theta n_t + \frac{d_{t+1} + \vartheta m_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1}R_{t+2}}$$

[21] For the sake of analytical simplicity, the analysis of dynamics relies here on the case of myopic anticipations about future factor prices.

[22] On the study of the optimal fertility timing in an OLG model with dynastic altruism *à la* Barro and Becker, see Pestieau and Ponthiere (2012).

[23] See also Pestieau and Ponthiere (2012) on another modelling, with time costs of children.

where $v(\cdot)$ is the welfare from having children, with $v'(\cdot) > 0$, $v''(\cdot) < 0$. FOCs yield:

$$\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1}, \quad \frac{u'(d_{t+1})}{u'(b_{t+2})} = \beta R_{t+2}, \quad \frac{v'(n_t)}{v'(m_{t+1})} = \beta R_{t+1}\frac{\theta}{\vartheta} \tag{13}$$

The last equation characterizes the trade-off faced by parents between early and late births. Impatience favours early births, while the interest rate favours late births. Moreover, the cost differential between the two types of children also affects the optimal fertility timing chosen by individuals.

Factor prices along the lifecycle (w_t, w_{t+1}, R_{t+1}, R_{t+2}) determine individual savings and fertility choices. As in the baseline model, we solve here the dynamics under myopic anticipations about factor prices. Under that postulate, we can rewrite second-period savings s_t and third-period savings z_{t+1}, as well as early and late fertility rates n_t and m_{t+1}, as functions of k_t: $s_t = \sigma(k_t)$, $z_{t+1} = \zeta(k_t)$, $n_t = \eta(k_t)$ and $m_{t+1} = \mu(k_t)$, where the form of $\sigma(\cdot)$, $\zeta(\cdot)$, $\eta(\cdot)$ and $\mu(\cdot)$ depends on the $u(\cdot)$, $v(\cdot)$, β, θ and ϑ.

5.2.1 Long-run dynamics

Substituting for s_t and z_t in the capital accumulation equation $K_{t+1} = N_{t-1}s_t + N_{t-2}z_t$, and substituting for n_t and m_t in $g_{t+1} = n_t + \frac{m_t}{g_t}$ allows us to describe the dynamics of the economy by means of the following system:

$$\begin{aligned} k_{t+1} &\equiv \hat{G}(k_t, \Omega_t, g_t) = \frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t} \\ \Omega_{t+1} &\equiv \hat{H}(k_t) = \frac{\zeta(k_t)}{g_t} \\ g_{t+1} &\equiv \hat{I}(k_t, \Omega_t, g_t) = \eta(G(k_t, \Omega_t, g_t)) + \frac{\mu(k_t)}{g_t} \end{aligned} \tag{C}$$

Proposition 6 *Denote* $\Xi_t \equiv \eta(G(k_t, \Omega_t, g_t) + \sqrt[2]{[\eta(G(k_t, \Omega_t, g_t))]^2 + 4\mu(k_t)}$.

Denote $\Pi_t \equiv \frac{2[\eta(G(k_t,\Omega_t,g_t))]\eta'(G(k_t,\Omega_t,g_t)G'(\cdot)+4'\mu(k_t)}{2[\Xi_t - \eta(G(k_t,\Omega_t,g_t)]}$.

- *Assuming that:*

  $lim_{k\to 0}\frac{\Xi_t}{2} + \frac{k_t\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot)+\Pi_t\right]}{2} - \sigma'(k_t) < lim_{k\to 0}\frac{2\zeta'(k_t)[\Xi_t] - 2\zeta(k_t)\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot)+\Pi_t\right]}{\Xi_t^2}$,

  $lim_{k\to +\infty}\frac{\Xi_t}{2} + \frac{k_t\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot)+\Pi_t\right]}{2} - \sigma'(k_t) > lim_{k\to +\infty}\frac{2\zeta'(k_t)[\Xi_t] - 2\zeta(k_t)\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot)+\Pi_t\right]}{\Xi_t^2}$

  there exists a stationary equilibrium with $k, \Omega, g > 0$.

- *That equilibrium is locally stable if and only if:*

  (i) $\left|\frac{\zeta'(k)\mu(k) - \zeta(k)\mu'(k)}{g^4}\right| < 1$;

  (ii) $1 - \frac{\zeta(k)\eta'(k)}{g^3} + \frac{\sigma'(k)\mu(k)}{g^3} - \frac{\mu'(k)[\sigma(k)+\Omega]}{g^3} + \frac{\zeta'(k)}{g^2}$

  $> -\left[\frac{\zeta'(k)\mu(k) - \zeta(k)\mu'(k)}{g^4}\right]\left[\frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{g^2}\right] - \frac{\mu(k)}{g^2}\right] + \left[\frac{\zeta'(k)\mu(k) - \zeta(k)\mu'(k)}{g^4}\right]^2$

(iii) $-\left[\frac{\zeta(k)\eta'(k)+\left[-\sigma'(k)\mu(k)+\mu'(k)[\sigma(k)+\Omega]\right]-\zeta'(k)g}{g^3}+1\right]$

$< \frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{g^2}\right] - \frac{\mu(k)}{g^2} + \frac{\zeta'(k)\mu(k)-\zeta(k_t)\mu'(k)}{g^4}$

$< \left[\frac{\zeta(k)\eta'(k)+\left[-\sigma'(k)\mu(k)+\mu'(k)[\sigma(k)+\Omega]\right]-\zeta'(k)g}{g^3}+1\right].$

Proof. See the Appendix. ■

As in the case with exogenous fertility, the conditions stated here can be used to examine whether the long-run dynamics of the economy is sensitive to the fertility timing. In order to examine how endogenous fertility affects our results, we focus here on the stability conditions when $\eta = 0$ at the equilibrium.

Corollary 3 *Assume that there exists a stationary equilibrium with* $\eta(k) = 0$ *and* $\mu(k) > 0$. *That equilibrium is locally stable if and only if:*

(i) $\left|\frac{\zeta'(k)\mu(k)-\zeta(k)\mu'(k)}{g^4}\right| < 1$;

(ii) $1 - \frac{\zeta(k)\eta'(k)}{g^3} + \frac{\sigma'(k)\mu(k)}{g^3} - \frac{\mu'(k)[\sigma(k)+\Omega]}{g^3} + \frac{\zeta'(k)}{g^2}$

$> -\left[\frac{\zeta'(k)\mu(k)-\zeta(k)\mu'(k)}{g^4}\right]\left[\frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{g^2}\right] - \frac{\mu(k)}{g^2}\right] + \left[\frac{\zeta'(k)\mu(k)-\zeta(k)\mu'(k)}{g^4}\right]^2$

(iii) $\frac{\sigma'(k)}{g} - \frac{\zeta(k)\eta'(k)}{g^3} - \frac{\mu'(k)\sigma(k)}{g^3} - \frac{\mu'(k)\zeta(k)}{g^4} + \frac{\zeta'(k)}{g^2} - 1$

$< \frac{\sigma'(k)}{g} - \frac{\zeta(k)\eta'(k)}{g^3} - \frac{\eta'(k)\sigma(k)}{g^2} - \frac{\mu'(k)\zeta(k)}{g^4} + \frac{\zeta'(k)}{g^2} - 1$

$< -\frac{\sigma'(k)}{g} + \frac{\zeta(k)\eta'(k)}{g^3} + \frac{\mu'(k)\sigma(k)}{g^3} + \frac{\mu'(k)\zeta(k)}{g^4} - \frac{\zeta'(k)}{g^2} + 1.$

Proof. Those conditions are obtained by fixing $\eta(k) = 0$ in the stability conditions of Proposition 6. ■

Ceteris paribus, condition (i) is, in comparison to the case where fertility is exogenous (i.e. $\mu'(k) = 0$), now strengthened (resp. weakend) when $\mu'(k) < 0$ (resp. $\mu'(k) > 0$). Regarding condition (ii), the effect of endogenous fertility is ambiguous: whether stability is more likely under endogenous fertility than under exogenous fertility depends on the level of $\eta'(k)$ and $\mu'(k)$. Finally, from condition (iii), it is straightforward to see that, under exogenous fertility (i.e. $\eta'(k) = \mu'(k) = 0$), stability never holds when $\eta = 0$. However, once fertility is endogenous, stability may prevail, depending on the relative levels of $\eta'(k)$ and $\mu'(k)$. When parents never have children early in their life (i.e. $\eta'(k) = 0$), condition (iii) can only be satisfied if $\frac{\mu'(k)\sigma(k)}{g^3} > 0$. Thus, when $\eta(k) = 0$ and $\eta'(k) = 0$, stability requires that late fertility grows when k increases.

In sum, introducing endogenous fertility has ambiguous effects on the necessary and sufficient conditions for the stability of a stationary equilibrium. There exists, under particular circumstances, a locally stable stationary equilibrium with all births being located during the second reproduction period. But fertility timing has still a major impact on the stability of the stationary equilibrium.

5.2.2 The long-run social optimum

The problem of the social planner can be written as:

$$\max_{c,d,b,k,n,m} u(c)+v(n)+\beta u(d)+\beta v(m)+\beta^2 u(b)$$
$$\text{s.t. } F\left(k,1+\frac{1}{g}\right)-kg=c+\theta n+\frac{d}{g}+\frac{\vartheta m}{g}+\frac{b}{g^2}$$

where $g=\frac{n+\sqrt[2]{n^2+4m}}{2}$. FOCs yield:

$$\frac{u'(c^*)}{\beta u'(d^*)}=\frac{u'(d^*)}{\beta u'(b^*)}=g^*=\frac{n^*+\sqrt[2]{n^{*2}+4m^*}}{2} \tag{14}$$

$$F_k\left(k^*,\cdot\right)=g^*=\frac{n^*+\sqrt[2]{n^{*2}+4m^*}}{2} \tag{15}$$

$$g_n^*\left(F_L\left(\cdot\right)\frac{1}{g^{*2}}+k^*\right)+\theta=\frac{v'(n^*)}{u'(c^*)}+g_n^*\left[\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}+\frac{\vartheta m^*}{g^{*2}}\right] \tag{16}$$

$$g_m^*\left(F_L\left(\cdot\right)\frac{1}{g^{*2}}+k^*\right)+\frac{\vartheta}{g^*}=\frac{\beta v'(m^*)}{u'(c^*)}+g_m^*\left[\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}+\frac{\vartheta m^*}{g^{*2}}\right] \tag{17}$$

As in the baseline model, the optimal consumption paths and capital are determined by the optimal long-run cohort growth factor g^*. But a major difference with respect to the baseline model is that the optimal age-specific fertility rates are here fully determined, since the FOCs for n^* and m^* cannot here be reduced to a condition characterizing g^*. That difference comes from the fact that parents care here directly about fertility, unlike in the baseline model, where fertility was valued only through consumption possibilities. One corollary of this is that, in this extended model, a government imposing the optimal fertility rates n^* and m^* will not make the competitive economy converge towards the long-run social optimum. Hence the Serendipity Theorem no longer holds here.

The optimal fertility profile depends on the form of preferences, and on the production process. To illustrate this, let us assume that $u(c)=log(c)$ and $v(n)=\varphi log(n)$, where $\varphi>0$ captures the parental taste for children, and that the production function is a Cobb-Douglas: $Y_t=AK_t^{\alpha}L_t^{1-\alpha}$. Various calibrations of parameters α, A, β, φ, θ and ϑ can rationalize the observed fertility profile (U.S.), where about 4/5th of births occur in the early reproduction period. Figures 7 and 8 show the long-run lifetime welfare under two distinct calibrations compatible with the observed fertility profile.

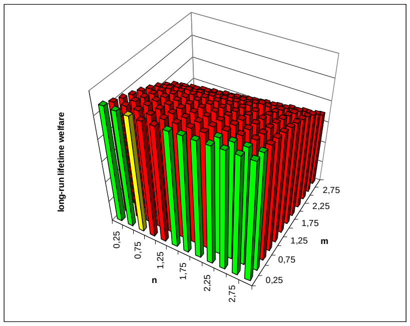

Fig. 7: $\beta = 0.80$, $\theta = 0.18$, $\vartheta = 2.10$.

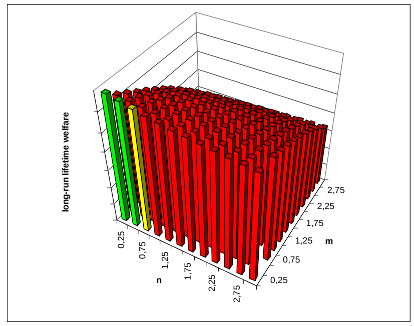

Fig. 8: $\beta = 0.60$, $\theta = 0.22$, $\vartheta = 1.95$.

Figure 7 shows that the existing fertility profile (in light color) is, from the perspective of lifetime welfare maximization at the long-run equilibrium, dominated by various alternative fertility profiles. Some dominating profiles include fewer early children ($n < 0.75$, same $m = 0.25$), whereas others include many more early children (i.e. $n \geq 1.5$), and, above $n = 2$, more late children ($m = 0.5 > 0.25$). That result comes from the existence of a local minimum at $n = 1$.[24] On the contrary, under the alternative parametrization with cheaper late children but larger impatience (Figure 8), the optimal fertility profile takes a quite distinct form: it involves the minimum number of early and late children ($n = 0.25, m = 0.25$). The reason is that, because of the larger impatience, the intergenerational redistribution effect is dominated by the capital dilution effect, which supports low fertility.

In sum, the optimal fertility profile depends strongly on the postulated preferences, since these determine the relative strengths of the capital dilution effect (supporting low fertility) and the intergenerational redistribution effect (supporting high fertility). But in any case, the postponement of births is not welfare-improving. Indeed, if a high cohort size growth is socially beneficial, it makes sense to concentrate births in the early in life, to obtain a large g. If, on the contrary, cohort growth is a bad thing for long-run welfare, then minimum fertility is the best, and postulating births does not help more.[25]

5.3 Three reproduction periods

The (im)possibility of asymptotic convergence of the age structure of an economy is significantly sensitive to how reproduction behavior is modelized. In a continuous time model, Lotka (1939) showed that, under a fixed vector of age-specific fertility rates and mortality rates, the age-structure of a closed economy

[24] The existence of an interior fertility rate that minimizes long-run lifetime welfare under logarithmic utility and Cobb-Douglas technology was studied by Deardorff (1976) and Michel and Pestieau (1993) in 2-period OLG models. The global optimum is achieved at $n = 4.75$, $m = 0.25$.

[25] Indeed, $n = 0$ would, by generating cyclical dynamics, lead to significant welfare losses.

will necessary converge towards a constant age structure in the long-run, which is independent from the initial one. In his attempt to derive the equivalent of Lotka Theorem in a discrete time model, MacFarland (1969) assumed that each cohort has descendants in at least two different cohorts.[26] That condition obviously rules out the case where $n = 0$ and $m > 0$.

In the light of this, it makes sense to explore the robustness of our results to the precise modelling of fertility. For that purpose, let us now extend our model to a 5-period OLG model with three reproduction periods (instead of two).[27] Agents have n children in period 2, m children in period 3, and o children in period 4.[28] The total number of births at t is:

$$N_t = N_{t-1}n + N_{t-2}m + N_{t-3}o \tag{18}$$

Dividing this by N_{t-1}, we obtain:

$$g_t = n + \frac{m}{g_{t-1}} + \frac{o}{g_{t-1}g_{t-2}} \tag{19}$$

Defining $\ell_{t+1} \equiv \frac{o}{g_t}$ allows us to study population dynamics through the system:

$$\begin{aligned} g_{t+1} &= n + \frac{m}{g_t} + \frac{\ell_t}{g_t} \\ \ell_{t+1} &= \frac{o}{g_t} \end{aligned} \tag{D}$$

Proposition 7 summarizes our results.

Proposition 7 *Assume three reproduction periods.*

- *If* $n > 0$*, there exists a stable equilibrium* g*.*
- *If* $n = 0$*,* $m = 0$*,* $o > 0$*, there exists no stable equilibrium* g*.*
- *If* $n = 0$*,* $m > 0$*,* $o = 0$*, there exists no stable equilibrium* g*.*
- *If* $n = 0$*,* $m > 0$*,* $o > 0$*, there exists a stable equilibrium* g*.*

Proof. See the Appendix. ■

The timing of births is, here again, not neutral for population dynamics. There exists no stable g under $n = 0$ when either $o = 0$ or $m = 0$. It is only when *both* $m > 0$ and $o > 0$ that asymptotic convergence towards some g is achieved. The existence of two strictly positive age-specific fertility rates is thus a necessary condition for the asymptotic convergence of g_t when $n = 0$. Note,

[26] See MacFarland (1969), Postulate 2, p. 305.

[27] The analysis focuses here exclusively on population dynamics, that is, on the identification of the formal conditions under which the age structure of the economy can stabilize in the long-run. The 'economic' side of the analysis is not explored in that section, since the dimension of the associated total eco-demographic dynamic system is too high (i.e. > 4).

[28] To insure a positive number of births at any time period, we assume initial conditions $N_{-2}, N_{-1}, N_0 > 0$.

however, that the results of Proposition 7 tend to reduce the importance of the first period of reproduction: when $m > 0$ and $o > 0$, there exists a stable equilibrium cohort growth factor despite $n = 0$. Hence, within that broader framework, it appears that the number of periods with strictly positive fertility seems to be more important than the timing of fertility.

It should also be noted that, even though the asymptotic convergence of g_t is achieved when both $m > 0$ and $o > 0$, the duration of the convergence can vary strongly, depending on the particular fertility timing. To illustrate this, Figure 9 compares the dynamics of g_t in two cases with at least two strictly positive age-specific fertility rates, with, in each case, a TFR equal to 1.05.

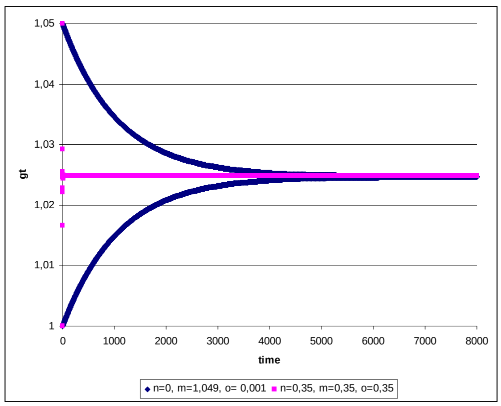

Figure 9: Convergence of g_t.

When the fertility profile is smoothed, i.e. $n = m = o = 0.35$, the convergence towards the equilibrium cohort growth factor is achieved after 16 periods, whereas, under $n = 0$, $m = 1.049$ and $o = 0.001$, the convergence is only achieved after 7,500 periods.[29] Thus the timing of births remains important here, despite the existence of asymptotic age structure convergence.

Because of space constraints, we do not provide here the resolution of the complete eco-demographic model with three reproduction periods.[30] But in the light of what was shown in the two-period case, there is little doubt that cohort growth fluctuations imply labour growth fluctuations, which prevent the convergence of the economy towards a stationary equilibrium. Hence, the observed sensitivity of economic dynamics to the timing of births for a given TFR is thus robust to the number of reproduction periods.

[29] Accuracy requirement: 4 decimal digits.

[30] The reason is that, as stated above, the associated dynamic system would have a size superior to 4.

6 Conclusions

Birth postponement is a key stylized fact of the last four decades. In this paper, we proposed to examine the consequences of that demographic trend on long-run economics dynamics, as well as the influence of fertility timing on the long-run social optimum. For that purpose, we developed a 4-period OLG model with physical capital, whose specificity is to include not one, but two reproduction periods. We firstly focused on an economy with exogenous age-specific fertility rates, and relaxed that assumption later on.

The study of the long-run dynamics in that economy revealed that the timing of births is not neutral at all for economic dynamics. Even for a given total fertility rate, the economy can exhibit quite different (stationary or cyclical) dynamics, depending on the location of births along the lifecycle. In particular, the economy exhibits long-run fluctuations when all births occur the late reproduction period.[31] The reason is that, under that fertility profile, labour force growth exhibits perpetual fluctuations. The only case where fertility timing does not affect the economy is under replacement fertility.

We also characterized the long-run utilitarian social optimum, and showed that the optimal consumptions and capital are determined by the optimal long-run cohort growth factor, in which there is no one-to-one substitutability between early and late births. We also showed that, when fertility does not directly affect parent's welfare, the only demographic variable characterizing the social optimum is the long-run cohort growth factor, whose level depends on how large the capital dilution effect is in comparison to the intergenerational redistribution effect. We also derived an extended version of Samuelson's Serendipity Theorem. Finally, the sensitivity of economic dynamics to the fertility timing was shown to be robust to the introduction of rational expectations, of endogenous fertility, and to the addition of a third reproduction period.

While those results highlight the importance of birth timing for the understanding of long-run economic dynamics, it should be stressed here that the present framework suffers from some simplifying assumptions, which invite further research. In the present model, the labour supply of each agent is inelastic, and does not vary with fertility and fertility timing. But in the real world, the labour supply of agents may be directly related to fertility behavior. In particular, the mere participation to the labour market may vary with fertility (e.g. female labour participation). Another important aspect that was not taken into account here consists of education. Undoubtedly, having children early in life can prevent higher education, with a negative effect on wages during the career. Moreover, our paper did not study the potential interactions between existing pensions systems, wealth accumulation and fertility timing.[32] Those additional

[31] Hence the present paper emphasizes that fertility behavior - in particular the timing of births - can be at the origin of economic fluctuations. Note that cycles can also emerge from the other end of the demographic chain (deaths), as recently shown by Goenka and Liu (2012).

[32] On wealth accumulation and PAYG pensions system in a dynamic OLG model under exogenous population growth, see Pestieau and Thibault (2012), where the population is heterogeneous in terms of preferences (altruism and taste for wealth).

links between fertility, fertility timing and the economy would also be worth being studied within a broader theoretical framework.

In sum, although often neglected by economists - who paid more attention to the number of births -, the timing of births is a major determinant of the evolution of economies over long periods of time. Moreover, given that there is, from the perspective of long-run social welfare, no one-to-one substitutability between early and late births, fertility timing also matters from a normative point of view. Focusing on the total number of births - i.e. the TFR - is thus a major simplification, whatever one is concerned with the study of long-run dynamics or with the characterization of the long-run social optimum.

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9 Appendix

9.1 Proof of Proposition 1

Existence of a stationary equilibrium From the first equation of system A, we can define the kk locus, along which k_t is constant. Imposing $k_{t+1} = k_t$ yields: $\Omega_t = g_t \left(k_t - \frac{\sigma(k_t)}{g_t} \right)$. From the second equation of system A, we can define the $\Omega\Omega$ locus: $\Omega_t = \frac{\zeta(k_t)}{g_t}$. Moreover, setting $g_{t+1} = g_t$ in the third equation of system A, we obtain $g_t = \frac{n+\sqrt[2]{n^2+4m}}{2}$, so that the gg locus is a horizontal plan in the (k_t, Ω_t, g_t) space, at a level $g_t = g^* = \frac{n+\sqrt[2]{n^2+4m}}{2}$.

Let us now study under which conditions the kk locus and the $\Omega\Omega$ locus intersect with each others at the cohort growth rate $g^* = \frac{n+\sqrt[2]{n^2+4m}}{2}$. The kk locus can, at that cohort growth rate, be rewritten as $\Omega_t = \left(\frac{n+\sqrt[2]{n^2+4m}}{2}\right)\left[k_t - \frac{2\sigma(k_t)}{n+\sqrt[2]{n^2+4m}}\right]$. The $\Omega\Omega$ locus can be rewritten as: $\Omega_t = \frac{2\zeta(k_t)}{n+\sqrt[2]{n^2+4m}}$. Note that, as $\sigma(0) = 0$ and $\zeta(0) = 0$, the two loci intersect at $k_t = 0$. Moreover, assuming that $lim_{k\to 0}\left(\frac{n+\sqrt[2]{n^2+4m}}{2}\right)\left[1 - \frac{2\sigma'(k_t)}{n+\sqrt[2]{n^2+4m}}\right] < lim_{k\to 0}\frac{2\zeta'(k_t)}{n+\sqrt[2]{n^2+4m}}$ and

$lim_{k\to +\infty}\left(\frac{n+\sqrt[2]{n^2+4m}}{2}\right)\left[1 - \frac{2\sigma'(k_t)}{n+\sqrt[2]{n^2+4m}}\right] > lim_{k\to +\infty}\frac{2\zeta'(k_t)}{n+\sqrt[2]{n^2+4m}}$, it follows that the kk locus lies below the $\Omega\Omega$ locus for low k_t levels, but lies above it for high k_t levels. Hence, by continuity, the kk and $\Omega\Omega$ loci must intersect along the gg locus. That intersection is a stationary equilibrium (k, Ω, g).

Stability of a stationary equilibrium The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial G(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial g_t} \\ \frac{\partial H(k_t)}{\partial k_t} & \frac{\partial H(k_t)}{\partial \Omega_t} & \frac{\partial H(k_t)}{\partial g_t} \\ \frac{\partial I(g_t)}{\partial k_t} & \frac{\partial I(g_t)}{\partial \Omega_t} & \frac{\partial I(g_t)}{\partial g_t} \end{pmatrix}$$

Estimating the entries of that matrix at the equilibrium, we obtain that the determinant and the trace are: $\det(J) = \frac{m\zeta'(k)}{g^4} \geq 0$ and $\mathrm{tr}(J) = \frac{\sigma'(k)}{g} - \frac{m}{g^2} \gtrless 0$.

Following Brooks's (2004) study of stability of first-order three-dimensional dynamic systems, we know that all eigenvalues of a 3x3 Jacobian matrix are lower than 1 in modulo (implying stability) if and only if the following three conditions are satisfied:

(i) $|\det(J)| < 1$

(ii) $1 > \left[\sum M_i(J)\right] - [tr(J)]\left[\det(J)\right] + \left[\det(J)\right]^2$

(iii) $-\left[\sum M_i(J) + 1\right] < tr(J) + \det(J) < \left[\sum M_i(J) + 1\right]$

where $\det(J)$, $tr(J)$ and $\sum M_i(J)$ denote respectively the determinant, the trace and the sum of the principal minors of the Jacobian matrix.

Brooks's (2004) conditions (i) is satisfied if and only if: $\frac{16m\zeta'(k)}{\left(n+\sqrt[2]{n^2+4m}\right)^4} < 1$.

Condition (ii) amounts to:

$1 > \left[-\frac{\zeta'(k)}{g^2} - \frac{m\sigma'(k)}{g^3}\right] - \left[\frac{\sigma'(k)}{g} - \frac{m}{g^2}\right]\left[\frac{m\zeta'(k)}{g^4}\right] + \left[\frac{m\zeta'(k)}{g^4}\right]^2$

Condition (iii) amounts to:

$-\left[-\frac{\zeta'(k)}{g^2} - \frac{m\sigma'(k)}{g^3} + 1\right] < \frac{\sigma'(k)}{g} - \frac{m}{g^2} + \frac{m\zeta'(k)}{g^4} < \left[-\frac{\zeta'(k)}{g^2} - \frac{m\sigma'(k)}{g^3} + 1\right]$

9.2 Proof of Corollary 1

The first part of Corollary 1 follows from imposing the restrictions $n > 0$ and $m = 0$ in the baseline model. When $m = 0$, the cohort growth factor is constant and equal to n, and the dynamic system becomes two-dimensional.

Regarding the existence of a stationary equilibrium, the conditions vanish to: $lim_{k\to 0}\left[1 - \frac{\sigma'(k_t)}{n}\right] < lim_{k\to 0}\frac{\zeta'(k_t)}{n^2}$ and $lim_{k\to +\infty}\left[1 - \frac{\sigma'(k_t)}{n}\right] > lim_{k\to +\infty}\frac{\zeta'(k_t)}{n^2}$. Those conditions guarantee, by continuity, that the kk locus intersect the $\Omega\Omega$ locus from below.

Regarding stability, condition (i) is satisfied when $m = 0$, since: $\frac{\zeta'(k)}{(n)^4} \times 0 < 1$. Condition (ii) amounts to: $1 > -\frac{\zeta'(k)}{n^2}$, which is also satisfied. Finally, condition (iii) amounts to: $-\left[-\frac{\zeta'(k)}{n^2} + 1\right] < \frac{\sigma'(k)}{n} < \left[-\frac{\zeta'(k)}{n^2} + 1\right]$.

Regarding the second part of Corollary 1, note first that, when $n = 0$, the gg locus becomes: $g_t = \frac{m}{g_t}$, so that $g = \sqrt[2]{m}$. The existence part is thus obtained by fixing $n = 0$ and $g = \sqrt[2]{m}$ in the existence conditions from Proposition 1. The stability part can be proved as follows. When $n = 0$, Brooks's condition (iii) is: $-\left[-\frac{\zeta'(k)}{m} - \frac{\sigma'(k)}{g} + 1\right] < \frac{\sigma'(k)}{g} - 1 + \frac{\zeta'(k)}{m} < \left[-\frac{\zeta'(k)}{m} - \frac{\sigma'(k)}{m^{1/2}} + 1\right]$. The first inequality is violated. Therefore there exists no stable stationary equilibrium.

9.3 Proof of Proposition 2

Existence of cycles To study the conditions under which a stable cycle arises, let us rewrite the variables as a function of their lagged past values. This gives us the following dynamic system:

$$\begin{aligned}
k_{t+2} &\equiv G(k_{t+1}, \Omega_{t+1}, g_{t+1}) = g_t \frac{\sigma\left(\frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t}\right)}{m} + \frac{\zeta(k_t)}{m} \equiv \Gamma\left(k_t, \Omega_t, g_t\right) \\
\Omega_{t+2} &\equiv H(k_{t+1}) = \frac{\zeta(\frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t})}{g_t} \equiv \Theta\left(k_t, \Omega_t, g_t\right) \\
g_{t+2} &\equiv I(g_{t+1}) = \frac{m}{g_{t+1}} = g_t \equiv \Lambda\left(g_t\right) \qquad \text{(A')}
\end{aligned}$$

Given that $g_{t+2} = g_t$ for all t, it is easy to see that g_t fluctuates between two levels, given by $g_0 = \frac{N_0}{N_{-1}}$ and $\frac{m}{g_0} = g_1$. Indeed, $g_2 = \frac{m}{g_1} = g_0$ and $g_3 = \frac{m}{g_2} = g_1 = \frac{m}{g_0}$. Hence, the gg locus takes, given $g_0 = \frac{N_0}{N_{-1}}$, the form of two horizontal planes in the (k_t, Ω_t, g_t) space, at $g_t = g_0$ and $g_t = \frac{m}{g_0}$.

The kk locus consists of all combinations (k_t, Ω_t) such that: $k_t = g_t \frac{\sigma\left(\frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t}\right)}{m} + \frac{\zeta(k_t)}{m}$. The $\Omega\Omega$ locus consists of all combinations (k_t, Ω_t) such that: $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t})}{g_t}$. At a stationary equilibrium, those two loci intersect. Moreover, we know that the equilibrium cohort growth rate is either $g_t = g_0$ or $g_t = \frac{m}{g_0}$.

At $g_t = g_0$, the two loci become: $\Omega_t = g_0 \left[\sigma^{-1}\left(\frac{m\left(k_t - \frac{\zeta(k_t)}{m}\right)}{g_0}\right) - \frac{\sigma(k_t)}{g_0}\right]$ and $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0})}{g_0}$. Let us denote $g_0 \left[\sigma^{-1}\left(\frac{m\left(k_t - \frac{\zeta(k_t)}{m}\right)}{g_0}\right) - \frac{\sigma(k_t)}{g_0}\right]$ as $\hat{D}(k_t)$. Let us assume that a non-negative solution Ω_t to the equality $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0})}{g_0}$ exists for any level of k_t. Let us denote that solution by $\Omega_t \equiv \hat{E}(k_t)$. Given that $\sigma(\cdot)$ and $\zeta(\cdot)$ are monotonically increasing, we have that $\hat{E}'(k_t) > 0$.

We know that, when $k_t = 0$, we have $\hat{D}(0) = g_0 \left[\sigma^{-1}\left(\frac{0}{g_0}\right) - 0\right]$ and $\hat{E}(0)$ is the solution to $\Omega_t = \frac{\zeta(\frac{\Omega_t}{g_0})}{g_0}$, which is assumed to be strictly positive. Given that $\sigma(0)=0$ and $\sigma'(k_t)>0$, it follows that $\hat{D}(0)=g_0\left[\sigma^{-1}\left(\frac{0}{g_0}\right)-0\right]=0$. We thus have $\hat{D}(0)<\hat{E}(0)$, that is, the kk locus lies below the $\Omega\Omega$ locus for low k_t. If we now suppose that $\lim_{k\to\infty}\hat{D}(k_t)>\lim_{k\to\infty}\hat{E}(k_t)$, it follows that the kk locus lies below the $\Omega\Omega$ locus for low k_t, but above the $\Omega\Omega$ locus for high k_t. Hence, by continuity of $\hat{D}(k_t)$ and $\hat{E}(k_t)$, the two loci must intersect at least once along the 1st part of the gg locus, along which $g_t=g_0$. We can denote that intersection as $\left(\hat{k},\hat{\Omega},g_0\right)$.

At $g_t=\frac{m}{g_0}$, the two loci become: $\Omega_t=\frac{m}{g_0}\left[\sigma^{-1}\left(\left(k_t-\frac{\zeta(k_t)}{m}\right)g_0\right)-\frac{g_0\sigma(k_t)}{m}\right]$ and $\Omega_t=g_0\frac{\zeta(\frac{g_0\sigma(k_t)}{m}+\frac{g_0\Omega_t}{m})}{m}$. Let us denote $\frac{m}{g_0}\left[\sigma^{-1}\left(\left(k_t-\frac{\zeta(k_t)}{m}\right)g_0\right)-\frac{g_0\sigma(k_t)}{m}\right]$ as $\check{D}(k_t)$. Let us assume that a non-negative solution Ω_t to the equality $\Omega_t=g_0\frac{\zeta(\frac{g_0\sigma(k_t)}{m}+\frac{g_0\Omega_t}{m})}{m}$ exists for any level of k_t. Let us denote that solution by $\Omega_t\equiv\check{E}(k_t)$. Given that $\sigma(\cdot)$ and $\zeta(\cdot)$ are monotonically increasing, we have that $\check{E}'(k_t)>0$.

We know that, when $k_t=0$, we have $\check{D}(0)=\frac{m}{g_0}\left[\sigma^{-1}\left((0)\,g_0\right)-0\right]$ and $\check{E}(0)$ is the solution to $\Omega_t=g_0\frac{\zeta(\frac{g_0\Omega_t}{m})}{m}$, which is assumed to be strictly positive. Given that $\sigma(0)=0$ and $\sigma'(k_t)>0$, it follows that $\check{D}(0)=\frac{m}{g_0}\left[\sigma^{-1}\left((0)\,g_0\right)-0\right]=0$. We thus have $\check{D}(0)<\check{E}(0)$, that is, the kk locus lies below the $\Omega\Omega$ locus for low k_t. If we now suppose that $\lim_{k\to\infty}\check{D}(k_t)>\lim_{k\to\infty}\check{E}(k_t)$, it follows that the kk locus lies below the $\Omega\Omega$ locus for low k_t, but above the $\Omega\Omega$ locus for high k_t. Hence, by continuity of $\check{D}(k_t)$ and $\check{E}(k_t)$, the two loci must intersect at least once along the 1st part of the gg locus, along which $g_t=\frac{m}{g_0}$. That intersection can be denoted as $\left(\check{k},\check{\Omega},\frac{g_0}{m}\right)$.

Stability of the cycle Let us now consider whether the two equilibria $\left(\hat{k},\hat{\Omega},g_0\right)$ and $\left(\check{k},\check{\Omega},\frac{g_0}{m}\right)$ are stable. The Jacobian matrix is:

$$J\equiv\begin{pmatrix}\frac{\partial\Gamma(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial\Gamma(k_t,\Omega_t,g_t)}{\partial\Omega_t} & \frac{\partial\Gamma(k_t,\Omega_t,g_t)}{\partial g_t}\\ \frac{\partial\Theta(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial\Theta(k_t,\Omega_t,g_t)}{\partial\Omega_t} & \frac{\partial\Theta(k_t,\Omega_t,g_t)}{\partial g_t}\\ \frac{\partial\Lambda(g_t)}{\partial k_t} & \frac{\partial\Lambda(g_t)}{\partial\Omega_t} & \frac{\partial\Lambda(g_t)}{\partial g_t}\end{pmatrix}$$

Computing those entries at the equilibrium, we obtain that the determinant of the Jacobian matrix is: $\det(J)=\frac{1}{g}\frac{\zeta'(k)}{m}\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}>0$, while the trace of the Jacobian matrix is: $\mathrm{tr}(J)=g\frac{\sigma'\left(\frac{\sigma(k)}{g}+\frac{\Omega}{g}\right)}{m}\frac{\sigma'(k)}{g}+\frac{\zeta'(k)}{m}+\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+1$.

Note that Brooks's (2004) condition (iii) is here:

$$-\left[\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+g\frac{\sigma'\left(\frac{\sigma(k)}{g}+\frac{\Omega}{g}\right)\sigma'(k)}{mg}+\frac{\zeta'(k)}{m}+\frac{\zeta'(k)}{m}\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+1\right]$$
$$<g\frac{\sigma'\left(\frac{\sigma(k)}{g}+\frac{\Omega}{g}\right)}{m}\frac{\sigma'(k)}{g}+\frac{\zeta'(k)}{m}+\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+1+\frac{1}{g}\frac{\zeta'(k)}{m}\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}$$
$$<\left[\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+g\frac{\sigma'\left(\frac{\sigma(k)}{g}+\frac{\Omega}{g}\right)\sigma'(k)}{mg}+\frac{\zeta'(k)}{m}+\frac{\zeta'(k)}{m}\frac{\zeta'(\frac{\sigma(k)}{g}+\frac{\Omega}{g})}{m}\frac{1}{g}+1\right]$$

The first inequality is satisfied, but the second inequality is not. As a consequence of this, the two equilibria $\left(\hat{k}, \hat{\Omega}, g_0\right)$ and $\left(\check{k}, \check{\Omega}, \frac{m}{g_0}\right)$ are not stable.

However, remind that the initial cohort growth factors g_0 and $g_1 = \frac{m}{g_0}$ coincide with the equilibrium levels of g_t. Therefore, the question of convergence of the economy amounts to investigate whether k_t and Ω_t converge towards $\hat{k}$ and $\hat{\Omega}$ when $g_t = g_0$, and whether k_t and Ω_t converge towards $\check{k}$ and $\check{\Omega}$ when $g_t = \frac{m}{g_0}$. The first issue can be studied on the basis of the system:

$$\begin{aligned}
k_{t+2} &\equiv G(k_{t+1}, \Omega_{t+1}) = g_0 \frac{\sigma\left(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0}\right)}{m} + \frac{\zeta(k_t)}{m} \equiv \Gamma\left(k_t, \Omega_t\right) \\
\Omega_{t+2} &\equiv H(k_{t+1}) = \frac{\zeta(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0})}{g_0} \equiv \Theta\left(k_t, \Omega_t\right)
\end{aligned}$$

The Jacobian matrix is: $J \equiv \begin{pmatrix} \frac{\partial\Gamma(k_t,\Omega_t)}{\partial k_t} & \frac{\partial\Gamma(k_t,\Omega_t)}{\partial \Omega_t} \\ \frac{\partial\Theta(k_t,\Omega_t)}{\partial k_t} & \frac{\partial\Theta(k_t,\Omega_t)}{\partial \Omega_t} \end{pmatrix}$.

Estimating the entries at the equilibrium $\left(\hat{k}, \hat{\Omega}, g_0\right)$, we obtain that the determinant and the trace are: $\det(J) = \frac{\zeta'(\hat{k})}{m} \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{g_0^2}$ and $\operatorname{tr}(J) = \frac{\sigma'\left(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0}\right)\sigma'(\hat{k})}{m} + \frac{\zeta'(\hat{k})}{m} + \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{(g_0)^2}$.

Hence the condition for stability $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are thus:

$$\left| D + \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{2(g_0)^2} + \frac{1}{2} \sqrt[2]{\left[2D + \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{(g_0)^2}\right]^2 - 4\frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})\zeta'(\hat{k})}{g_0^2 m}} \right| < 1$$

and

$$\left| D + \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{2(g_0)^2} - \frac{1}{2} \sqrt[2]{\left[2D + \frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})}{(g_0)^2}\right]^2 - 4\frac{\zeta'(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0})\zeta'(\hat{k})}{g_0^2 m}} \right| < 1$$

where $D \equiv \frac{\sigma'\left(\frac{\sigma(\hat{k})}{g_0} + \frac{\hat{\Omega}}{g_0}\right)\sigma'(\hat{k}) + \zeta'(\hat{k})}{2m}$.

Regarding whether k_t and Ω_t converge towards $\check{k}$ and $\check{\Omega}$ when $g_t = \frac{m}{g_0}$, that issue can be discussed on the basis of the system:

$$\begin{aligned}
k_{t+2} &\equiv G(k_{t+1}, \Omega_{t+1}) = \frac{\sigma\left(\frac{g_0\sigma(k_t)}{m} + \frac{g_0\Omega_t}{m}\right)}{g_0} + \frac{\zeta(k_t)}{m} \equiv \Gamma\left(k_t, \Omega_t\right) \\
\Omega_{t+2} &\equiv H(k_{t+1}) = g_0 \frac{\zeta(\frac{g_0\sigma(k_t)}{m} + \frac{g_0\Omega_t}{m})}{m} \equiv \Theta\left(k_t, \Omega_t\right)
\end{aligned}$$

A rationale similar to the one developed in the first part of this proof can be used to lead to the stability conditions:

$$\left| V + \frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{2m^2} + \frac{1}{2} \sqrt[2]{\left[2V + \frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{m^2}\right]^2 - 4\frac{\zeta'(\check{k})}{m} \frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{m^2}} \right| < 1$$

$$\left| V + \frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{2m^2} - \frac{1}{2}\sqrt[2]{\left[2V + \frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{m^2}\right]^2 - 4\frac{\zeta'(\check{k})}{m}\frac{\zeta'(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m})}{m^2}} \right| < 1$$

where $V \equiv \frac{\sigma'\left(\frac{g_0\sigma(\check{k})}{m} + \frac{g_0\check{\Omega}}{m}\right)\sigma'\left(\check{k}\right)+\zeta'(\check{k})}{2m}$.

9.4 Proof of remark 1

When $N_{-1} = N_0$, we have $g_0 = 1$. Hence the cycle in g_t has, as two values, $g_0 = 1$ and $g_1 = \frac{m}{1}$. Thus, if $m = 1$, we have $g_0 = g_1 = 1$. As a consequence, the dynamic system becomes two-dimensional:

$$\begin{aligned} k_{t+1} &\equiv G(k_t, \Omega_t) = \sigma\left(k_t\right) + \Omega_t \\ \Omega_{t+1} &\equiv H\left(k_t\right) = \zeta(k_t) \end{aligned}$$

From the first equation, we can define the kk locus. Imposing $k_{t+1} = k_t$ yields: $\Omega_t = k_t - \sigma\left(k_t\right)$. From the second equation, we can define the $\Omega\Omega$ locus: $\Omega_t = \zeta(k_t)$. Note that, as $\sigma(0) = 0$ and $\zeta(0) = 0$, the two loci intersect at $k_t = 0$. Moreover, assuming that $lim_{k\to 0}\ 1 - \sigma'\left(k_t\right) < lim_{k\to 0}\ \zeta'(k_t)$ and $lim_{k\to +\infty}\ 1 - \sigma'\left(k_t\right) > lim_{k\to +\infty}\ \zeta'(k_t)$, it follows that the kk locus lies below the $\Omega\Omega$ locus for low k levels, but lies above it for high k levels. Hence, by continuity, the kk and $\Omega\Omega$ loci must intersect at some point.

Regarding the stability of that equilibrium, the Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial G(k_t,\Omega_t)}{\partial k_t} & \frac{\partial G(k_t,\Omega_t)}{\partial \Omega_t} \\ \frac{\partial H(k_t)}{\partial k_t} & 0 \end{pmatrix}$$

When computing the entries of that matrix at the equilibrium, we obtain that the determinant and the trace of the Jacobian matrix are: $\det(J) = -\zeta'(k) < 0$ and $\mathrm{tr}(J) = \sigma'\left(k\right) > 0$. Hence it is straightforward to deduce that the eigenvalues are: $\lambda_1 = \frac{\sigma'(k_t) + \sqrt[2]{(\sigma'(k_t))^2 + 4\zeta'(k_t)}}{2} > 0$ and $\lambda_2 = \frac{\sigma'(k_t) - \sqrt[2]{(\sigma'(k_t))^2 + 4\zeta'(k_t)}}{2} < 0$. We are thus in a case where $\Delta > 0$, and where the two eigenvalues are of opposite signs.

The condition for $|\lambda_1| < 1$ is: $\sqrt[2]{\left(\sigma'\left(k\right)\right)^2 + 4\zeta'(k)} < 2 - \sigma'\left(k\right)$. The above condition can be rewritten as: $\zeta'(k) < 1 - \sigma'\left(k\right)$. We know that, at the stationary equilibrium, the $G(k_t, \Omega_t)$ curve intersects the $H(k_t)$ curve from below, which means that: $1 - \sigma'\left(k^*\right) > \zeta'(k^*)$. Hence the condition for $|\lambda_1| < 1$ is satisfied.

The condition for $|\lambda_2| < 1$, this can be rewritten as: $\zeta'(k) < 1 + \sigma'\left(k\right)$. Given that, at the equilibrium, the $G(k_t, \Omega_t)$ curve intersects the $H(k_t)$ curve from below, we have: $1 - \sigma'\left(k^*\right) > \zeta'(k^*)$. Hence the condition for $|\lambda_2| < 1$ is also satisfied at our equilibrium. Note that, while stability is guaranteed, the convergence towards the stationary equilibrium takes a non-monotonic form, due to the opposite signs of the eigenvalues.

9.5 Proof of Proposition 3

As shown above, the equation:

$$g^3 + \frac{(F_L(k,\cdot) - d)}{k} g - \frac{2b}{k} = 0$$

characterizes the *interior* optimal cohort growth rate g. Given that this equation takes the form of a so-called "depressed cubic" equation, we can use the resolution method develop by Cardano (1545). That method consists in first introducing two new variables, whose sum equals g: $s + t = g$. We substitute for it in the depressed cubic equation, and obtain:

$$\begin{aligned} (s+t)^3 + (\frac{F_L(k,\cdot) - d}{k})(s+t) - \frac{2b}{k} &= 0 \\ s^3 + t^3 + (\frac{F_L(k,\cdot) - d}{k} + 3st)(s+t) - \frac{2b}{k} &= 0 \end{aligned}$$

Then, imposing the constraint $\frac{F_L(k,\cdot)-d}{k} + 3st = 0$, we get:

$$\begin{aligned} s^3 + t^3 &= \frac{2b}{k} \\ st &= -\frac{F_L(k,\cdot) - d}{3k} \implies s^3 t^3 = -\frac{(F_L(k,\cdot) - d)^3}{27k^3} \end{aligned}$$

Thus s^3 and t^3 are the roots of the equation: $m^2 + m\left(-\frac{2b}{k}\right) - \frac{(F_L(k,\cdot)-d)^3}{27k^3} = 0$. Note that

$$\Delta \equiv \left(-\frac{2b}{k}\right)^2 + \frac{4\left(\frac{F_L(k,\cdot)-d}{k}\right)^3}{27} = \frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot) - d)^3}{27k^3} \gtrless 0$$

If $\Delta > 0$, we have the two roots: $m_1 = s^3 = \frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}$ and $m_2 = t^3 = \frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}$. Hence it follows that the optimal g is given by:

$$g = s + t = \sqrt[3]{\frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}} + \sqrt[3]{\frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}}$$

If $\Delta = 0$, we need to choose a cubic root for s^3. As there is no direct way to choose the cubic root of t^3, we need to use the relation $t = -\frac{F_L(k,\cdot)-d}{3ks}$, which yields: $s = \sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot)-d)^3}{27k^3}}}$. Hence the optimal cohort growth rate $g = s + t$ is:

$$g = \sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot) - d)^3}{27k^3}}} - \frac{F_L(k,\cdot) - d}{3k\sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot)-d)^3}{27k^3}}}}$$

If $\Delta < 0$, one can obtain the complex cubic roots by multiplying one of the two above cubic roots by $\frac{-1}{2} + i\frac{\sqrt[2]{3}}{2}$, and the other by $\frac{-1}{2} - i\frac{\sqrt[2]{3}}{2}$. This yields: $m_1 = s^3 = \frac{-\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} + i\sqrt[2]{3}\frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}$ and $m_2 = t^3 = \frac{-\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} - i\sqrt[2]{3}\frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}$. Hence $g = s + t$ is:

$$g = \begin{array}{l} \sqrt[3]{\frac{-\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}} + \frac{2bi\sqrt[2]{3}}{k} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}} \\ + \sqrt[3]{\frac{-\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}} - \frac{2bi\sqrt[2]{3}}{k} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}} \end{array}$$

9.6 Proof of Proposition 5

Existence of a stationary equilibrium

Fixing $g_{t+1} = g_t$ in the third equation of system B leads to the long-run cohort growth factor $g = \frac{n + \sqrt[2]{n^2+4m}}{2}$. Fixing $k_{t+1} = k_t$ in the first equation of system B allows us to write the kk locus as: $X_t = \frac{k_t^{2-\alpha}\frac{g_t m}{g_t - n}\left[\left(1+\beta+\beta^2\right)\alpha(1+g_t)+(1-\alpha)\right]}{\beta^2 A^2\alpha^2\left(\frac{m}{g_t-n+m}\right)^{\alpha-1}\left(\frac{m(g_t-n)}{m-ng_t+n^2+m(g_t-n)}\right)^{\alpha}(1+g_t)} - \frac{\frac{m}{g_t-n}k_t(1-\alpha)\left(\frac{m}{g_t-n+m}\right)\left[1+\beta-\frac{g_t-n}{m}\right]}{A\alpha\left(\frac{m(g_t-n)}{m-ng_t+n^2+m(g_t-n)}\right)^{\alpha}}$.

The XX locus is: $X_t = (1-\alpha)k_t^{\alpha}$. Hence the existence of a stationary equilibrium depends on whether the kk locus and XX locus intersect when g_t equals its long-run value $g = \frac{n+\sqrt[2]{n^2+4m}}{2}$. That intersection occurs when, for some k_t, we have: $(1-\alpha)k_t^{\alpha}\left(\frac{g}{1+g}\right)^{\alpha} + \frac{k_t(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)} = \frac{k_t^{2-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)}$.

The LHS and the RHS are thus equal for $k_t = 0$, which is a stationary equilibrium. In order to examine whether there exists another equilibrium, note that, given $\alpha < 1$: $\lim_{k\to 0}(1-\alpha)\alpha k_t^{\alpha-1}\left(\frac{g}{1+g}\right)^{\alpha} + \frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)} = +\infty > \lim_{k\to 0}\frac{(2-\alpha)k_t^{1-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)} = 0$, and that $\lim_{k\to\infty}(1-\alpha)\alpha k_t^{\alpha-1}\left(\frac{g}{1+g}\right)^{\alpha} + \frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)} = \frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)} \lesseqgtr \lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)}$.

Hence, provided: $\frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)} < \lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)}$, there exists a stationary equilibrium with $k > 0$.

Stability of a stationary equilibrium The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial G(k_t,X_t,g_t)}{\partial k_t} & \frac{\partial G(k_t,X_t,g_t)}{\partial X_t} & \frac{\partial G(k_t,X_t,g_t)}{\partial g_t} \\ \frac{\partial H(k_t)}{\partial k_t} & 0 & 0 \\ 0 & 0 & \frac{\partial I(g_t)}{\partial g_t} \end{pmatrix}$$

Estimating the entries at the equilibrium (k, X, g), we obtain that the determinant and the trace are: $\det(J) = \frac{\beta^2 A^2 \alpha^3 (1-\alpha) k^{2\alpha-2} \left(\frac{m}{g-n+m}\right)^{\alpha-1} \left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha} (1+g)}{\frac{g^3}{g-n}[(1+\beta+\beta^2)\alpha(1+g)+(1-\alpha)]}$

and $\mathrm{tr}(J) = \left[\begin{array}{c} \frac{A\alpha^2 k^{\alpha-1}(1-\alpha)\left(\frac{m}{g-n+m}\right)^{\alpha}(1+g)\left[(\beta+\beta^2)+\frac{\beta^2(g-n)}{m}\right]}{g\left[(1+\beta+\beta^2)\alpha(1+g_t)+(1-\alpha)\right]} \\ +\frac{\beta^2 A^3(\alpha-1)(1-\alpha)\alpha^2 k^{2\alpha-2}\left(\frac{m}{g-n+m}\right)^{\alpha-1}\left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha}(1+g)}{\frac{gm}{g-n}\left[(1+\beta+\beta^2)\alpha(1+g)+(1-\alpha)\right]} - \frac{m}{g^2} \end{array}\right]$.

Hence Brooks's condition (i) is:

$$\left|\frac{\beta^2 A^2 \alpha^3 (1-\alpha) k^{2\alpha-2} \left(\frac{m}{g-n+m}\right)^{\alpha-1} \left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha} (1+g)}{\frac{g^3}{g-n}[(1+\beta+\beta^2)\alpha(1+g)+(1-\alpha)]}\right| < 1.$$

Reminding that, at equilibrium, we have: $1 = \frac{Ak^{\alpha-1}(1-\alpha)\alpha\left(\frac{m}{g_t-n+m}\right)^{\alpha}(1+g_t)}{g_t[(1+\beta+\beta^2)\alpha(1+g_t)+(1-\alpha)]}\left[(\beta+\beta^2)+\frac{\beta^2(g_t-n)}{m}\right] + \frac{\beta^2\alpha^2 A^3(1-\alpha)k^{2\alpha-2}\left(\frac{m}{g_t-n+m}\right)^{\alpha-1}\left(\frac{m(g_t-n)}{m-ng_t+n^2+m(g_t-n)}\right)^{\alpha}(1+g_t)}{\frac{g_t m}{g_t-n}[(1+\beta+\beta^2)\alpha(1+g_t)+(1-\alpha)]}$,

it follows that condition (ii) can be written as:

$$1 > \frac{-\alpha m}{g^3}[1-A\Lambda] + \Lambda\left(\frac{Am(1-\alpha)}{g_t^2} - \alpha\right) - \left[\alpha - \Lambda A - \frac{m}{g^2}\right]\left[\frac{\Lambda\alpha m}{g^2}\right] + \left[\frac{\Lambda\alpha m}{g^2}\right]^2,$$

where $\Lambda \equiv \frac{\beta^2 A^2 (1-\alpha)\alpha^2 k^{2\alpha-2}\left(\frac{m}{g-n+m}\right)^{\alpha-1}\left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha}(1+g)}{\frac{gm}{g-n}[(1+\beta+\beta^2)\alpha(1+g)+(1-\alpha)]}$.

Using the same simplification, condition (iii) can be written as:

$$\frac{m\alpha[1-\Lambda A]}{g^2} - \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) - 1 < \frac{\alpha g^2 - A\Lambda g^2 - m(1-\alpha\Lambda)}{g^2} < \frac{-m\alpha[1-\Lambda A]}{g^2} + \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) + 1$$

9.7 Proof of Corollary 2

Existence of a stationary equilibrium Fixing $g_{t+1} = g_t$ in $g_{t+1} = n + \frac{m}{g_t}$ in system B under $n = 0$ leads to the long-run cohort growth factor, equal to $g = \sqrt[2]{m}$. The kk locus can be rewritten as: $X_t = \frac{k_t^{2-\alpha} m\left[(1+\beta+\beta^2)\alpha(1+g_t)+(1-\alpha)\right]}{\beta^2 A^2\alpha^2\left(\frac{m}{g_t+m}\right)^{\alpha-1}\left(\frac{m(g_t)}{m+m(g_t)}\right)^{\alpha}(1+g_t)} - \frac{\frac{m}{g_t}k_t(1-\alpha)\left(\frac{m}{g_t+m}\right)\left[1+\beta-\frac{g_t}{m}\right]}{A\alpha\left(\frac{m(g_t)}{m+m(g_t)}\right)^{\alpha}}$.

The XX locus is $X_t = (1-\alpha)k_t^{\alpha}$. The existence of a stationary equilibrium depends on whether the kk locus and XX locus intersect when g_t equals its long-run value $g = \sqrt[2]{m}$. That intersection occurs when, for some k_t, we have:

$$(1-\alpha)k_t^{\alpha}\left(\frac{\sqrt[2]{m}}{1+\sqrt[2]{m}}\right)^{\alpha} + \frac{k_t(1-\alpha)\sqrt[2]{m}\left[\sqrt[2]{m}+\sqrt[2]{m}\beta-1\right]}{A\alpha\left(1+\sqrt[2]{m}\right)} = \frac{k_t^{2-\alpha}m\left[(1+\beta+\beta^2)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+\sqrt[2]{m}}{\sqrt[2]{m}}\right)^{\alpha-1}\left(1+\sqrt[2]{m}\right)}.$$

The LHS and the RHS are equal for $k_t = 0$. In order to examine whether there exists another stationary equilibrium, note that, given $\alpha < 1$:

$$\lim_{k\to 0}(1-\alpha)\alpha k_t^{\alpha-1}\left(\frac{\sqrt[2]{m}}{1+\sqrt[2]{m}}\right)^{\alpha} + \frac{(1-\alpha)\sqrt[2]{m}\left[\sqrt[2]{m}+\sqrt[2]{m}\beta-1\right]}{A\alpha\left(1+\sqrt[2]{m}\right)} = +\infty$$

$$> \lim_{k\to 0}\frac{(2-\alpha)k_t^{1-\alpha}m\left[(1+\beta+\beta^2)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+\sqrt[2]{m}}{\sqrt[2]{m}}\right)^{\alpha-1}\left(1+\sqrt[2]{m}\right)} = 0.$$

Hence, provided: $\frac{(1-\alpha)\sqrt[2]{m}\left[\sqrt[2]{m}+\sqrt[2]{m}\beta-1\right]}{A\alpha\left(1+\sqrt[2]{m}\right)} < \lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}m\left[(1+\beta+\beta^2)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+\sqrt[2]{m}}{\sqrt[2]{m}}\right)^{\alpha-1}\left(1+\sqrt[2]{m}\right)}$, there exists a stationary equilibrium with strictly positive k, X, g.

Stability of a stationary equilibrium Brooks's condition (i) is:
$\left|\frac{\beta^2 A^2 \alpha^3 (1-\alpha) k^{2\alpha-2} \left(\frac{m}{\sqrt[2]{m}+m}\right)^{\alpha-1} \left(\frac{m\sqrt[2]{m}}{m+m\sqrt[2]{m}}\right)^{\alpha} \left(1+\sqrt[2]{m}\right)}{m\left[(1+\beta+\beta^2)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}\right| < 1.$

Condition (ii) is: $1 > \frac{-\alpha m}{\left(\sqrt[2]{m}\right)^3} [1 - A\Lambda] + \Lambda \left(\frac{Am(1-\alpha)}{\left(\sqrt[2]{m}\right)^2} - \alpha\right) - [\alpha - \Lambda A - 1] [\Lambda\alpha] + [\Lambda\alpha]^2$, where $\Lambda \equiv \frac{\beta^2 A^2 (1-\alpha) \alpha^2 k^{2\alpha-2} \left(\frac{m}{\sqrt[2]{m}+m}\right)^{\alpha-1} \left(\frac{m\sqrt[2]{m}}{m+m\sqrt[2]{m}}\right)^{\alpha} \left(1+\sqrt[2]{m}\right)}{m\left[(1+\beta+\beta^2)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}$.

Condition (iii) is, after simplifications:
$\alpha - A\Lambda - 1 + \alpha\Lambda < \alpha - A\Lambda - (1 - \alpha\Lambda) < -\alpha [1 - \Lambda A] + \Lambda (A(1-\alpha) - \alpha) + 1$
The first inequality is not satisfied. Therefore condition (iii) breaks, and the equilibrium is not stable.

9.8 Proof of Proposition 6

Existence of a stationary equilibrium Imposing $k_{t+1} = k_t$ in the first equation of system C yields the kk locus: $k_t = \frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t}$, which can be rewritten as: $\Omega_t = g_t k_t - \sigma(k_t)$. Imposing $\Omega_{t+1} = \Omega_t$ in $\Omega_{t+1} \equiv \hat{H}(k_t)$ yields the $\Omega\Omega$ locus: $\Omega_t = \frac{\zeta(k_t)}{g_t}$. Imposing $g_{t+1} = g_t$ in $g_{t+1} \equiv \hat{I}(k_t, \Omega_t, g_t)$ yields the gg locus: $g_t = \eta(G(k_t, \Omega_t, g_t)) + \frac{\mu(k_t)}{g_t}$. Hence: $g_t - \frac{\mu(k_t)}{g_t} = \eta(G(k_t, \Omega_t, g_t))$.

Assuming that $\eta(k_t)$ and/or $\mu(k_t)$ is positive, it follows that, along the gg locus, the cohort growth factor satisfies: $g_t^2 - \eta(G(k_t, \Omega_t, g_t))g_t - \mu(k_t) = 0$, from which it follows that: $g_t = \frac{\eta(G(k_t,\Omega_t,g_t) + \sqrt[2]{[\eta(G(k_t,\Omega_t,g_t))]^2 + 4\mu(k_t)}}{2}$.

At that sustainable cohort growth factor, the kk locus can be rewritten as:
$\Omega_t = \frac{\eta(G(k_t,\Omega_t,g_t) + \sqrt[2]{[\eta(G(k_t,\Omega_t,g_t))]^2 + 4\mu(k_t)}}{2} k_t - \sigma(k_t)$,
while the $\Omega\Omega$ locus is:
$\Omega_t = \frac{2\zeta(k_t)}{\eta(G(k_t,\Omega_t,g_t) + \sqrt[2]{[\eta(G(k_t,\Omega_t,g_t))]^2 + 4\mu(k_t)}}$

Note that the kk locus takes a value of 0 at $k_t = 0$. The same is also true for the $\Omega\Omega$ locus. Denote $\Xi_t \equiv \eta(G(k_t, \Omega_t, g_t) + \sqrt[2]{[\eta(G(k_t, \Omega_t, g_t))]^2 + 4\mu(k_t)}$ and $\Pi_t \equiv \frac{2[\eta(G(k_t,\Omega_t,g_t))]\eta'(G(k_t,\Omega_t,g_t)G'(\cdot) + 4'\mu(k_t)}{2[\Xi_t - \eta(G(k_t,\Omega_t,g_t)]}$.

Assuming that:
$lim_{k\to 0} \frac{\Xi_t}{2} + k_t \frac{\eta'(G(k_t,\Omega_t,g_t)G'(\cdot) + \Pi_t}{2} - \sigma'(k_t) < lim_{k\to 0} \frac{2\zeta'(k_t)[\Xi_t] - 2\zeta(k_t)\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot) + \Pi_t\right]}{[\Xi_t]^2}$,
$lim_{k\to +\infty} \frac{\Xi_t}{2} + k_t \frac{\eta'(G(k_t,\Omega_t,g_t)G'(\cdot) + \Pi_t}{2} - \sigma'(k_t) > lim_{k\to +\infty} \frac{2\zeta'(k_t)[\Xi_t] - 2\zeta(k_t)\left[\eta'(G(k_t,\Omega_t,g_t)G'(\cdot) + \Pi_t\right]}{[\Xi_t]^2}$

it follows that the kk locus lies below the $\Omega\Omega$ locus for low k_t, but that the kk locus lies above the $\Omega\Omega$ locus for high k_t. Hence, by continuity, the kk and $\Omega\Omega$ loci must intersect at some point along the gg locus.

Stability of a stationary equilibrium The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial \hat{G}(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial \hat{G}(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial \hat{G}(k_t,\Omega_t,g_t)}{\partial g_t} \\ \frac{\partial \hat{H}(k_t)}{\partial k_t} & 0 & \frac{\partial \hat{H}(k_t)}{\partial g_t} \\ \frac{\partial \hat{I}(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial \hat{I}(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial \hat{I}(k_t,\Omega_t,g_t)}{\partial g_t} \end{pmatrix}$$

Estimating the entries at the equilibrium, we obtain that: $\det(J) = \frac{\zeta'(k)\mu(k)-\zeta(k_t)\mu'(k)}{g^4}$ and that the trace is: $\operatorname{tr}(J) = \frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{(g)^2}\right] - \frac{\mu(k)}{(g)^2}$.

Brooks's condition (i) can be written here as: $\left|\frac{\zeta'(k)\mu(k)-\zeta(k)\mu'(k)}{g^4}\right| < 1$.

Brooks's condition (ii) is:

$$1 > \frac{\frac{\zeta(k_t)\eta'(k)}{g^3} + \frac{-\sigma'(k)\mu(k)}{g^3} + \frac{\mu'(k)[\sigma(k)+\Omega]}{g^3} - \frac{\zeta'(k)}{g^2}}{-\left[\frac{\zeta'(k)\mu(k)-\zeta(k_t)\mu'(k)}{g^4}\right]\left[\frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{(g)^2}\right] - \frac{\mu(k)}{(g)^2}\right] + \left[\frac{\zeta'(k)\mu(k)-\zeta(k_t)\mu'(k)}{g^4}\right]^2}$$

Brooks's condition (iii) is here:

$$\begin{aligned} &-\left[\frac{\zeta(k_t)\eta'(k)}{g^3} + \frac{-\sigma'(k)\mu(k)+\mu'(k)[\sigma(k)+\Omega]}{g^3} - \frac{\zeta'(k)}{g^2} + 1\right] \\ &< \frac{\sigma'(k)}{g} - \eta'(k)\left[\frac{\sigma(k)+\Omega}{(g)^2}\right] - \frac{\mu(k)}{(g)^2} + \frac{\zeta'(k)\mu(k)-\zeta(k_t)\mu'(k)}{g^4} \\ &< \left[\frac{\zeta(k_t)\eta'(k)}{g^3} + \frac{-\sigma'(k)\mu(k)+\mu'(k)[\sigma(k)+\Omega]}{g^3} - \frac{\zeta'(k)}{g^2} + 1\right]. \end{aligned}$$

9.9 Proof of Proposition 7

From the first equation of system D, and fixing $g_{t+1} = g_t$, one can write the gg locus as: $g_t\left(g_t - n\right) - m = \ell_t$. From the second equation, and fixing $\ell_{t+1} = \ell_t$, one can write the $\ell\ell$ locus as: $\ell_t = \frac{o}{g_t}$. Hence the two loci intersect when: $g_t\left(g_t - n\right) - m = \frac{o}{g_t}$. Let us denote the LHS by the function $\Phi\left(g_t\right) \equiv g_t\left(g_t - n\right) - m$. We have $\Phi(0) = -m$, $\Phi'(g_t) = 2g_t - n$ and $\Phi''(g_t) = 2$. Given that $2g_t - n = n + 2\frac{m}{g_t} + 2\frac{\ell_t}{g_t} \geq 0$, the LHS is an increasing convex curve starting at $\ell_t = -m$. Regarding the $\ell\ell$ locus, denoted by the function $\Upsilon\left(g_t\right) \equiv \frac{o}{g_t}$, we have $\Upsilon(0) = +\infty$, $\Upsilon'(g_t) = \frac{-o}{g_t^2} < 0$ and $\Upsilon''(g_t) = \frac{o2}{g_t^3} > 0$. Hence the $\ell\ell$ locus is a decreasing convex curve. Given that $\Upsilon(0) > \Phi(0)$ and $\Upsilon(\infty) = 0 < \Phi(\infty) > 0$, it follows, by continuity, that the two loci must intersect. From the strict convexity of the two loci, that intersection is unique.

Regarding the stability, the Jacobian matrix is: $\begin{pmatrix} \frac{-(m+\ell)}{g^2} & \frac{1}{g} \\ -\frac{o}{g^2} & 0 \end{pmatrix}$.

Hence the determinant and the trace are: $\det(J) = \frac{o}{g^3}$ and $\operatorname{tr}(J) = \frac{-(m+\ell)}{g^2} = \frac{-m}{g^2} - \frac{o}{g^3}$. Thus the eigenvalues are: $\lambda_{1,2} = \frac{\left(-\frac{m}{g^2} - \frac{o}{g^3}\right) \pm \sqrt[2]{\left(-\frac{m}{g^2} - \frac{o}{g^3}\right)^2 - \frac{4o}{g^3}}}{2}$. Stability requires $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

Let us consider different reproduction schemes.

- $n > 0$, $m = o = 0$: $|\lambda_1| = 0 < 1$ and $|\lambda_2| = 0 < 1$.
- $n > 0$, $m > 0$, $o = 0$: $|\lambda_1| = 0 < 1$ and $|\lambda_2| = \left|-\frac{4m}{2n^2+4m+2n\sqrt[2]{n^2+4m}}\right| < 1$.

- $n=0$, $m>0$, $o>0$. Note that, when $n=0$, we have, at the equilibrium: $1=\frac{m}{g^2}+\frac{o}{g^3}$ and $4-\frac{4m}{g^2}=\frac{4o}{g^3}$. Hence stability is achieved if and only if:

$$|\lambda_1| = \left| \frac{-1+\sqrt[2]{-3+\frac{4m}{g^2}}}{2} \right| < 1 \quad \text{and} \quad |\lambda_2| = \left| \frac{-1-\sqrt[2]{-3+\frac{4m}{g^2}}}{2} \right| < 1$$

When $m=0$, we have: $|\lambda_1| = \left|\frac{-1+\sqrt[2]{3}i}{2}\right|$ and $|\lambda_2| = \left|\frac{-1-\sqrt[2]{3}i}{2}\right| < 1$, that is, complex eigenvalues. Those two eigenvalues can be written as: $\lambda_{1,2}=\frac{-1}{2}\pm\frac{\sqrt[2]{3}}{2}i$. Denoting $r=\left|\frac{-1}{2}\pm\frac{\sqrt[2]{3}}{2}i\right|$, we have: $r=\sqrt[2]{\left(\frac{-1}{2}\right)^2+\left(\frac{\sqrt[2]{3}}{2}\right)^2}=\sqrt[2]{1}$. Hence, the two eigenvalues lie exactly on the unit circle. Hence the equilibrium is unstable.

When $m>0$, two cases can arise. If $3>\frac{4m}{g^2}$, then eigenvalues are complex numbers. We have: $\lambda_{1,2}=\frac{-1\pm i\sqrt[2]{\left(3-\frac{4m}{g^2}\right)}}{2}$. Denoting $r=\left|\frac{-1}{2}\pm\frac{\sqrt[2]{3-\frac{4m}{g^2}}}{2}i\right|$, we have: $r=\sqrt[2]{\left(\frac{-1}{2}\right)^2+\left(\frac{\sqrt[2]{3-\frac{4m}{g^2}}}{2}\right)^2}=\sqrt[2]{1-\frac{m}{g^2}}$. The condition $r<1$ is satisfied when $3>\frac{4m}{g^2}$. If $3\leq\frac{4m}{g^2}$, the eigenvalues are real numbers. Thus the conditions for stability are: $\sqrt[2]{-3+\frac{4m}{g^2}}<3$ for $|\lambda_1|<1$ and $\sqrt[2]{-3+\frac{4m}{g^2}}<1$ for $|\lambda_2|<1$. Hence we need: $1>\frac{m}{g^2}$. When $o=0$, that condition is not satisfied, since $g=\sqrt[2]{m}$. When $o>0$, that condition is satisfied. Indeed, at the equilibrium, we have: $g=\frac{m}{g}+\frac{o}{g^2}\implies g^2=m+\frac{o}{g}$, so that $g^2>m$, insuring convergence.