On the Number of Abelian Bordered Words

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Université de Liège

A word is *bordered*, if it has a proper prefix that is also a suffix of that word. Otherwise, it is *unbordered*.

A word $v = wu$ is a *nontrivial Duval extension* of the unbordered word w if v contains no unbordered factors of length $> |w|$.

Duval's conjecture (Harju–Nowotka J. ACM 2007)

If wu is a nontrivial Duval extension of w, then $|u| < |w|$.

$$w.u = abaabbabaababb.aaba$$

$\underline{a}baabbabaababb\underline{a}$ $\underline{baa}bbabaabab\underline{baa}$ $\underline{aab}babaababb\underline{aab}$ $\underline{a}bbabaababbaab\underline{a}$

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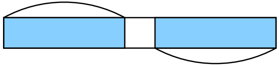

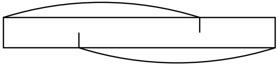

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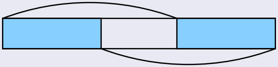
A word is *abelian bordered*, if it has a proper prefix that is a permutation of a suffix of that word.
Otherwise, it is *abelian unbordered*.

$$ababaabbbaabb$$

Remark

If w is abelian bordered, then it has a border of length $\leq |w|/2$.



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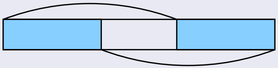

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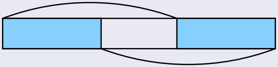

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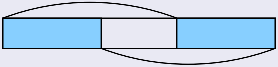

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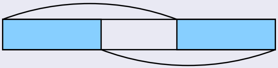

Let $A = \{a, b\}$. Consider the language B of abelian bordered words.

B is not context-free:

$$B \cap a^+ b^+ a^+ b^+ = \{a^i b^j a^k b^\ell : k \geq i \text{ and } j \geq \ell\}.$$

$$B = \{\overbrace{aa, bb}^{2}, \overbrace{aaa, aba, bab, bbb}^{4},$$

$$\underbrace{aaaa, aaba, abaa, abab, abba, baab, baba, babb, bbab, bbbb}_{10}, \ldots\}$$

$$(\#(B \cap A^n))_{n \geq 0} = 0, 0, 2, 4, 10, 20, 44, 88, 186, \ldots$$

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0, 0, 2, 4, 10, 20, 44, 88, 186, 372, 772, 1544, 3172, 6344, 12952, 25904, 5 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:0,0,2,4,10,20,44,88,186,372,772,1544,3172,6344,12952,25904,52666**

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If we consider instead, the language U of abelian unbordered words $U \cap aA^* = \{\ a,\ ab,\ aab,\ abb,\ aaab,\ aabb,\ abbb,\ aaaab,\ aaabb,\ aabbb,\ abbbb,\ aabab,\ ababb, \ldots\}$

 # A063886 Number of n-step walks on a line starting from the origin but not returning to it.

$(\#(U \cap A^n))_{n\geq 0}$ =1, 2, 2, 4, 6, 12, 20, 40, 70, 140, 252, 504, 924, 1848, 3432, 6864, 12870, 25740, 48620, 97240, 184756, 369512, 705432, 1410864, 2704156, 5408312, 10400600, 20801200, 40116600, 80233200, 155117520, 310235040, 601080390, 1202160780,...

We have to find a one-to-one correspondence...

Consider a word $w = u_1 \cdots u_n v_n \cdots v_1 \in \{a, b\}^*$ of length $2n$.

$$m(w) := \begin{pmatrix} u_1 & \cdots & u_n & v_n & \cdots & v_1 \\ v_1 & \cdots & v_n & u_n & \cdots & u_1 \end{pmatrix}$$

$$c : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto rr, \quad \begin{pmatrix} b \\ a \end{pmatrix} \mapsto \ell\ell, \quad \begin{pmatrix} a \\ a \end{pmatrix} \mapsto \ell r, \quad \begin{pmatrix} b \\ b \end{pmatrix} \mapsto r\ell.$$

Example

$$w = abaaaabb, \ m(w) = \begin{pmatrix} a & b & a & a & a & a & b & b \\ b & b & a & a & a & a & b & a \end{pmatrix}$$

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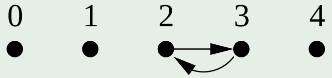

01232

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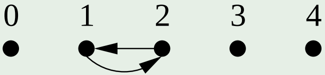

0123212

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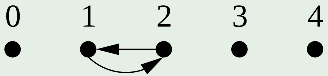

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Consider a word $w = u_1 \cdots u_n \, \alpha \, v_n \cdots v_1 \in \{a, b\}^*$ of length $2n+1$.

$$m(w) = \begin{pmatrix} u_1 & \cdots & u_n & \alpha & v_n & \cdots & v_1 \\ v_1 & \cdots & v_n & \alpha & u_n & \cdots & u_1 \end{pmatrix}$$

After the path of length $2n$, add an extra step to the left (resp. right) if $\alpha = a$ (resp. $\alpha = b$).

$$abab\,aabbb\,aabb, \quad m(w) = \begin{pmatrix} a\,ba\,b\,aabbbaabb \\ b\,ba\,a\,bbbaababa \end{pmatrix}$$

Easy observation

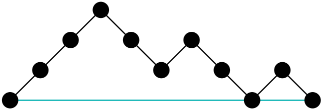
A word $w \in A^+$ is abelian bordered if and only if there exists a proper prefix p of $m(w)$ such that the numbers of occurrences of $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} b \\ a \end{pmatrix}$ in p are the same.

$$c : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto rr, \quad \begin{pmatrix} b \\ a \end{pmatrix} \mapsto \ell\ell, \quad \begin{pmatrix} a \\ a \end{pmatrix} \mapsto \ell r, \quad \begin{pmatrix} b \\ b \end{pmatrix} \mapsto r\ell.$$

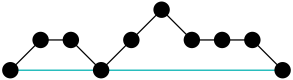
Corollary

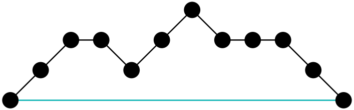
One-to-one correspondence between n-step walks on a line starting from the origin but not returning to it and abelian unbordered words of length n (over a 2-letter alphabet).

Dyck path

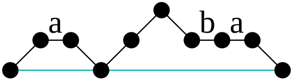


Motzkin path





Irreducible (or elevated)
Motzkin path

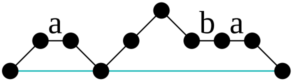


2−colored
Motzkin path

$$h : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto R, \ \begin{pmatrix} b \\ a \end{pmatrix} \mapsto F, \ \begin{pmatrix} a \\ a \end{pmatrix} \mapsto a, \ \begin{pmatrix} b \\ b \end{pmatrix} \mapsto b.$$

$$m(w) = \begin{pmatrix} a & a & b & a & a & b & b & a & b \\ b & a & a & b & b & a & b & a & a \end{pmatrix}$$

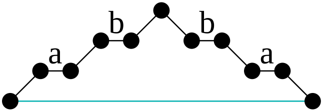
$$h(m(w)) = RaFRRFbaF$$



Lemma

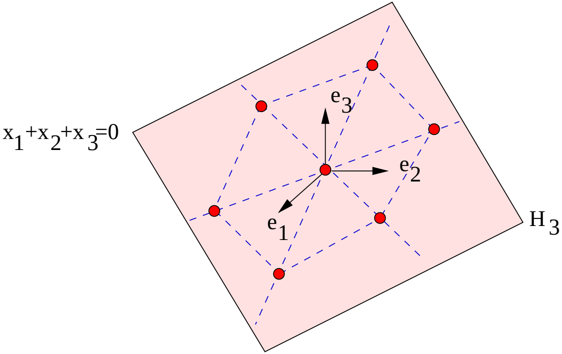
A word w starting with a is abelian unbordered if and only if $h(m(w))$ is a symmetric and irreducible Motzkin path.

$$\begin{pmatrix} a & a & a & b & a & b & b & b & a & b \\ b & a & b & b & b & a & b & a & a & a \end{pmatrix}$$



For larger alphabets

Consider the alphabet $A = \{1, \ldots, k\}$



Walks on a graph G_k where the set of vertices is

$\mathbf{H}_k = \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i, j \leq k\}^*$(i.e., all finite sums of these vectors)

and each vertex has $k(k-1)$ neighbors connected with the edges $\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$.

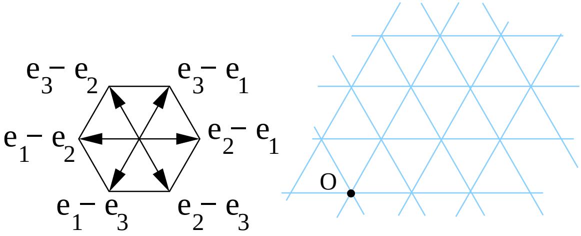
For larger alphabets

$$h_k : (A \times A)^* \to \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \le i, j \le k\}^* \subset (\mathbb{Z}^k)^*$$

$$h_k \begin{pmatrix} a_i \\ a_j \end{pmatrix} = \mathbf{e}_i - \mathbf{e}_j, \quad \forall i, j \in \{1, \dots, k\}.$$

k-colored path associated with a word, e.g., $w = 23321211$

$$\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$



For $k = 3$, we have the *triangular lattice*.

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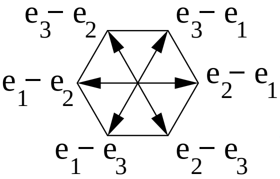

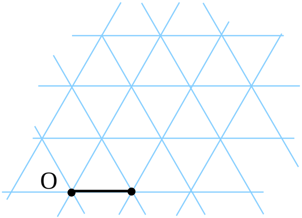

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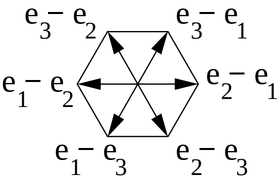

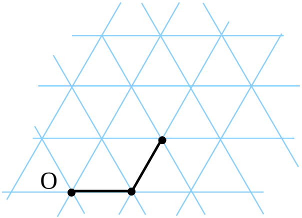

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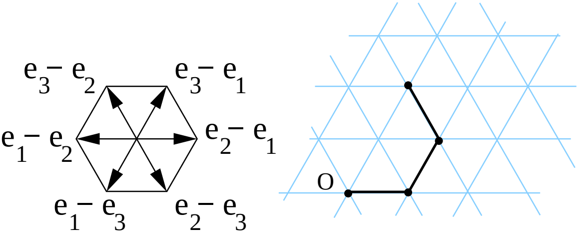
k-colored path associated with a word, e.g., $w = 23321211$

$$\begin{pmatrix} 2 & 3 & \color{red}{3} & 2 \\ 1 & 1 & \color{red}{2} & 1 \end{pmatrix}$$



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k-colored path associated with a word, e.g., $w = 23321211$

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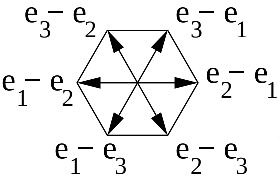
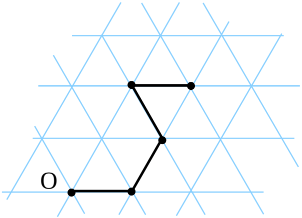
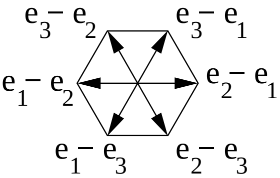


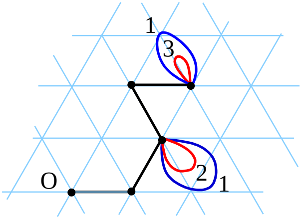


For $k = 3$, we have the *triangular lattice*.

$w' = 2321323113121211.$

$$\begin{pmatrix} 2 & 3 & 2 & 1 & 3 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 3 & 1 \end{pmatrix}.$$





Proposition

- A word of length $2n$ is abelian unbordered if and only if the k-colored path of length n starts from the origin but does not return to it.
- A word of length $2n+1$ is abelian unbordered if and only if the k-colored path starts from the origin but does not return to it. In particular, such a path ends with a loop whose color is the one corresponding to the central letter of w.

Corollary

To count abelian unbordered words, one has to count *these paths in the graph* G_k.

For a fixed k

$$\boxed{p(n) = \sum_{j=1}^{n} s(j)\, k^{n-j} \binom{n-1}{n-j}.}$$

$p(n)$: number of paths of length n (not returning to $\mathbf{0}$)
$s(n)$: number of *simple* paths of length n (not returning to $\mathbf{0}$)

$$s(n+1) = k(k-1)(s(n) - \gamma_n)$$

$\gamma_0 = 0$ and

$$\gamma_{n+1} = \frac{\mathcal{S}_{n+2,\mathbf{0}}}{k(k-1)} - \sum_{i=0}^{n} \gamma_i \mathcal{S}_{n-i+1,\mathbf{0}}$$

where $\mathcal{S}_{n,\mathbf{0}}$ is the number of (simple) paths of length n starting at $\mathbf{0}$ and going back at $\mathbf{0}$

homomorphism $\chi : (\mathbf{H}_k, +) \to (\mathbb{Z}((z_1, \ldots, z_{k-1})), \cdot)$ defined by the images of a basis of $\mathbf{H}_k$

$$\chi : \begin{cases} \mathbf{e}_1 - \mathbf{e}_k & \mapsto & z_1 \\ \mathbf{e}_2 - \mathbf{e}_k & \mapsto & z_2 \\ & \vdots & \\ \mathbf{e}_{k-1} - \mathbf{e}_k & \mapsto & z_{k-1}. \end{cases}$$

$$T = \sum_{i=1}^{k-1} \left(z_i + \frac{1}{z_i} \right) + \sum_{i \neq j} \frac{z_i}{z_j} = \left(1 + \sum_{i=1}^{k-1} z_i \right) \left(1 + \sum_{i=1}^{k-1} \frac{1}{z_i} \right) - k.$$

Let $\mathbf{x} \in \mathbf{H}_k$ be such that $\chi(\mathbf{x}) = z_1^{j_1} \cdots z_{k-1}^{j_{k-1}}$.

- The number of paths of length n from $\mathbf{0}$ to $\mathbf{x}$ is given by the coefficient of $z_1^{j_1} \cdots z_{k-1}^{j_{k-1}}$ in T^n.
- The constant term is the number of paths of length n starting at $\mathbf{0}$ and going back at $\mathbf{0}$, i.e., $\mathcal{S}_{n,\mathbf{0}}$

Unbordered abelian factors for Thue–Morse

$$\mathbf{t} = 0110100110010110100101100110 1001\cdots$$

Currie and Saari 2009

If $n \not\equiv 1 \pmod 6$, then $\mathbf{t}$ has an unbordered factor of length n.

Goč, Henshall and Shallit 2012

$\mathbf{t}$ has an unbordered factor of length n IFF $(n)_2 \notin 1(01^*0)^*10^*1$.

For abelian unbordered factors of length n:
we obtain a strict subset of the one described above.
For $n = 9$, $(n)_2 = 1001 \notin 1(01^*0)^*10^*1$ but
all factors of length 9 occurring in $\mathbf{t}$ are abelian bordered, e.g.,

001100101 is unbordered but abelian bordered.

By a computer search, the first few values of $n \in \{0, 2000\}$ such that $\mathbf{t}$ has an unbordered abelian factor of length n are

$$0, 1, 2, 3, 5, 8, 10, 12, 14, 16, 22, 50, 54, 66, 70, 194, 198,$$

$$258, 262, 770, 774, 1026, 1030$$

CONJECTURE

$\mathbf{t}$ has an unbordered abelian factor of length $n \geq 50$ if and only if

$$(n)_2 \in 110(00)^*\{01, 11\}0 \cup 10(00)^+\{01, 11\}0.$$

Whether or not this conjecture holds true, *we know precisely how to derive the correct expression*.

We take verbatim *Büchi's theorem* as stated by Charlier, Rampersad and Shallit expressing that *k-automatic sequences are exactly the sequences definable in the first order structure* $\langle \mathbb{N}, +, V_k \rangle$.

Theorem

If we can express a property of a k-automatic sequence $\mathbf{x}$ using quantifiers, logical operations, integer variables, the operations of addition, subtraction, indexing into $\mathbf{x}$, and comparison of integers or elements of $\mathbf{x}$, then this property is decidable.

"This algorithmic methodology does not seem to be applicable to any questions concerning "abelian" properties of words..."

L. Schaeffer (2013)

The set of occurrences of abelian squares occurring in the (2-automatic) paper folding word is not definable in $\langle \mathbb{N}, +, V_k \rangle$ for any $k \geq 2$.

Let $\mathbf{x}$ be a k-automatic sequence.

- Same factor of length n occurring in position i and j

$$F_{\mathbf{x}}(n,i,j) \equiv (\forall k < n)(\mathbf{x}(i+k) = \mathbf{x}(j+k))$$

- First occurrence of a factor of length n occurring in position i

$$P_{\mathbf{x}}(n,i) \equiv (\forall j < i)\neg F_{\mathbf{x}}(n,i,j)$$

The set $\{(n,i) \mid P_{\mathbf{x}}(n,i) \text{ true}\}$ is k-recognizable and

$$\forall n \geq 0, \quad \#\{i \mid P_{\mathbf{x}}(n,i) \text{ true}\} = p_{\mathbf{x}}(n).$$

Let $\mathbf{x}$ a k-automatic sequence.

- Two factors of length n occurring in position i and j are abelian equivalent

$$A_{\mathbf{x}}(n,i,j) \equiv (\exists \nu \in S_n)(\forall k < n)(\mathbf{x}(i+k) = \mathbf{x}(\nu(j+k)))$$

  The length of the formula is $\simeq n!$ and grows with n.

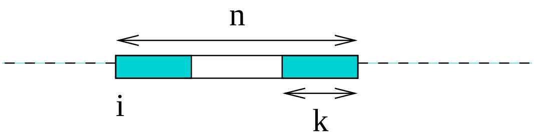
- First occurrence (up to abelian equivalence) of a factor of length n occurring in position i

$$AP_{\mathbf{x}}(n,i) \equiv (\forall j < i) \neg A_{\mathbf{x}}(n,i,j)$$

For a constant n. The set $\{i \mid AP_{\mathbf{x}}(n,i) \text{ true}\}$ is k-recognizable and

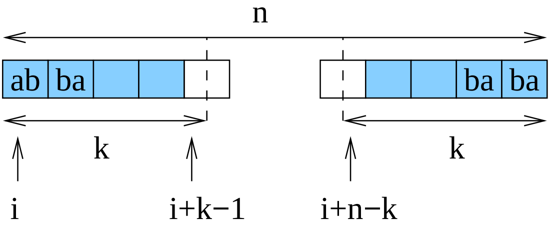
$$\#\{i \mid AP_{\mathbf{x}}(n,i) \text{ true}\} = a_{\mathbf{x}}(n).$$

$B(i,n,k)$ is true if and only if $\mathbf{t}$ has an abelian bordered factor of length n occurring at i with a border of length k:



$$
\begin{array}{lll}
 & & (e(i) \wedge e(n) \wedge e(k)) \\
\vee & & (e(i) \wedge e(n) \wedge o(k) \wedge \mathbf{t}(i+k-1) = \mathbf{t}(i+n-k)) \\
\vee & & (e(i) \wedge o(n) \wedge e(k) \wedge \mathbf{t}(i+n-k) \neq \mathbf{t}(i+n-1)) \\
\vee & & (e(i) \wedge o(n) \wedge o(k) \wedge \mathbf{t}(i+k-1) = \mathbf{t}(i+n-1)) \\
\vee & & \{o(i) \wedge e(n) \wedge e(k) \\
 & \wedge & [(\mathbf{t}(i) = \mathbf{t}(i+n-k) \wedge \mathbf{t}(i+k-1) = \mathbf{t}(i+n-1)) \\
 & & \vee(\mathbf{t}(i) = \mathbf{t}(i+n-1) \wedge \mathbf{t}(i+k-1) = \mathbf{t}(i+n-k))]\} \\
\vee & & (o(i) \wedge e(n) \wedge o(k) \wedge \mathbf{t}(i) = \mathbf{t}(i+n-1)) \\
\vee & & (o(i) \wedge o(n) \wedge e(k) \wedge \mathbf{t}(i) \neq \mathbf{t}(i+k-1)) \\
\vee & & (o(i) \wedge o(n) \wedge o(k) \wedge \mathbf{t}(i) = \mathbf{t}(i+n-k))
\end{array}
$$

$$(e(i) \wedge e(n) \wedge o(k) \wedge \mathbf{t}(i+k-1) = \mathbf{t}(i+n-k))$$



Now the Thue–Morse word has an abelian unbordered factor of length n if and only if the formula

$$\varphi(n) \equiv (\exists i)(\forall k)(k \geq 1 \wedge 2k \leq n) \rightarrow \neg B(i, n, k)$$

holds true.

For the Fibonacci word

Exactly two unbordered factors of length n occurring in $\mathbf{f}$ IFF n is a Fibonacci number. Otherwise, no unbordered factors of length n in $\mathbf{f}$.

The same result holds for abelian unbordered factors.

- K. Saari, Lyndon words and Fibonacci numbers, `arXiv:1207.4233`.
- W.-F. Chuan, Unbordered factors of the characteristic sequences of irrational numbers, *Theoret. Comput. Sci.* **205** (1998), 337–344.