Reliability: Hypothesis testing

Maarten Arnst and Marco Lucio Cerquaglia

December 6, 2017
Statistical inference
Stochastic model (random variable, stochastic process, ...)

**Theory**

Probabilistic characterization (PDF, quantiles, ...) of "test statistics" (sample mean, sample variance, ...).

**Hypothesis tests:** can we go in the opposite direction? Can we use "test statistics" to infer conclusions about the stochastic model?
Suppose that we observe $\nu$ statistically independent trajectories up to time $t$ of a failure counting process. Then, the setting is as

- **data:** the numbers of failures in each interval $[0, t]$ and the time instants at which the failures occurred,

- **candidate stochastic model:** Poisson process.

In this case, the **goodness of fit of the Poisson distribution** can be tested:

- **Null hypothesis:** Poisson distribution is suitable,

- **Alternative hypothesis:** Poisson distribution is not suitable.

The **chi-squared test** is a particular goodness-of-fit test in which under the null hypothesis, the test statistic is a sample of a chi-squared distribution.
**Chi-squared hypothesis test:**

- **The chi-squared test statistic** measures the goodness of fit in terms of the sum of the squares of the differences between the observed and calculated outcome frequencies, divided by the calculated outcome frequencies:

\[
d = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i}
\]

with \(f_i\) : observed frequency for i-th value/bin, \(e_i\) : calculated frequency for i-th value/bin.

- Under the null hypothesis, the chi-squared test statistic is a sample from, approximately, the **chi-squared distribution** with \(k - 2\) degrees of freedom.

- **Accept** the null hypothesis if the test statistic lies within the acceptance region and **reject** the null hypothesis otherwise.
Let us consider an example involving failures of 66 machines over one day:

- 20 with zero failure, 23 with 1 failure, 15 with 2 failures, 6 with 3 failures, and 2 with 4 failures.

- Parameter estimation:

\[
\hat{m} = 0 \times \frac{20}{66} + 1 \times \frac{23}{66} + 2 \times \frac{15}{66} + 3 \times \frac{6}{66} + 4 \times \frac{2}{66} = 1.197 \text{ failure/day}.
\]

- Chi-squared test statistic:

<table>
<thead>
<tr>
<th>Number of failures</th>
<th>( f_i )</th>
<th>( e_i )</th>
<th>( \frac{(f_i - e_i)^2}{e_i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>19.94</td>
<td>1.18e^{-4}</td>
</tr>
<tr>
<td>1</td>
<td>23</td>
<td>23.86</td>
<td>0.0310</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>14.29</td>
<td>0.0353</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>5.69</td>
<td>0.0169</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.70</td>
<td>0.0528</td>
</tr>
<tr>
<td>&gt;4</td>
<td>0</td>
<td>0.52</td>
<td>0.52</td>
</tr>
</tbody>
</table>

\[
d = 0.65
\]

- Since \( d = 0.65 < c^{-1}_{\chi^2} (0.95; 4) = 9.48 \), we accept null hypothesis at \( \gamma = 95\% \) significance.
Suppose that we observe a trajectory up to time $t$ of a Poisson process. Then, the setting is as:

- **data:** $n$, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the failures occurred,

- **candidate stochastic model:** Poisson process.

In this case, the homogeneity of the Poisson process can be tested:

- **Null hypothesis:** the Poisson process is homogeneous.

- **Alternative hypothesis:** the rate of occurrence of failures decreases (increases).

The **logarithm test** is a particular **trend test** in which under the null hypothesis, the test statistic is a sample of a chi-squared distribution.
Hypothesis testing

- Logarithm hypothesis test:
  - The **logarithm test statistic** measures homogeneity by
    \[
    \nu = -2 \sum_{i=1}^{n} \log \frac{t_i}{t}.
    \]
  - Under the null hypothesis, the logarithm test statistic is a sample from the **chi-squared distribution** with \( n \) degrees of freedom.

Indeed, under the null hypothesis, the time instants at which the failure occurred are statistically independent and uniformly distributed in the interval \([0, t]\). It can be shown that the sign-reversed double of the sum of the logarithms of \( n \) statistically independent uniform random variables with values in \([0, 1]\) is a chi-squared random variable with \( n \) degrees of freedom.

- **Accept** the null hypothesis if the test statistic lies in the acceptance region \([0, c_{\chi^2}(\gamma, n)]\)
  \[
  \left(\left[c_{\chi^2}(1 - \gamma, n), +\infty\right]\right)
  \]
  and **reject** otherwise.
References

Suggested reading material


Additional references also consulted to prepare this lecture


