Multifractality of quantum wave packets

J. Martin[1], I. García-Mata[2], O. Giraud[3], B. Georgeot[4]

[1]Institut de Physique Nucléaire, Atomique et de Spectroscopie, Université de Liège, 4000 Liège, Belgium
[2]Instituto de Investigaciones Físicas de Mar del Plata, CONICET–UNMdP, Funes 3350, B7602AYL Mar del Plata, Argentina
[3]CNRS, LPTMS, UMR8626, Université Paris-Sud, 91405 Orsay, France
[4]CNRS, Université de Toulouse, UPS, Laboratoire de Physique Théorique (IRSAMC), 31062 Toulouse, France

Introduction

Fractality and multifractality have been studied in various areas of science. These ideas have found a new field of application in quantum mechanics. Multifractal wave functions characterize systems intermediate between quantum chaos and integrability, and also show up at the Anderson metal-insulator transition [1]. Experimental progress opens the way to direct observation of multifractality. It is therefore important to have a detailed analysis of the multifractal properties of wave packets in order to characterize and interpret experimental results.

The model (Ruijsenaars-Schneider)

We consider a periodically kicked system with period T and Hamiltonian $H(p,q) = p^2 + V(q)\sum_n \delta(t-nT)$, with potential $V(q) = \gamma\{q\}$ where $\{q\}$ denotes the fractional part of q, and (p,q) are the conjugated momentum and position variables. In order to allow for long spreading times for a wave packet, we fix $\hbar$ and let the classical phase space grow with N ("open" phase space). The quantization of this system gives the unitary evolution operator ($0 \leqslant P, P' \leqslant N-1$)

$$U_{PP'} = \frac{e^{i\Phi_P}}{N}\frac{1-e^{2i\pi\gamma}}{1-e^{2i\pi(P-P'+\gamma)/N}}$$

with $\Phi_P = -\pi T P^2$. In the following, we shall replace the kinetic phase Φ_P by random phases in order to get averaged quantities while keeping the same physics.

Analytical results

We consider the evolution of a wave packet initially localized on one single momentum state, $\Psi_P^{(0)} = \delta_{PP_0}$. Iterations of the map make the wave packet spread out. When γ is close to an integer, namely, $\gamma = k + \epsilon$, with k being an integer, analytical calculations show that the average wave packet over random phases Φ_P is given by [2]

$$\langle|\Psi_P^{(t)}|^2\rangle = \frac{\epsilon^2\pi^2 t}{N^2}\frac{1}{\sin^2\frac{\pi}{N}(P-P_0+kt)} \quad (P \neq P_0 - kt)$$

This formula implies that close to integer values of γ, the wave packet displays a single peak moving at speed k.

Numerical computation of the multifractal exponents of quantum wave packets

Different methods can be used in order to extract multifractal exponents. With the method of moments, multifractal exponents D_q ($q \in \mathbb{R}$) of wave functions living in a N-dimensional Hilbert space are computed from the scaling of their moments $\mathcal{P}_q = \sum_P |\Psi_P|^{2q}$ with N through $\langle\mathcal{P}_q\rangle \propto N^{-D_q(q-1)}$. However, this method is not suited for wave packets as it assumes scale invariance of the system. Another method (box-counting method) relies on the scaling with the box size N_{box} of the moments

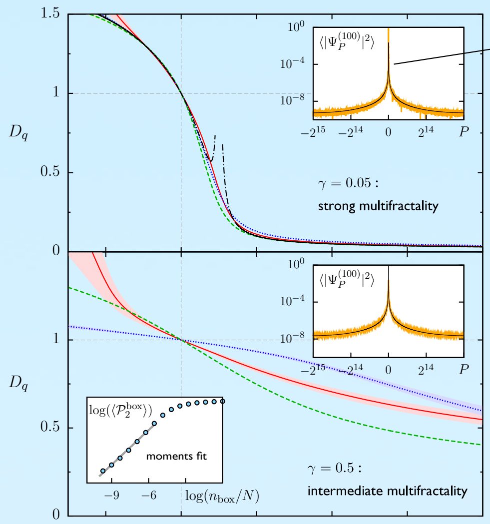
$$\mathcal{P}_q^{\text{box}} = \sum_{\text{boxes}}\left(\sum_{P\in\text{box}}|\Psi_P|^2\right)^q \sim N_{\text{box}}^{-D_q(q-1)}$$

In the case of wave packets, an average is made over the positions of the box centers to eliminate a threshold effect linked to the relative position of the wave packets and the boxes. Besides, as long as the wave packet remains localized, the multifractal exponents are extracted from scales smaller than the typical wave packet size.

We recall that the exponents D_q are positive and decrease for $q > 0$ from $D_0 = 1$; at a fixed $q > 0$, the smaller D_q is, the stronger the multifractality is. For an ergodic wave function one has $D_q = 1\ \forall q$.

[1] F. Evers and A. D. Mirlin, RMP 80, 1355 (2008).
[2] I. García-Mata et al., PRE 86, 056215 (2012).
[3] E. Bogomolny and O. Giraud, PRE 84, 036212 (2011).

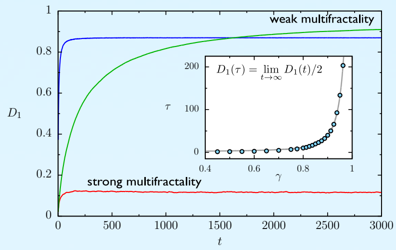
Multifractal exponents for wave packets and eigenvectors



Caption
Red solid curves: wave packet at $t = 100$; blue dotted curves: $t = \infty$ limit; green dashed curves: eigenvectors.

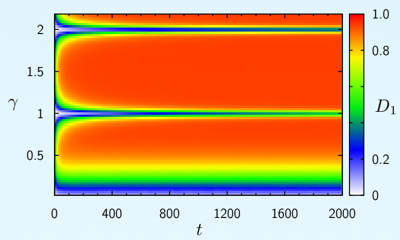
Studies of eigenvectors of the Ruijsenaars-Schneider map have shown that their multifractality is the strongest close to $\gamma = 0$ (localized eigenvectors) and decreases when γ gets close to nonzero integers (delocalized eigenvectors) [3].

Our results show that the multifractality of wave packets evolves with γ in the same way as for eigenvectors. Our interpretation is that the initial wave packet has significant components on more eigenvectors when these are delocalized (less multifractal), which in turn leads to an overall decrease of the multifractality as time evolution mixes these eigenfunctions.



Time evolution of the multifractal exponent D_1 for kick strengths $\gamma = 0.05, 0.5, 0.95$ (red, blue and green curves).

Density plot of D_1 as a function of kick strength γ and time t.



Conclusions

We identified the box counting method as the most efficient method to measure the multifractality of wave packets. In our periodically kicked system, the multifractality of wave packets was shown to typically decrease with time until it reaches an asymptotic limit. This asymptotic multifractality is different from the one of eigenvectors more commonly studied, but is related to it. The rate at which the asymptotic limit is reached can also be related to the multifractality of eigenvectors.