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REVISITING EARLY WORKS OF THE AEROSPACE LABORATORY OF LIEGE
Dual analysis, 60's and 70's

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SUMMARY

The goal of this presentation is to describe the main scientific contribution of the Aerospace Laboratory of the Liège University (LTAS) during the sixties and seventies. This period was very important for the development of the finite element techniques and the major part of the concepts and the algorithms were defined at that time. The contribution of the LTAS was especially original because it was the only team in the world who was proposing the dual method and included it in a Finite Element code for all the applications : bars, membrane and plates, using for both types of models the same displacement solution algorithm. After the presentation of this method a short description is made about recent improvements proposed by the authors.

1. INTRODUCTION

It seems strange to present thirty years later a scientific contribution that was well known at that time. Moreover, young searchers use to read only recent papers and generally do not try to study the origin of a method or the steps of its effective implementation.

Today with the necessity of ensuring the quality and the reliability of the finite element computations the foundations of the method have to be well understood and major contributions of the first period of development reveal to be very useful especially in the context of the estimation of the discretization error which is the main ingredient of the quality control of the analysis.

2. THE DUAL ANALYSIS CONCEPT

2.1.- Historical sketch

The dual analysis concept was obviously the main guideline of the research performed at the LTAS during the 60's and 70's. The words "dual analysis" or "upper and lower bounds" appear in the title of 16 papers and there are a lot of other papers where the concept was used.

The first mention to the dual analysis appeared in a paper in french language, due to B. Fraeijs de Veubeke [1], in 1961. It is of a general nature, and not yet related to finite elements which were not used at that time in Liège.

A second paper [2] was presented in 1962, where the dual analysis procedure is applied in matrix structural analysis. Indeed, it is also the first presentation of an equilibrium membrane element, but the words "finite elements" never appear.

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The major contribution was certainly the famous paper entitled "Displacement and equilibrium models in the finite element method" presented at Swansea in 1964. Here a complete treatment of all possible variational formulations is given. Special emphasis is set on pure models, where compatibility or equilibrium is verified exactly, because these models are the key of the dual analysis allowing an *a posteriori* convergence measurement.

All subsequent papers consisted to apply these concepts by developing new elements or covering new fields, such as thermal analysis, plasticity, and so on.

2.2.- An elementary physical approach

It is perhaps useful to begin with a very simplified approach insisting on physical ideas which are less apparent in a mathematical analysis. Let us consider a structure with one single load F, and let u be the corresponding displacement. The stiffness coefficient is then (fig. 1)

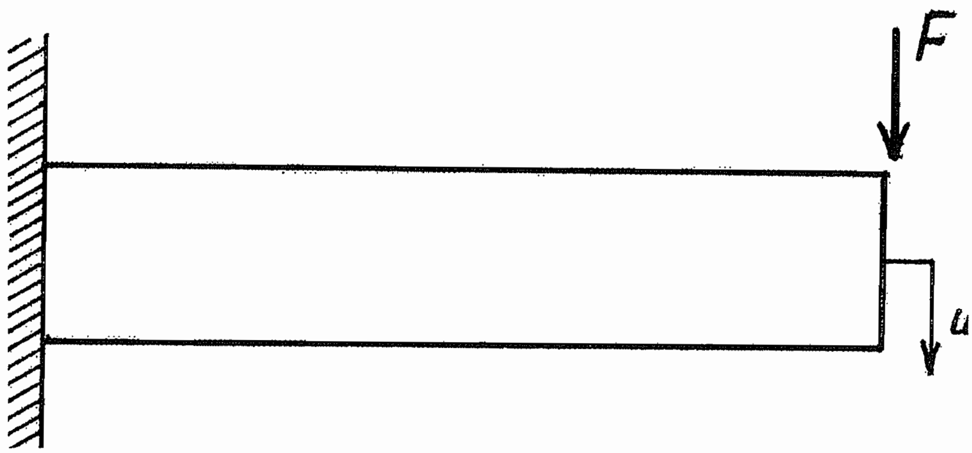

Figure 1 : Simplified Approach

$$k = F / u, \tag{1}$$

and the work of the load is given by

$$T = Fu = ku^2 = F^2 / k \tag{2}$$

When using a compatible displacement model, restrictions on the possible displacements are done, so that the computed stiffness is overestimated.

$$k_{dep} \geq k \tag{3}$$

Conversely, with an equilibrium models, possible stress modes are restricted, so that the stiffness is too low :

$$k_{éq} \leq k \tag{4}$$

Let us first suppose that the load is given. In this case, the displacement model leads to a lower bound of the work

$$T_{dép} = \frac{F^2}{k_{dep}} \leq \frac{F^2}{k} = T \tag{5}$$

and the equilibrium model leads to an upper bound of the work,

$$T_{éq} = \frac{F^2}{k_{éq}} \geq \frac{F^2}{k} = T, \tag{6}$$

If, now, the displacement is given, one has

$$T_{dep} = k_{dep} u^2 \geq ku^2 = T \tag{7}$$

and

$$T_{eq} = k_{éq} u^2 \leq ku^2 = T \tag{8}$$

that means that the bounds are reversed.

In each case, by refining the finite element mesh, both approaches converge to the true value of the stiffness,

$$k_{dep} \downarrow k,\ k_{éq} \uparrow k$$

and the comparison of both results lead to a suitable convergence test.

2.3.- The general elasticity problem

The general elasticity problem may be described as follows. A domain Vol., with a boundary Γ is submitted to surface loads of the form $f_i dV$, and to side loads of the form $t_i\, d\,\Gamma$ on a part Γ_2 of its boundary. The complementary part $\Gamma_1 = \Gamma - \Gamma_2$ is subjected to prescribed displacements $\overline{u}_i$.

The displacement approach consists of finding, among each displacement field that comply on Γ_1 with the prescribed values, the particular one that makes the total energy minimum,

$$TE(u) = U(u) + P(u) \text{ min}, \tag{9}$$

with

$$U(u) = strain\ energy = \int_{Vol} W(E)dV, \tag{10}$$

$$W(E) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl},\ \varepsilon_{ij} = strain = \frac{1}{2}(D_i u_j + D_j u_i) \tag{11}$$

$$P(u) = load\ potential = -\int_{Vol} f_i u_i dV - \int_{\Gamma_2} t_i u_i d\Gamma \tag{12}$$

As well known, the minimum corresponds to equilibrium, in the form

$$\left. \begin{array}{l} D_j\sigma_{ij} + f_i = o \ in \ Vol \\ n_j\sigma_{ji} = t_i \ on \ \Gamma_2 \end{array} \right\} \tag{13}$$

where the stresses σ_{ij} are related to the strains by

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{14}$$

The equilibrium approach consists of finding, among each stress fields verifying the equilibrium conditions (13), the particular one that makes the total complementary energy minimum,

$$CE(\sigma) = V(\sigma) + Q(\sigma) \text{ min} \tag{15}$$

with

$$V(\sigma) = stress \ energy = \int_{Vol} \Phi(\sigma)\, dV, \tag{16}$$

$$\Phi(\sigma) = \frac{1}{2}C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl} \tag{17}$$

$$Q(\sigma) = reaction \ potential = -\int_{\Gamma_1} n_j\sigma_{ji}\bar{u}_i d\Gamma \tag{18}$$

Here, the minimum corresponds to compatibility.

2.4.- Kinematical admissibility

A displacement field is kinematically admissible if

- it leads to a finite strain energy
- it verifies $u_i = \bar{u}_i$ on Γ_1.

The set of kinematically admissible displacements fields will be noted K.
Any difference

$$\delta\, u_i = u_i - v_i \tag{19}$$

between two kinematically admissible displacements is called a displacement variation. On Γ_1, it verifies

$$\delta\, u_i = \bar{u}_i - \bar{u}_i = 0 \tag{20}$$

The set of displacement variations is a linear space δK. Moreover, if a particular displacement field, which complies with the condition on S_1, is known, it may be noted $\bar{u}$ without risk of confusion, and one has

$$K = \bar{u} + \delta\, K \tag{21}$$

Displacement finite element models consist of seeking a subset K_h of K, to which corresponds the variation subspace δK_h. For the purpose of the dual analysis, it is essential that K_h contains displacements that exactly verify $u_h = \overline{u}$ on S_1, that means that, the displacement field $\overline{u}$ in (21) can be chosen in K_h. The exact problem is to find $u \in \overline{u} + \delta K$ such that TE(u) is minimum. Its F.E. approximation is

$$\text{Find } u_h \in \overline{u} + \delta K_h \text{ such that } TE(u_h) \text{ is minimum.} \quad (22)$$

Using this approximation, the local equilibrium is lost and replaced by more global conditions of equilibrium.

2.5.- Static admissibility

A stress field is statically admissible if

- it leads to a finite stress energy
- it verifies the equilibrium equations (13).

The set of statically admissible stress fields will be noted S.

Any difference

$$\delta \sigma_{ij} = \sigma_{ij} - \tau_{ij} \quad (23)$$

between two statically admissible stress fields is called a stress variation.
It verifies

$$D_j \sigma_{ji} = 0 \text{ in Vol}, \; n_j \sigma_{ji} = 0 \text{ on } \Gamma_2 \quad (24)$$

It is thus a self-stress. The set of stress variations is a linear space δS.
Assuming that a particular solution $\overline{\sigma}$ of equilibrium is known, one has

$$S = \overline{\sigma} + \delta S \quad (25)$$

Equilibrium finite element models consist of seeking a subset S_h of S, to which corresponds the variation subspace δS_h. Here also, the dual analysis needs that S_h contains an exact solution of equilibrium (it is a restriction). The finite element approximation consists of

$$\text{Find } \sigma_h \in \overline{\sigma} + \delta S_h \text{ such that } CE(\sigma) \text{ is minimum.} \quad (26)$$

2.6.- A property of the exact solution

The exact solution verifies

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl},$$

from which

$$W(\varepsilon) + \Phi(\sigma) = \sigma_{ij} \varepsilon_{ij} \quad (27)$$

Now, by the classical virtual work theorem,

$$\int_{Vol} \sigma_{ij}\varepsilon_{ij}dV = \int_{Vol} f_i u_i dV + \int_{\Gamma_2} t_i u_i dS + \int_{\Gamma_1} n_j \sigma_{ji}\bar{u}_i dS$$

and making use of property (27), one has

$$TE(u) + CE(\sigma) = 0 \tag{28}$$

2.7.- Bounds on the total energies

Because the finite element subsets verify

$$K_h \subset K, S_h \subset S,$$

for a kinematically admissible F.E.M.,

$$TE(u_h) = \min_{K_h} TE \geq \min_{K} TE = TE(u) \tag{29}$$

and, for a statically admissible F.E.M.,

$$CE(\sigma_h) = \min_{S_h} CE \geq \min_{S} CE = CE(\sigma) \tag{30}$$

Therefore, from (28),

$$TE(u_h) + CE(\sigma_h) \geq 0 \tag{31}$$

This result was known by Fraeijs de Veubeke [1] but surprisingly, never used as stated here.

2.8.- First problem - zero prescribed displacements

In this case, Q = 0 and CE = V. Moreover, K = δK, K_h = δK_h and the solution itself is a variation. Setting δu = u in the first variation of U leads to

$$\delta U = \int_{Vol} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV = 2U(u)$$

so that the condition δCE = 0 gives

$$2U(u) = -P(u),$$

and

$$TE(u) = U(u) + P(u) = -U(u) \tag{32}$$

From (28), one obtains

$$0 = TE(u) + CE(\sigma) = -U(u) + V(\sigma),$$

so that

$$U(u) = V(\sigma) \tag{33}$$

Now, result (32) is also true for u_h, by the same way, and

$$-U(u_h) = TE(u_h) \geq TE(u) = -U(u),$$

from which

$$U(u_h) \leq U(u) \tag{34}$$

Concerning the equilibrium model,

$$V(\sigma_h) = CE(\sigma_h) \geq CE(\sigma) = V(\sigma) \tag{35}$$

As U(u) = V(σ), the energy bounds are obtained.

2.9.- Second problem - zero applied loads

Here, P = 0, and by the same way as for the first problem, the converse result is obtained.

$$U(u_h) \geq U(u) = V(\sigma) \geq V(\sigma_h) \tag{36}$$

3. WORKABILITY

After establishing the concept, it remained to solve a lot of problems.

3.1.- Elements

At that time, equilibrium elements were not available, if one excepts the old shear panels and connected bars. Conforming plates were not available.

3.2.- Solution algorithms

If displacement elements naturally suggest a stiffness matrix process (displacement solution algorithm), the same is not true for equilibrium elements, for which a force method, with structural self-stresses determination, seemed more natural. But the force method is very difficult to automate. Here, Fraeijs de Veubeke showed from the earliest stage that it is possible to use equilibrium elements in a stiffness formulation [2]. It is also a major contribution of the LTAS.

3.3.- F.E. code

Another difficulty appeared at the coding level. In practice, most F.E. programs were based on the nodes at which the degrees of freedom are attached. But in the equilibrium elements, the degrees of freedom are mean displacements not related to nodes. It was thus a necessity to build a code without any node concept, the displacements being related to the elements.

4. STIFFNESS OBTENTION IN EQUILIBRIUM ELEMENTS

The way to obtain the stiffness matrix in an equilibrium element will be illustrated by the simple case of a constant stress triangle. The stresses are given by

$$\sigma_x = \alpha_1, \sigma_y = \alpha_2, \tau = \alpha_3 \tag{37}$$

This may be summarized as

$$s = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{bmatrix} = N \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = Na \tag{38}$$

Of course, with such a stress field, no body load is admitted.

On the boundary, one has to transmit the interface loads on each side. On side 1-2, whose length is l_{12},

$$g_x^{12} = l_{12}(n_x^{12}\sigma_x + n_y^{12}\tau)$$

$$g_y^{12} = l_{12}(n_x^{12}\tau + n_y^{12}\sigma_y) \tag{39}$$

and so on. This leads to a relation of the form

$$g = \begin{bmatrix} g_x^{12} \\ g_y^{12} \\ g_x^{23} \\ g_y^{23} \\ g_x^{31} \\ g_y^{31} \end{bmatrix} = Ca \tag{40}$$

If H is the Hooke matrix, the stress energy is obtained from

$$V = \frac{1}{2}\int_{elt} s^T H^{-1} s \, dxdy, = \frac{1}{2} a^T \left(\int_{elt} N^T H^{-1} N \, dxdy \right) a = \frac{1}{2} a^T sa \tag{41}$$

Supposing that the side load vector is given, one has to minimize this energy with the condition (40), and this may be done by making use of a Lagrange multiplier vector q : we thus make the expression

$$\frac{1}{2} a^T Ja + q^T (g - Ca) \tag{42}$$

stationary. Varying a, one obtains

$$Ja = C^T q , \tag{43}$$

from which a direct interpretation of q is available. For a variation δa, one obtains

$$\delta V = \delta a^T Ja = \delta a^T C^T q = \delta g^T q$$

where use is made of (40). The multipliers are thus displacements working with the side loads. (more precisely, they are mean interface displacements). Now, from (43) and (40),

$$a = J^{-1} C^T q$$

and

$$g = Ca = CJ^{-1} C^T q = Kq \tag{44}$$

This is the stiffness matrix, relating the surface tractions to the conjugated displacements.

Two points have to be noted. Firstly, the J matrix is always invertible, as the stress energy is positive definite. The due singularities of the stiffness matrix come from the connection matrix C. Unfortunately, for higher degrees, the rank of this matrix is too low and kinematical modes appear.

Secondly, the degrees of freedom of this element are interfacial displacements, not nodal displacements (fig. 2). This is why a code using such elements may not be based on nodal definitions.

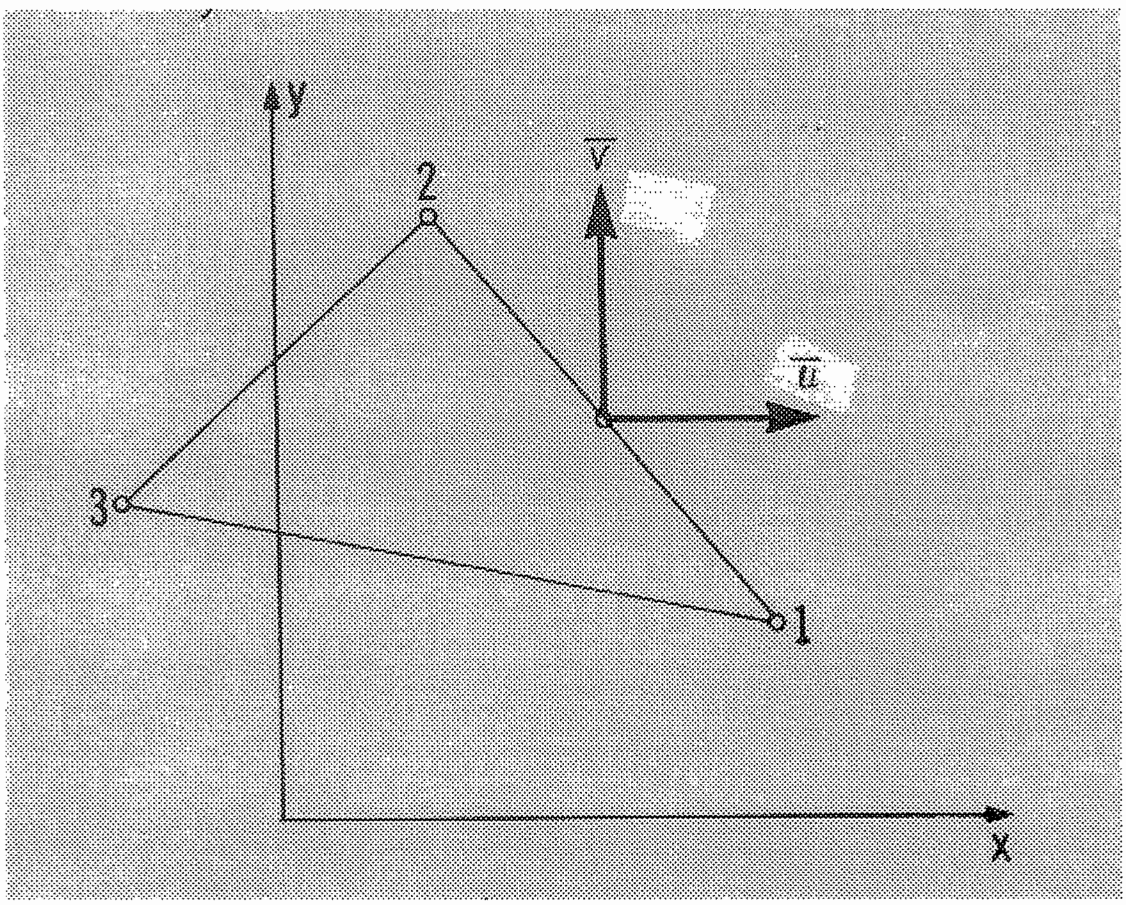

Figure 2 : Connections of the constant stress element

5. KINEMATICAL MODES

The constant stress element having somewhat deficient connections, it was natural to turn to higher degree elements. The first degree element is characterized by 4 displacements on each side, which are conjugated to the resultants

$$g_x^{(1)} = \int_o^l \left(1 - \frac{s}{l}\right) t_x ds, \qquad g_x^{(2)} = \int_o^l \frac{s}{l} t_x ds \tag{45}$$

and so on. Unfortunately, it exhibits two kinematical modes, corresponding to the possible displacements of the hatched "skeleton" represented on fig. 3. However, the superelement obtained by assembling three such elements has no more kinematical modes, as illustrated on fig. 4. Moreover, the superelement obtained by assembling the four elements defined on a quadrilateral by its diagonals has only one kinematical mode, which is purely internal and does not work for interface loads (fig. 5).

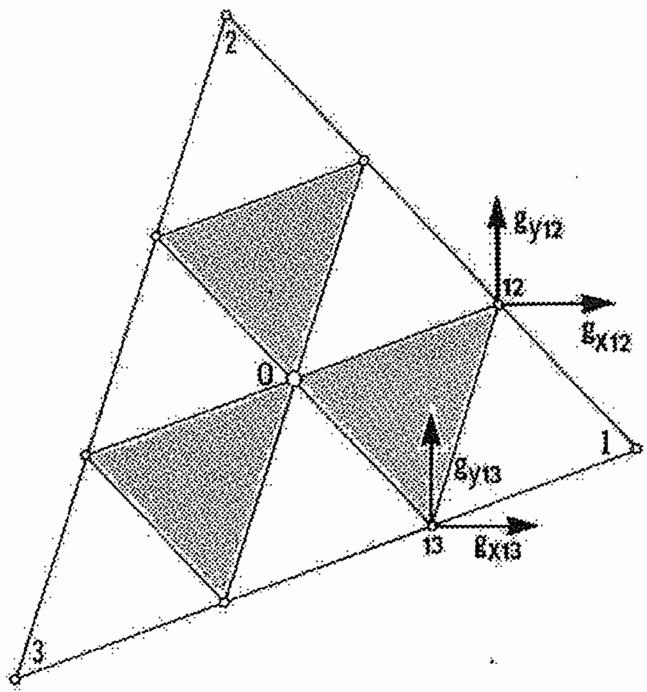

Figure 3 : Kinematical mode of the linear stress element

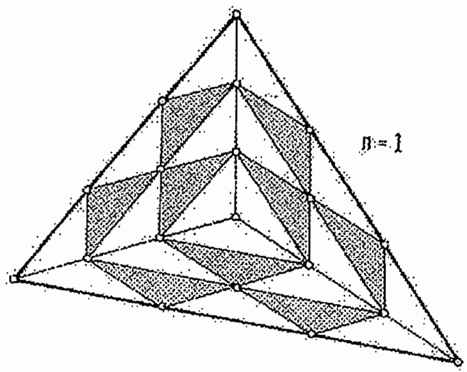

Figure 4 : 3-triangle superelement without kinematic modes

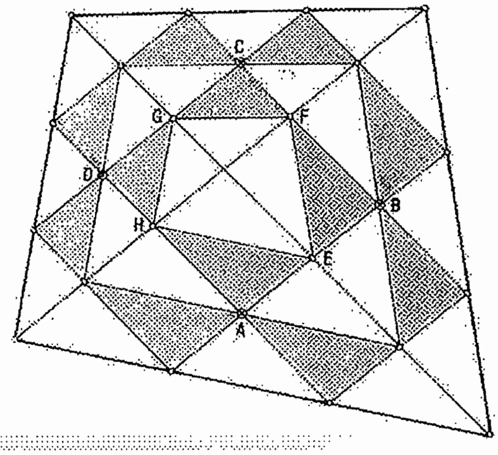

Figure 5 : Quadrilateral Superelement n = 1

6. PLATE ELEMENTS

For plates, the situation is exactly reversed, a property that is not so surprising if one refers to the Southwell analogies between plates and membranes [18]. There is a formal correspondence between membrane strains and plate moments and between membrane stresses and plate curvatures. The result is that equilibrium plate elements are easy to obtain, and conforming plate elements difficult to obtain. Here, C^1- connections of the displacements are necessary. For a third degree triangular element, the following degrees of freedom are needed :

- displacement and slopes at each node (9)
- normal slope at mid-sides (3),

So, 12 d.o.f. are necessary while a third degree field only contains 10 parameters. The connection is thus not possible.

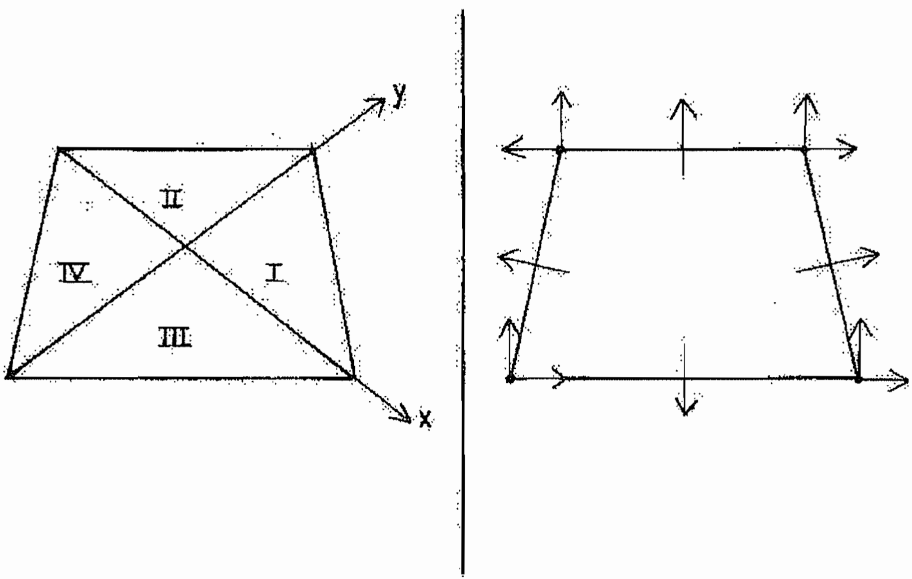

Figure 6 : The CQ element

Comparing to the equilibrium membrane, it seems that superelements could work. It is the case, as we will see on a quadrilateral divided by its diagonals. The number of required connectors is then 16 (fig. 6). Consider the following displacement field :

Subelement I : $w_I = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3$
Subelement II : $w = w_I + x^2 (\alpha_{11} + \alpha_{12} x + \alpha_{13} y)$
Subelement III : $w = w_I = y^2 (\alpha_{14} + \alpha_{15} x + \alpha_{16} y)$
Subelement IV : $w = w_I = x^2 (\alpha_{11} + \alpha_{12} x + \alpha_{13} y) + y^2 (\alpha_{14} + \alpha_{15} x + \alpha_{16} y)$ (46)

It is obviously internally conforming and possesses 16 d.o.f., as required. This element, known as CQ[20] (conforming quadrilateral) was developed in 1964 [4]. Another possibility is to assemble three triangles, and was presented by HSIEH, CLOUGH and TOCHER in 1966 [19].

7. THE SAMCEF CODE

It was a necessity to have a finite element code working independently of nodes. Therefore, the LTAS worked on a self-made code, whose static version was called ASEF. Progressive extensions to nodal analysis, dynamic response, fluid-structure interactions, nonlinear analysis and optimization appeared during the sixties and seventies, and finally led to a commercial code called SAMCEF.

One of the greatest merits of the early LTAS is to have succeeded in the challenge of producing a high level research with partially industrial funds. This is well reflected by the ten Ph.D. thesis which appeared between 1969 and 1980 [25 to 34].

8. BACK TO THE DUAL ANALYSIS

The dual analysis concept certainly acted as a motor of the largest part of the LTAS works during two decades. But industrial users of finite elements were never convinced by this approach. At that time, the fact that two analyses were needed was considered as too expensive. Moreover, the idea of evaluating the quality of results had not emerged. Results were obtained, and it was sufficient.

A second drawback was the fact that dual analysis was seen as a convergence sketch, but not explicitly as an error measure.

Finally, the upper and lower bounds are reversed when passing from a given load problem to a given displacement problem, which is paradoxical, and mixed problems were not treated.

A recent re-examination of the fundamental principles of dual analysis by the authors led to the following results [22, 23, 24].

1) The most fundamental result is (28 and 31)

$$CE(\sigma_h) \geq CE(\sigma) = -TE(u) \geq -TE(u_h), \tag{47}$$

so that if one defines a unique function

$$C^* = \begin{cases} CE(\sigma) \; \textit{for an equilibrium model} \\ -TE(u) \; \textit{for a displacement model,} \end{cases} \tag{48}$$

the convergence curves are always on the same side.

2) These curves are related to error measures, since

$$CE(\sigma_h) - CE(\sigma) = energy\ of\ (\sigma_h - \sigma) \quad (49)$$

$$TE(u_h) - TE(u) = energy\ of\ (u_h - u) \quad (50)$$

3) All these results are valid for any kinematically admissible displacement field u_h and for any statically admissible stress field, without necessity of advocating a Rayleigh-Ritz process. This includes procedures such as the so-called error in constitutive equations developed by LADEVEZE [21] where a non Rayleigh-Ritz procedure is used to obtain the equilibrium models.

Such an extension and the observation that by now, a second analysis is no more considered as prohibitive, let us hope that a possible future would be a systematization of dual analysis in practical cases.

REFERENCES

[1] B. FRAEIJS de VEUBEKE. *Sur certaines inégalités fondamentales et leur généralisation dans la théorie des bornes supérieures et inférieures en élasticité.*
Revue Universelle des Mines, 9e série, tome XVII, n° 5, 1961, pp. 305-314.

[2] B. FRAEIJS de VEUBEKE. *Upper and lower bounds in matrix structural analysis.*
14th meeting of the Material Structures task force of AGARD, July 1962, AGARDograph nr72, Pergamon Press, 1964.

[3] B. FRAEIJS de VEUBEKE. *Displacement and equilibrium models in the finite element method.*
Symposium on "Numerical methods in elasticity", Swansea, January 1964. Chapter 9 of Stress Analysis, edited by ZIEKIEWICZ and HOLISTER, John Wiley & Sons, 1965.

[4] G. SANDER. *Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion.*
Bulletin de la Société Royale des Sciences de Liège, 33e année, n° 7-8, 1964, pp. 456-494.

[5] G. SANDER. *Upper and lower bounds in matrix analysis of plates in bending.*
LTAS - Report SA2, University of Liège, 1964.

[6] G. SANDER. *Upper and lower bounds in spar matrix analysis.*
LTAS - Report SA3, University of Liège, 1965.

[7] B. FRAEIJS de VEUBEKE. *Bending and stretching of plates. Special models for upper and lower bounds.*
U.S. Air Force Report AFFDL-TR-66-80, Wright Patterson Air Force Base, 1966, pp. 863-866.

[8] G. SANDER, B. FRAEIJS de VEUBEKE. *Upper and lower bounds to structural deformation by dual analysis in finite elements.*
U.S. Air Force Report AFFDL-TR-66-199, Wright Patterson Air Force Base, 1967.

[9] G. SANDER. *Dual analysis of a multiweb sweptback wing model.*
Aircraft Engineering, vol. XL, nr 2, Feb. 1968, pp. 6-16.

[10] B. FRAEIJS de VEUBEKE, O.C. ZIENKIEWICZ. *Strain energy bounds in finite element analysis by slab analogy.*
Journal of Strain Analysis, vol. 2, n^r 4, 1967, pp. 265-271.

[11] G. SANDER. *Application of the dual analysis principle.*
Proc. IUTAM Colloquium on High Speed Computing of Elastic Structures, Liège, 1970.

[12] B. FRAEIJS de VEUBEKE, G. SANDER, P. BECKERS. *Dual analysis by finite elements - Linear and non linear applications.*
U.S. Air Force Report AFFDL-TR-72-93, Wright-Patterson Air Force Base, 1972.

[13] B. FRAEIJS de VEUBEKE. *The dual principles in elastodynamics finite element application.*
NATO Advanced Studies Institute, Lecture Seriez on Finite Element Methods in Continuum Mechanics, Lisbon, 1971, ed. by ODEN & OLIVEIRA, UAH Press, 1973.

[14] B. FRAEIJS de VEUBEKE. *Duality in structural analysis by finite elements.*
NATO Advanced Studies Institute, Lecture Series on Finite Element Methods in Continuum Mechanics, Lisbon, 1971, ed. ODEN & OLIVEIRA, UAH Press, 1973.

[15] B. FRAEIJS de VEUBEKE, M. HOGGE. *Dual analysis for heat conduction problems by finite elements.*
International Journal for Numerical Methods in Engineering, vol. 5, 1972, pp. 65-82.

[16] C. NYSSEN, P. BECKERS. *Dual analysis in elastoplasticity.*
Proc. of the International Conference on "Finite Elements in Non Linear Solid Structural Mechanics", Geils, Norway, Aug. 1977, Tapir Publishers, vol. 1, 1978, pp. 41-60.

[17] P. BECKERS, C. NYSSEN. *Linear and non linear dual analysis of axisymmetrical structures.*
Proc. Of the 2d International Conference on "Applied Numerical Modelling" Madrid, Spain, Sept. 11-15, 1978.

[18] R.V. SOUTHWELL. *On the Analogues Relating Flexure and Extension of Flat Plates.*

[19] R. CLOUGH, J. TOCHER. *Finite element stiffness matrices for the analysis of plate bending.*
Proc. Conf. Matrix Methods in Struct. Mech.
Report AFFDL-TR-66-80, 1966.

[20] B. FRAEIJS de VEUBEKE. *A conforming finite element for plate bending.*
Int. Jl. Solids Structures, 4, pp. 95-108, 1968.

[21] P. LADEVEZE, D. LEGUILLON. *Error estimate procedure in the finite element method and applications.*
SIAM J. Numer. Anal., 20 (3), 1983, pp. 483-509.

[22] J.F. DEBONGNIE. *A general theory of dual error bounds by finite elements.*
Report LMF/D5, University of Liège, 1983.

[23] J.F. DEBONGNIE, H.G. ZHONG, P. BECKERS. *Dual analysis with general boundary conditions.*
Comp. Meth. Applied Mech & Engineering, 122, 1995, pp. 183-192.

[24] J.F. DEBONGNIE, P. BECKERS. *Recent advances in the dual analysis theory.*
IV Congreso Métodos Numéricos in Ingenieria, Sevilla, 7-10 junio, 1999.

Ph.D. theses :

[25] G. SANDER. *Applications de la méthode des éléments finis à la flexion des plaques*, 1969.

[26] P. BECKERS. *Les fonctions de tension dans la méthode des éléments finis*, 1972.

[27] M. GERADIN. *Analyse dynamique duale des structures par la méthode des éléments finis*, 1972.

[28] S. IDELSOHN. *Analyse statique et dynamique des coques par la méthode des éléments finis,* 1974.

[29] M. HOGGE. *Transfert de chaleur et contraintes thermiques dans les structures par la méthode des éléments finis*, 1976.

[30] A. HUCK. *Méthodes numériques en réponse dynamique avec amortissement structural*, 1977.

[31] J.F. DEBONGNIE. *Modélisation de problèmes hydro-élastiques par éléments finis. Application aux lanceurs aérospatiaux*, 1978.

[32] C. FLEURY. *Le dimensionnement automatique des structures élastiques*, 1978.

[33] C. NYSSEN. *Modélisation par éléments finis du comportement non linéaire de structures aérospatiale*, 1979.

[34] E. CARNOY. *Etude de la stabilité élastique des coques par éléments finis*, 1980.