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On the use of the σ-coordinate system in regions of large bathymetric variations

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ABSTRACT

The σ-transformation is a widely used coordinate change that maps the actual depth-varying sea onto a computational domain, the depth of which is constant. The advantages of this technique are numerous. It permits an efficient use of computer resources, a simple treatment of the surface and bottom boundary conditions, and an accurate representation of the bathymetry. However, if the range of the depth is too large, or when the depth varies too rapidly, as in the shelf break region, it may be shown that the σ-transformation leads to severe numerical errors. In the application of GHER's three-dimensional model to the Western Mediterranean, the occurrence of those numerical errors is avoided by the introduction of a two-fold σ-coordinate system in the deep sea.

Introduction

One of the first tasks in the preparation of a marine model is to list the length scales of the phenomena that are expected to be represented. It is necessary to distinguish between horizontal scales ($L_h^{(1)},..., L_h^{(i)},..., L_h^{(I)}$) and vertical scales ($L_v^{(1)},..., L_v^{(j)},..., L_v^{(J)}$), the former being, in most cases, much longer than the latter, i.e., the aspect ratio of marine processes is generally very small.

In the hope that the model will prove to be accurate, the mesh sizes are chosen small compared with the corresponding length scales. Accordingly, one has

$$\Delta x_h = \varepsilon_h L_h \quad (1)$$

and

$$\Delta x_v = \varepsilon_v L_v \quad (2)$$

with

$$(L_h, L_v) = \left[\min_i \left(L_h^{(i)}\right), \min_j \left(L_v^{(j)}\right)\right] \quad (3)$$

$$\varepsilon_h, \varepsilon_v \ll 1. \quad (4)$$

In the above formulas, Δx_h and Δx_v denote the horizontal and the vertical space increments, respectively. In general, L_h and L_v must be considered functions of time and position.

Accuracy requirement (4) is readily taken into account when one uses a coordinate system of which the coordinate surfaces, delimiting the grid boxes, are horizontal or vertical, i.e., non oblique. When this condition is not satisfied, implementing eqn. (4) becomes a delicate matter, for model coordinates then intertwine vertical and horizontal directions, to which correspond length scales whose ratio is very small. Consequently, terrain-following coordinates, such as the σ-coordinate, may not be utilized without precautions. This has been known for a long time in meteorology (Gary, 1973; Janjic, 1977; Mesinger, 1982, Arakawa and Suarez, 1983), and was recently discussed in the scope of oceanic modelling by Haney (1991). So far, the attention mostly focused on the representation of the pressure gradient force.

In the present note, we show that limitations to the use of the σ-coordinate pertain not only to the pressure gradient force but also to most of the terms of the governing equations of marine models. In addition, it is seen that, in the region of the shelf break, a two-fold σ-coordinate system helps preserve the model's accuracy. This is exemplified with the application of the three-dimensional model of GHER to the Western Mediterranean.

The σ-coordinate system and its advantages

To take into account the effect of the bottom and the surface of the sea, many authors used a change of coordinates in which the surface and the bottom become coordinate surfaces (Freeman et al., 1972; Owen, 1980; Nihoul et al., 1986; Blumberg and Mellor, 1987; Davies, 1987; Deleersnijder, 1989; Spall and Robinson, 1990;

De Kok, 1991). All of the existing formulations of such a change of coordinates basically stem from Phillips (1957). The σ-transformation is probably the most simple of them. It reads:

$$\left(\tilde{t},\ \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3 = L\sigma\right)$$

$$= \left(t,\ x_1,\ x_2,\ L\frac{x_3+h}{\eta+h} = L\frac{x_3+h}{H}\right), \qquad (5)$$

where "~" denotes new or "transformed" variables; t is the time; x_1 and x_2 stand for horizontal coordinates; x_2 represents the vertical coordinate, positive upwards; η and h are the sea surface elevation and the unperturbed water height, respectively, so that $H = \eta + h$ represents the total water height; L is the—constant—sea depth in the σ-space.

Along with eqn. (5), it is necessary to introduce a new vertical velocity defined as

$$\tilde{u}_3 = L\left(\frac{\partial\sigma}{\partial t} + u_1\frac{\partial\sigma}{\partial x_1} + u_2\frac{\partial\sigma}{\partial x_2} + u_3\frac{\partial\sigma}{\partial x_3}\right), \qquad (6)$$

where u_i (i = 1, 2, 3) denotes the velocity components along the x_i-axis—in the real space.

Several advantages of the σ-coordinate system are usually acknowledged.

It permits an efficient use of computer resources. The impermeability of the sea surface and the sea bottom is easily accounted for by

$$[\tilde{u}_3]_{\sigma=0,1} = 0. \qquad (7)$$

In addition, the model bathymetry available in the σ-system is probably much closer to the real bathymetry than the "staircase" resulting from the use of the cartesian "x_3-coordinate" system which implies that the effect of the sea bottom is probably better represented. Finally, the Jacobian of the σ-transformation, H/L, is independent of the vertical coordinate. There is thus no need to locally evaluate this Jacobian, as is the case with more general terrain-following coordinates (Kasahara, 1974; Gal-Chen and Somerville, 1975; Dutton, 1976). Hence, the typical evolution equation for a quantity a in cartesian coordinates,

$$\frac{\partial a}{\partial t} + \frac{\partial(u_1 a)}{\partial x_1} + \frac{\partial(u_2 a)}{\partial x_2} + \frac{\partial(u_3 a)}{\partial x_3}$$

$$= Q^a + D^a + \frac{\partial}{\partial x_3}\left(\lambda_T^a \frac{\lambda a}{\partial x_3}\right), \qquad (8)$$

leads, in σ-coordinate, to an equation which is not significantly more complex than the original one,

$$\frac{\partial(Ha)}{\partial\tilde{t}} + \frac{\partial(Hu_1 a)}{\partial\tilde{x}_1} + \frac{\partial(Hu_2 a)}{\partial\tilde{x}_2} + \frac{\partial(H\tilde{u}_3 a)}{\partial\tilde{x}_3}$$

$$= HQ^a + H\tilde{D}^a + \frac{\partial}{\partial\tilde{x}_3}\left(\tilde{\lambda}_T^a \frac{\partial(Ha)}{\partial\tilde{x}_3}\right), \qquad (9)$$

where Q^a is a source/sink term; $\lambda_T{}^a$ is the turbulent diffusivity of a; and $\tilde{\lambda}_T{}^a = (L/H)^2\lambda_T{}^a$. In eqn. (8), D^a represents a horizontal diffusion term, the functional form of which generally involves the horizontal Laplacian of a. Whether or not $\tilde{D}^a$ should be equal to D^a is far from clear. As a matter of fact, the determination of the functional form of the "horizontal" diffusion in σ-coordinate is a very controversial matter (Pielke and Martin, 1981; Mellor and Blumberg, 1985; Deleersnijder and Wolanski, 1990). This is, however, not the subject of the present note. Hence, no detailed discussion of that problem will be done here.

Limitations to the use of the σ-coordinate system

When solving a differential equation by means of a finite difference method, it is customary to evaluate the truncation error to determine the precision of the scheme. For a derivative of m-th order with respect to an independent variable, y, one may write

$$N_{m,n}(y,\ a) = \frac{\partial^m a}{\partial y^m}(1+e), \qquad (10)$$

where $N_{m,n}$ denotes the numerical operator used to approximate $\partial^m a/\partial y^m$ with n-th order accuracy. Accordingly, the relative truncation error e may be evaluated by an asymptotic expression stemming from the Taylor expansion used to define $N_{m,n}$ (Roache, 1982),

$$e \sim \alpha_{m,n} \frac{\dfrac{\partial^{m+n} a}{\partial y^{m+n}}}{\dfrac{\partial^m a}{\partial y^m}} \Delta y^n, \ \Delta y \to 0, \tag{11}$$

where Δy is the mesh size associated with the discretization of the independent variable y. One generally has $\alpha_{m,n} < 1$.

Truncation error in x_3-coordinate

When no coordinate change is used, i.e., when the equations of the model are of the form (8), one must evaluate the order of magnitude of space derivatives that are either purely horizontal or purely vertical. By virtue of eqn. (3), one has

$$\frac{\partial^m a}{\partial x_i^m} \approx \frac{A_{\mathrm{h}}}{(L_{\mathrm{h}})^m}, \ i = 1, 2, \ m = 1, 2, 3, \ldots \tag{12}$$

and

$$\frac{\partial^m a}{\partial x_3^m} \approx \frac{A_{\mathrm{v}}}{(L_{\mathrm{v}})^m}, \ m = 1, 2, 3, \ldots \tag{13}$$

where A_{h} and A_{v} stand for the characteristic scales of variation of a in the horizontal plane and in the vertical plane, respectively.

Introducing eqns. (1), (2), (12) and (13) into eqn. (11), one obtains the order of magnitude of the relative truncation error for horizontal derivatives,

$$e_{\mathrm{h}} \approx \alpha_{m,n} (\varepsilon_{\mathrm{h}})^n, \tag{14}$$

and for vertical derivatives,

$$e_{\mathrm{v}} \approx \alpha_{m,n} (\varepsilon_{\mathrm{v}})^n. \tag{15}$$

Thus, as expected, ε_{h} and ε_{v} are relevant measures of the truncation error of the numerical scheme used to solve an equation of the form (8).

Truncation error in σ-coordinate

When turning to the σ-coordinate system, it is tempting to repeat the above analysis, straightforwardly replacing x_i by $\tilde{x}_i$ (i = 1, 2, 3) and concluding that eqns. (12)-(13), and thus eqns. (14)-(15) are still valid. As will be shown, doing this would be a major mistake.

In the remainder of this text, it will be assumed that $|\eta| \ll h$ so that

$$h \sim H, \tag{16}$$

which is almost always true.

For vertical derivatives, one uses the following scaling

$$\frac{\partial^m a}{\partial \tilde{x}_3^m} \approx \left(\frac{h}{L}\right)^m \frac{A_{\mathrm{v}}}{(L_{\mathrm{v}})^m}, \ m = 1, 2, 3, \ldots \tag{17}$$

so that the relative truncation error reads

$$e_{\mathrm{v}} \approx \alpha_{m,n} \frac{(h\Delta\sigma)^n}{(L_{\mathrm{v}})^n}, \tag{18}$$

which is equivalent to eqn. (15), because $h\Delta\sigma \sim \Delta x_3$.

Scaling derivatives in iso-σ surfaces as

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$$\frac{\partial^m a}{\partial \tilde{x}_i^m} \approx \frac{A_\mathrm{h}}{(L_\mathrm{h})^m},\ i=1,2,\ m=1,2,3,\ldots \tag{19}$$

is, in general, incorrect, because iso-σ surfaces are not horizontal. Hence, it is necessary to revert to an expression involving only horizontal and vertical derivatives so as to allow a proper scaling. Taking into account that

$$\frac{\partial}{\partial \tilde{x}_i} = \frac{\partial}{\partial x_i} - \left[(1-\sigma)\frac{\partial h}{\partial x_i} - \sigma\frac{\partial \eta}{\partial x_i}\right]\frac{\partial}{\partial x_3},\ i=1,2, \tag{20}$$

and assuming that eqn. (16) holds true, one gets

$$\frac{\partial^m a}{\partial \tilde{x}_i^m} \sim \left[\frac{\partial}{\partial x_i} - (1-\sigma)\frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_3}\right]^m a,$$
$$i=1,2,\ m=1,2,3,\ldots, \tag{21}$$

the order of magnitude of which is roughly given by

$$\frac{\partial^m a}{\partial \tilde{x}_i^m} \approx \max\left[\frac{A_\mathrm{h}}{(L_\mathrm{h})^m}, \left|\frac{\partial h}{\partial x_i}\right|^m \frac{A_\mathrm{v}}{(L_\mathrm{v})^m}\right],$$
$$i=1,2,\ m=1,2,3,\ldots. \tag{22}$$

It is appropriate to introduce the scale of variation of the depth as

$$L^d = \min_{i=1,2}\left(L_i^d = h\Big/\left|\frac{\partial h}{\partial x_i}\right|\right). \tag{23}$$

With the above definition, one may evaluate the magnitude of the relative error concerning derivatives in iso-σ surfaces:

$$e_\mathrm{h} \approx \alpha_{m,n}\max\left[(\varepsilon_\mathrm{h})^n, \left(\frac{h}{L_\mathrm{v}}\right)^n\left(\frac{\Delta x_\mathrm{h}}{L^d}\right)^n\right]. \tag{24}$$

Thus, requiring $\varepsilon_\mathrm{h} << 1$ is not sufficient to ensure accuracy. An additional condition on the bathymetry must be satisfied, i.e.

$$\left(\frac{h}{L_\mathrm{v}}\right)^n\left(\frac{\Delta x_\mathrm{h}}{L^d}\right)^n \ll 1, \tag{25}$$

or, equivalently,

$$\left(\frac{\Delta x_\mathrm{h}}{L^d}\right)^n \ll \left(\frac{L_\mathrm{v}}{h}\right)^n. \tag{26}$$

Since $(L_\mathrm{v}/h) \lesssim 1$, eqn. (26) means that the variations of the sea depth must be properly resolved by the computational grid.

Practically, it is generally difficult to determine beforehand the value of L_v/h. For simplicity, it is proposed to only require

$$\frac{\Delta x_\mathrm{h}}{L^d} \ll 1. \tag{27}$$

Thus, the bathymetry length scale, L^d, may be considered as any other length scale, and may be added to the list of horizontal length scales. Doing so, L_h, defined by eqn. (3), will account for depth variations and eqn. (27) will be part of eqn. (4).

The pressure gradient term

The evaluation of the relative truncation error holds for space derivatives of any variable, including the pressure. It is however desirable to work with a properly defined reduced pressure, as suggested in meteorology by Gary

(1973) and confirmed by Haney (1991) for marine applications.

If all the terms of an equation have the same order of magnitude, looking at the relative truncation error is satisfactory. By contrast, when there are terms that are much larger than others one must examine their absolute truncation error.

As is well known, the density of seawater, ρ, is at any instant and at any location, very close to a reference value ρ_0, i.e.

$$\left|\frac{\rho-\rho_0}{\rho_0}\right| \ll 1. \tag{28}$$

Since the aspect ratio is assumed to be small, the hydrostatic approximation holds (Pedlosky, 1979). One may decompose the pressure p as follows

$$p = p_0(x_3) + p'(t, x_1, x_2, x_3) \tag{29}$$

with

$$\frac{\mathrm{d}p_0}{\mathrm{d}x_3} = -\rho_0 g \tag{30}$$

and

$$\frac{\partial p'}{\partial x_3} = -(\rho-\rho_0)g, \tag{31}$$

where g denotes the gravitational acceleration. When the integration constant of eqn. (30) is well chosen, because of eqn. (28), the reduced pressure p' is much smaller than p_0,

$$|p'| \ll |p_0| \approx |p|. \tag{32}$$

In the momentum equations in x_3-coordinate, the pressure force reads

$$\frac{\partial p}{\partial x_i} = \frac{\partial p'}{\partial x_i}, \ i = 1, 2, \tag{33}$$

and, because of the relative smallness of p', no particular numerical problem is to be expected from this formulation.

In σ-coordinate, the horizontal pressure derivatives transform to

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \tilde{x}_i} + \frac{L}{H}\left[(1-\sigma)\frac{\partial h}{\partial x_i} - \sigma\frac{\partial \eta}{\partial x_i}\right]\frac{\partial p}{\partial \tilde{x}_3}, \quad i = 1, 2. \tag{34}$$

The right-hand member of eqn. (34) represents the sum of two large contributions that roughly cancel, resulting in a much smaller term, $\partial p'/\partial x_i$. Numerically, even if the relative error on each of the two large terms of eqn. (34) is small, it is far from certain that the absolute errors will roughly cancel each other, possibly resulting in an important error on $\partial p/\partial x_i$ (Gary, 1973; Janjic, 1977; Mesinger, 1982; Arakawa and Suarez, 1983; Haney, 1991).

Taking advantage of eqn. (33), one may write, in the σ-space, the horizontal pressure derivatives as

$$\frac{\partial p}{\partial x_i} = \frac{\partial p'}{\partial \tilde{x}_i} + \frac{L}{H}\left[(1-\sigma)\frac{\partial h}{\partial x_i} - \sigma\frac{\partial \eta}{\partial x_i}\right]\frac{\partial p'}{\partial \tilde{x}_3}, \quad i = 1, 2, \tag{35}$$

where all of the terms hopefully have similar magnitude. Thus, the latter form, though mathematically equivalent to eqn. (34), is, from a numerical point of view, better conditioned than eqn. (34) (Haney, 1991).

In agreement with the present discussion, the GHER three-dimensional marine model uses $q = p'/\rho_0$ and $b = -g(\rho - \rho_0)/\rho_0$ as state variables (Nihoul, 1984; Nihoul and Djenidi, 1987; Deleersnijder and Nihoul, 1988; Nihoul et al., 1989; Beckers, 1991).

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It must be stressed that, with the pressure gradient term, the problem is not just the truncation error discussed above. In the words of Haney (1991):" Rousseau and Pham (1971), Janjic (1977), and Mesinger (1982) have identified a problem of 'hydrostatic consistency' associated with the σ-coordinate system". The condition for hydrostatic consistency reads

$$\frac{\Delta x_{\mathrm{h}}}{L^{d}} < \frac{\Delta \sigma}{1-\sigma}, \tag{36}$$

which may be more restrictive than eqn. (27), depending on the vertical grid spacing and on the way "<<" in eqn. (27) is interpreted.

Discussion

There is no need to distinguish between the truncation error of time derivatives in the physical space and in the σ-space. Indeed, those time derivatives are related to one another by

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} + \sigma \frac{\partial \eta}{\partial t} \frac{\partial}{\partial x_{3}}, \tag{37}$$

so that, in general,

$$\frac{\partial}{\partial \tilde{t}} \sim \frac{\partial}{\partial t}, \tag{38}$$

because $|\eta| << L_{\mathrm{v}}$.

The above discussion rests on the underlying assumptions that $L_{\mathrm{v}} << L_{\mathrm{h}}$ and that L_{h} corresponds to variations that are strictly horizontal. The former assumption may not be questioned. By contrast, the latter is rather arbitrary because it suggests that the most natural coordinates to be used to describe marine processes are the physical cartesian coordinates. By analogy with boundary layer theory, Mellor and Blumberg (1985) assumed that the most appropriate coordinates are, in fact, terrain-following coordinates having one family of coordinate surfaces perpendicular to the bottom. It may be argued that the σ-coordinate system is very close to such a coordinate system, because the bottom slope is usually very small ($\lesssim 10^{-3}$). As a result, the scaling eqn. (19) becomes valid and, when analyzing the error in the x_3-coordinate system, one should revert to derivatives in the σ-space. In this case, it would be the x_3-coordinate system that should be subject to an accuracy requirement concerning the bathymetric variations.

What is the correct view? Near the bottom, for instance, the σ-coordinate system is clearly more appropriate than the cartesian one (Mellor and Blumberg, 1985; Deleersnijder and Wolanski, 1990). Away from the bottom boundary layer, it is not obvious that the iso-σ surfaces, which are not horizontal, correspond to preferential directions of variation of the dependent variables. Thus, assuming that L_{h} is associated with variations in horizontal planes is certainly safer than considering the iso-σ surfaces as the primary reference for the definition of L_{h}. Hence, condition (27) is justified. But this does not mean that the x_3-coordinate system is, overall, better suited to the modelling of marine processes, at least because it provides a less accurate representation of bottom topography.

If one considers the minimum of L^d over the computational domain, condition (27) prescribes an overall upper limit to the mesh size. But, eqn. (27) may also be used locally, serving as a guide line to variable mesh refinement. If, for some reason, it is not possible to define a grid that is fine enough to satisfy eqn. (27), another attitude must be adopted. In this case, the grid size is considered fixed and it is the bathymetry that must accomodate to the grid. In other words, the model bathymetry must be filtered to smear out the sharpest depth gradients—that are not properly resolved by the grid. Nevertheless, if the computational domain covers the shelf break, a region of large depth variations between shallow and deep waters, it is clear that no filtering of the bathymetry will be sufficient, unless the filtering procedure is so strong that the shelf break is completely smoothed out, which is definitely not appropriate. In this case, modifications to the standard σ-coordinate system must be considered. Those modifications should retain the advantages of the σ-coordinate while allowing a proper representation of the bathymetry in regions of large depth variations.

The two-fold σ-coordinate system

The GHER three-dimensional model is used to represent the general circulation in the Western Mediterranean (Beckers, 1988, 1991). The computational domain encompasses continental shelves of small extent ($h \lesssim 200$ m), the shelf break region with steep bottom slopes ($|\nabla h| \gtrsim 10^{-2}$), and the deep sea ($h \approx 3000$ m) (Fig. 1). From the sea surface to the sea bottom, one usually distinguishes three water masses: light Atlantic water from the surface to ≈ 200 m, a very sharp pycnocline, intermediate Levantine water to ≈ 500 m, a moderate pycnocline, and dense deep water from ≈ 500 m to the bottom. The general circulation in the Mediterranean is generally considered to

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be induced mainly by wind stress and spatially varying surface heating.

***Fig. 1.** Bathymetric map of the Western Mediterranean where the path of a plane of section is shown (AB).*

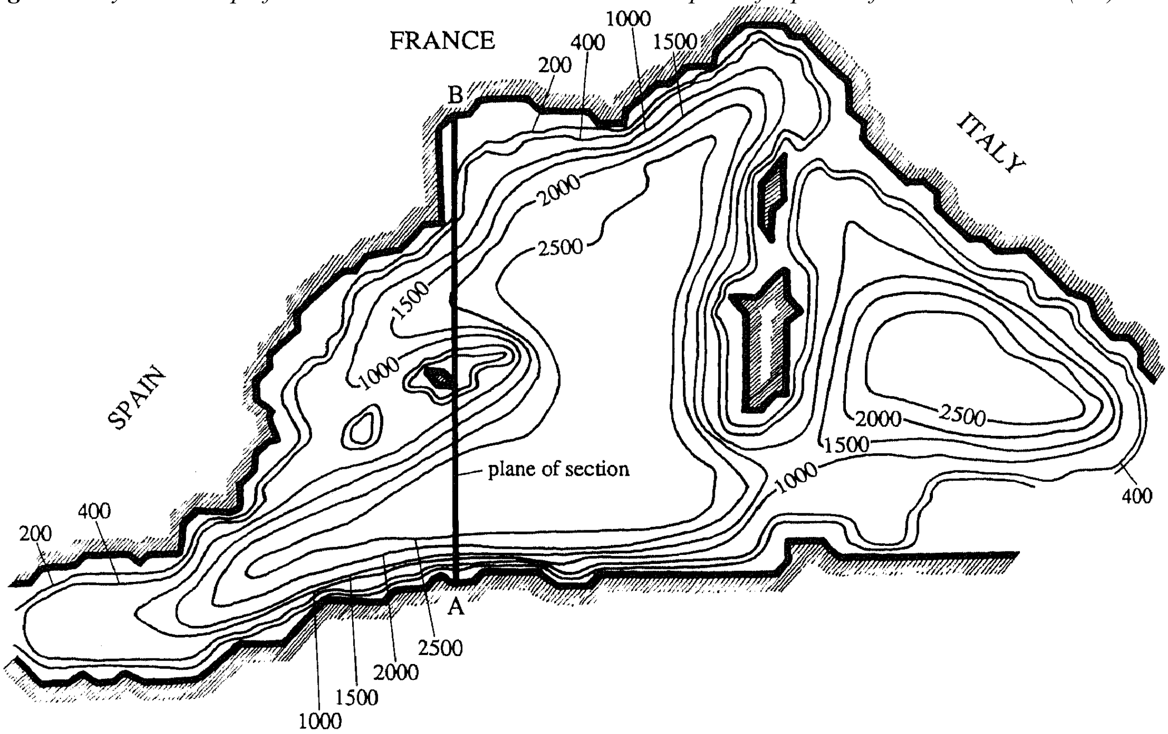
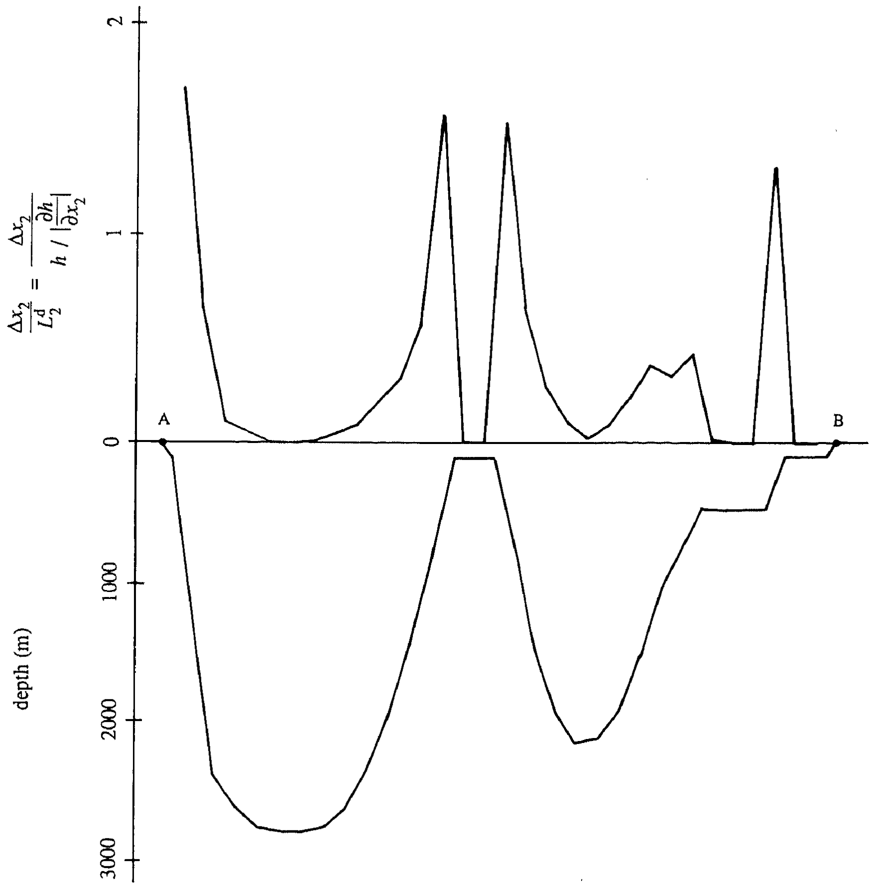


***Fig. 2.** Sea depth (lower curve) and ratio of the mesh size to the bathymetry length scale (upper curve) in the section AB (see Fig. 1). The lack of resolution of the bathymetric variations in the shelf break region is clearly shown.*



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It is thus necessary that the model have high resolution in the top hundreds meters so as to ensure a proper representaion of air-sea exchanges, of the associated turbulent processes, and of the pycnocline between Atlantic and Levantine waters. Consequently, the vertical length scale L_v must be considered a function of x_3 having its minimum value near the surface and the pycnocline. As $\Delta x_3 \sim h\Delta\sigma$, $h\Delta\sigma$ should also have its minimum in the same regions. However, if the pycnocline is horizontal—in the real space —, it will cross iso-σ surfaces in the σ-space, requiring thus $h\Delta\sigma$ to be small everywhere. Since the range of the bathymetry is large (50 m $\lesssim h \lesssim$ 3500 m), this would demand many vertical levels, which would be a waste of computer resources.

On the other hand, in the region of the shelf break L^d is much smaller than at any other location in the domain. Thus, a rather small horizontal mesh size is, in principle, required in order to avoid the occurence of severe numerical errors, as indicated by eqns. (27) and (36).

Applying the classical σ-coordinate system to the Western Mediterranean, where the range of the depth is large and where the minimum of the bathymetry length scale is small (Fig. 2), would require so fine vertical and horizontal grid sizes that the computer resources needed are unlikely to be afforded.

The above restrictions may be, to some extent, circumvented. Beckers (1988, 1991) suggests introducing a two-fold σ-coordinate system: the sea is divided into two sub-domains by a horizontal plane, the equation of which is $x_3 = -h_{\text{lim}}$. In each sub-domain, the σ-transformation is applied. Thus, in the upper sub-domain, one defines

$$\sigma^{\mathrm{U}} = \frac{x_3 + h_{\text{lim}}}{\eta + h_{\text{lim}}}, \tag{39a}$$

and, in the lower sub-domain,

$$\sigma^{\mathrm{L}} = \frac{x_3 + h}{h - h_{\text{lim}}}. \tag{39b}$$

Spall and Robinson (1990) used a similar technique, but did not introduce the σ-coordinate in the upper sub-domain. The vertical velocities are defined according to eqns (6) and (39),

$$\tilde{u}_3^{\mathrm{D}} = L^{\mathrm{D}}\left(\frac{\partial\sigma^{\mathrm{D}}}{\partial t} + u_1\frac{\partial\sigma^{\mathrm{D}}}{\partial x_1} + u_2\frac{\partial\sigma^{\mathrm{D}}}{\partial x_2} + u_3\frac{\partial\sigma^{\mathrm{D}}}{\partial x_3}\right), \tag{40}$$

where "D" stands for "U" or "L". At the interface $x_3 = -h_{\text{lim}}$, the vertical fluxes of the state variables of the model arising from the two sub-domains are simply matched. For instance, the matching of the water fluxes requires

$$\frac{h_{\text{lim}} + \eta}{L^{\mathrm{U}}}\left[\tilde{u}_3^{\mathrm{U}}\right]_{\sigma^{\mathrm{U}}=0} = \frac{h - h_{\text{lim}}}{L^{\mathrm{L}}}\left[\tilde{u}_3^{\mathrm{L}}\right]_{\sigma^{\mathrm{L}}=1}. \tag{41}$$

In the practice, matching the relevant fluxes is easily accomplished since the numerical method is based on the finite volume technique (Peyret and Taylor, 1983).

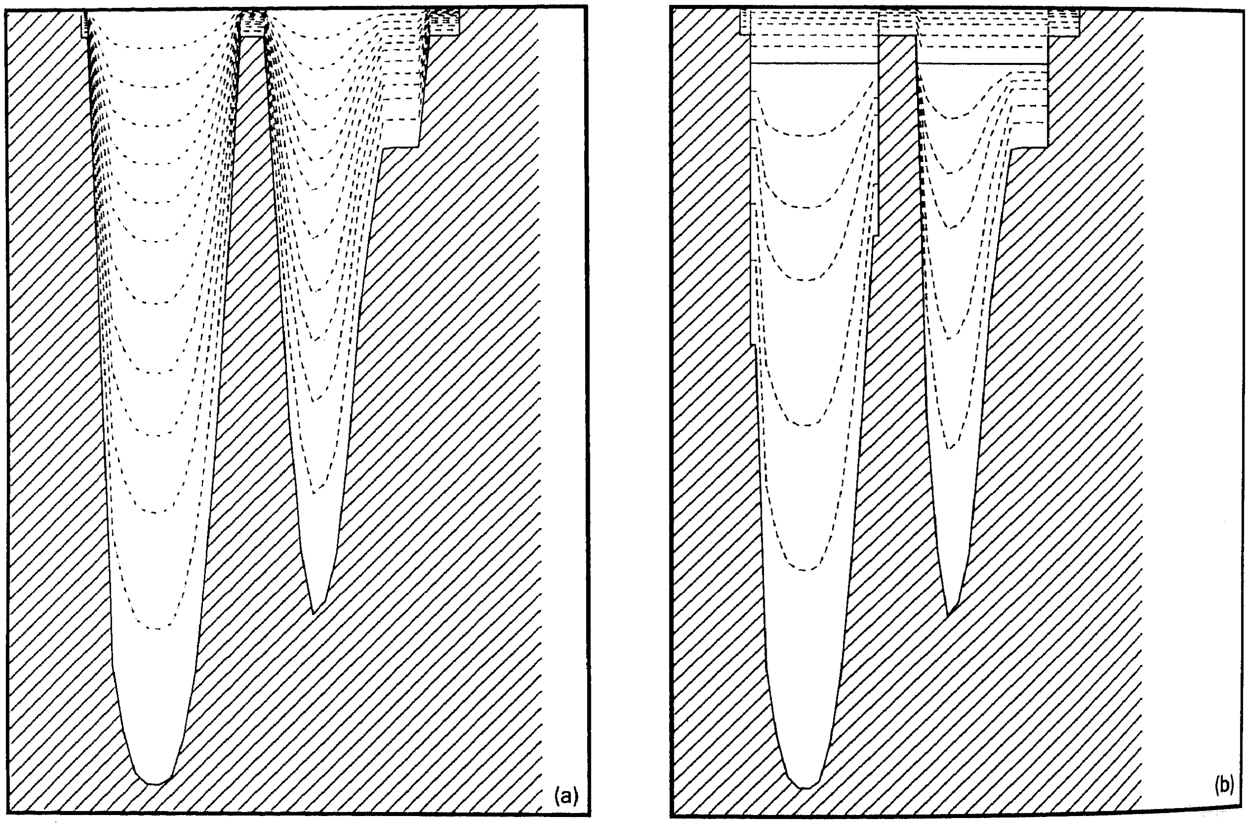
Of course, when $h < h_{\text{lim}}$, no lower sub-domain is needed. Beckers (1988, 1991) took

$$h_{\text{lim}} = 200 \text{ m} \tag{42}$$

so that the shelves involve only the upper sub-domain. The vertical mesh sizes were non-uniformly distributed in order to increase the resolution near the surface.

With the introduction of the two-fold σ-transformation, appropriate resolution in the top hundreds meters is guaranteed, without having recourse to a prohibitive number of grid points on the vertical. Furthermore, the steep slope of the shelf break region, where $\Delta x_h/L^d$ is maximum (Fig. 2), is partly replaced by a vertical wall (Fig. 3) so that conditions (27) and (36) are to some extent satisfied. It must be emphasized that it is the introduction of the vertical walls that reduces $\Delta x_h/L^d$ and not the two-fold σ-coordinate system per se. On the other hand, the vertical walls distort the bathymetry (Fig. 3) but it is deemed that this drawback is much less serious than the problems encountered when trying to represent steep slopes with the classical σ-coordinate system.

Fig. 3. *Iso-σ- surfaces in the section AB (see Fig. 1) for the classical σ-system (a) and the two-fold σ-system (b).*



Conclusion

The use of the σ-coordinate system, despite its many advantages, is subject to conditions ensuring the accuracy of the numerical derivatives. These conditions pertain to the variation of the sea depth and the main one states that a suitable bathymetry length scale must be introduced and plays the same role as any "classical" length scale in the determination of the mesh size of the model (27).

When the computational domain covers regions of very large bathymetric variations, such as the shelf break zone, the σ-coordinate system should not be used. However, in the case of the Western Mediterranean, the introduction of the twofold σ-transformation hopefully provides the model with all of the advantages of the classical σ-transformation, while avoiding some of the accuracy problems associated with large bathymetric irregularities. The model results (Beckers, 1991) indicate that the proposed vertical discretization is valid.

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