The Problem of a Drillstring Inside a Curved Borehole: a Perturbed Eulerian Approach

Vincent Denoël[1], Emmanuel Detournay[2]

[1] National Fund for Scientic Research & University of Liège, Belgium
[2] University of Minnesota, USA & CSIRO Petroleum, Australia

1 Introduction

This paper is concerned with the calculation of the deformed configuration of a drillstring during drilling, which critically hinges on accurately identifying the contacts between the drillstring and the borehole walls. For this study, we assume that the position of the bit relative to rig is fixed and that a known axial force is imposed at the rig. A mathematically related problem is the insertion (or the pulling) of the drillstring into (or out of) the borehole, as the nature of the axial boundary conditions at both ends of the drillstring is exchanged.

This subject matter is part of a larger class of problems involving *a priori* unknown contacts between an elastica and a rigid boundary. These problems are computationally challenging, especially in the context of the drilling applications. Indeed, the large deflections of the drillstring from a stress-free configuration require consideration of a geometrically non-linear model. Furthermore, application of standard numerical tools to this problem results in an ill-conditioned system of equations, owing mainly to the narrowness of the borehole compared to its length, but also to the large flexibility of the drillstring and the assumed rigid nature of the borehole walls.

We propose here a novel mathematical formulation of this problem, which takes advantage of the extreme slenderness of the borehole and which is based on expressing the deformed configuration of the drillstring as a perturbation of the borehole axis.

2 Problem Definition

We consider a borehole of length L and radius A, assumed to be contained in a vertical plane. Its *known* geometry is completely defined by the inclination $\Theta(S)$ of the borehole on the vertical axis $\mathbf{e}_1$, where S $(0 \leq S \leq L)$ is the

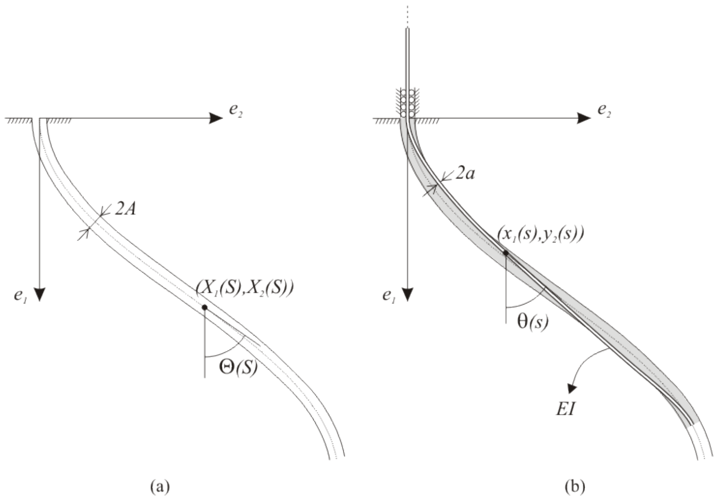

Figure 1: Problem definition.

borehole curvilinear coordinate with origin on the ground surface, see Fig. 1. A drillstring of external radius a, bending stiffness EI, and weight per unit length w is contained within the wellbore, with the bit at the hole bottom. Assuming that the position of the bit is fixed, we seek to determine the length ℓ and the deformed configuration of the drillstring, defined by its inclination $\theta_*(s)$ on $\mathbf{e}_1$, where s $(0 \leq s \leq \ell)$ is the drillstring curvilinear coordinate. For simplicity, we refer to S as the Eulerian coordinate and to s as the Lagrangian coordinate. The constraint on the drillstring to deform inside the borehole leads to the appearance of contacts between the borehole and the drillstring, either discrete or continuous. The contacts, which can be assumed to be frictionless as the drillstring is rotating, impose conditions on the distance Δ between the borehole and the drillstring axes, on the inclination θ_* and also on the curvature for continuous contacts, namely $\Delta = A - a$, $\theta_* = \Theta$, and $\theta_*' = \Theta'$.

The deformation of the drillstring (assumed to be inextensible) is governed by the classical geometrically nonlinear beam equations, which outside the contacts read

$$\begin{aligned} F_{1*}\theta_*' + F_{2*}' - w\sin\theta_* &= 0, \\ F_{2*}\theta_*' - F_{1*}' - w\cos\theta_* &= 0, \\ M_*' + F_{2*} &= 0, \\ EI\theta_*' &= M_*, \end{aligned} \tag{1}$$

where $F_{1*}(s)$, $F_{2*}(s)$ and $M_*(s)$ denote the axial force, transverse force and bending moment, respectively. This system of equations can be reduced to a 4th order nonlinear differential equation

$$EI\left(\theta_*'\theta_*'''' - \theta_*''\theta_*''' + \theta_*'^3\theta_*''\right) = w\left(\theta_*''\sin\theta_* - 2\theta_*'^2\cos\theta_*\right). \tag{2}$$

The formulation of this problem is closed, with the imposition of the boundary conditions at the rig ($s = S = 0$) and at the bit ($s = \ell$, $S = L$), besides the conditions at the *a priori* unknown contacts. At the rig, the boundary conditions take the form $F_{1*} = H$, $\Delta = 0$, and $\theta_* = \Theta$, while at the bit, $\Delta = 0$, $\theta_* = \Theta$ (for example) in addition to an integral constraint on $\sin\theta_*$ and on $\cos\theta_*$ that express that bit is positioned at the hole bottom. The problem of determining $\theta_*(s)$, ℓ, and the contacts is thus well posed, in principle [1]. However, semi-analytical or numerical methods that are directly based on solving the non-linear differential equation (2) result in ill-conditionned sets of equations that fail to converge when the dimensionless parameter EI/wl^3, where l is the distance between two contacts, becomes too small ($\simeq 0.2$).

3 Perturbed Eulerian Formulation

The approach proposed in this paper overcomes the above issues by expressing the drillstring configuration as a perturbation from the geometry of the borehole using the variable Δ, rather than in terms of θ_*, and by reformulating the problem in terms of the Eulerian coordinate S. Furthermore, as already proposed in [1], both the drillstring and the borehole are divided into segments limited by contacts and the global problem is expressed as a connected set of elementary problems. The number of elementary problems is *a priori* unknown, however. The critical aspect of these computations involve the determination of the positions of the contact points, which are used to segment the original problem into elementary ones. Each elementary problem is solved by assuming the positions of the contacts to be given; these positions are then recalculated at the reconnection stage in order to satisfy some continuity conditions at the contacts. The solution of the global problem requires therefore iterations to solve for the positions of the contacts, and each iteration requires the solution of a succession of elementary problems.

Evidently, all the elementary problems can be treated similary, by means of what we refer to as the auxiliary problem, namely the problem of finding the deformed configuration of the drillstring in a segment of the borehole between two contact points. First, we introduce the following dimensionless quantities: $\xi = (S - S_{i-1})/L_i$ where $L_i = S_i - S_{i-1}$ is the length of the borehole segment situated between contacts $i-1$ and i, $\alpha = (A-a)/L_i$, $\epsilon^2 = EI/wL_i^3$, and the scaled distance $\delta(\xi) = \Delta\left[S\left(\xi\right)\right]/(A-a)$. With the introduction of $\delta(\xi)$ as the fundamental unknown, we have expressed the drillstring deformed configuration as a perturbation of the borehole geometry. We also introduce the borehole inclination $\vartheta(\xi) = \Theta\left[S\left(\xi\right)\right]$, which is readily deduced from $\Theta(S)$.

Formulated in terms of $\delta(\xi)$, the differential equation (2) becomes after dropping terms of order $O(\alpha^2)$ and above

$$\alpha D\left[\delta\left(\xi\right);\vartheta\left(\xi\right);\epsilon\right] + F\left[\vartheta\left(\xi\right);\epsilon\right] = 0 \tag{3}$$

where D is a 5th order linear differential operator on $\delta\left(\xi\right)$ and F is a functional of $\vartheta(\xi)$ given by

$$F = \epsilon^2\left(\vartheta'\vartheta'''' - \vartheta''\vartheta''' + \vartheta'^3\vartheta''\right) - \vartheta''\sin\vartheta + 2\vartheta'^2\cos\vartheta. \tag{4}$$

It can readily be seen by setting $\delta\left(\xi\right) = 0$ in (3), that F is actually a measure of the out-of-balance forces that need to be applied on the drillstring so that it is espouses exactly the borehole geometry. Because (3) results from the consideration that θ is a small perturbation of ϑ, the function $\alpha D(\delta)$ is necessarily of the same order as $F(\vartheta)$, as otherwise the deviation of θ from ϑ would be too large and there would be an intermediate contact between the two ends $\xi = 0$ and $\xi = 1$. This is an application of the so-called method of dominant balance [2]. The boundary conditions for the differential equation (3) are that $\delta = \delta' = 0$ at both ends. Furthermore the axial force at one end is known, which provides a supplementary condition on a linear combination of δ'' and δ'''' .

With the perturbed Eulerian formulation, the integro-restrained nonlinear boundary value problem in θ_* (2) has been transformed into a classical linear boundary value problem in δ (3). The advantages of the new formulation are therefore obvious but are further clarified next by solving (2) and (3) with a similar shooting method.

4 Examples

The shooting method consists in transforming the 2-point boundary value problem into an initial value one by collecting the boundary conditions at the second end and, eventually, the restraining conditions in the form of objective functions $G(\mathbf{\Upsilon})$, where Υ represents the assumed initial conditions. Enforcement of the conditions that have been discarded in the formulation of the initial value problem is done by imposing that $G(\mathbf{\Upsilon}) = 0$. This method is used to solve the auxiliary problem expressed in Lagrangean coordinates (2) and in Eulerian coordinates with the perturbed formulation (3).

As an example, let us consider the auxiliary problem with $\Theta\left(S\right) = S/R$, $S_0 = 0$, $S_1 = L = R\pi/2$, which corresponds to a quadrant of a circular borehole.

Since there are only two contact points, we dispense of the subscript 1, when referring to the borehole or beam segment; i.e., $\ell = \ell_1$ and $L = L_1$.

In the Lagrangean formulation (2), the problem is solved with the following boundary conditions

$$\theta_*(0) = 0 \quad ; \quad \theta_*(L) = \frac{\pi}{2}$$
$$\int_0^\ell \sin\theta_*(s)\,ds = R \quad ; \quad \int_0^\ell \cos\theta_*(s)\,ds = R \tag{5}$$

expressing the compliance of inclination between the drillstring and the borehole at both ends as well as the constraints related to the offset between both beam ends. Conditions (5) are expressed as functions of the unknown beam length ℓ. A supplementary condition, related to the axial force $F_{1*} = \pi_1 w L$ (with a given number π_1) at $s = 0$, or equivalently to $\theta_*'''(0)$, is therefore added to obtain a closed set of equations. With this approach, the augmented initial conditions vector Υ_* collects $\theta_*'(0)$ and $\theta_*''(0)$, as well as the unknown beam length ℓ, while the objective function $G_*(\boldsymbol{\Upsilon}_*)$ gathers the second end condition and both constraints in (5).

In the perturbed Eulerian formulation, these boundary conditions are simply

$$\delta(0) = \delta(L) = \delta'(0) = \delta'(L) = 0. \tag{6}$$

They need also to be complemented, in order to close the system of equations, by a fifth condition on the axial force at $S = 0$, which is equivalently written as a function of $\delta''''(0)$. In this case, the augmented initial condition vector $\boldsymbol{\Upsilon}$ contains $\delta''(0)$ and $\delta'''(0)$, whereas the objective function expresses both second end conditions in (6).

Figure 2 shows contour levels of functions $G_*(\boldsymbol{\Upsilon}_*)$ and $G(\boldsymbol{\Upsilon})$ for $\pi_1 = 1$, $\varepsilon = 1$ and $\alpha = 0.001$. The solution of the problems with the shooting method is geometrically illustrated as the computation of the intersections of zero level curves of $G_*(\boldsymbol{\Upsilon}_*)$ and $G(\boldsymbol{\Upsilon})$. This is typically performed with a non-linear solver. The complexity of the level curves is a reflection of the convergence rate. Figure 2 illustrates therefore the advantages of the Eulerian approach of the problem, combined with a perturbation formulation.

5 Outlook

The Eulerian view of the drillstring flow into the borehole is especially advantageous within the context of a propagating borehole, when this model is used

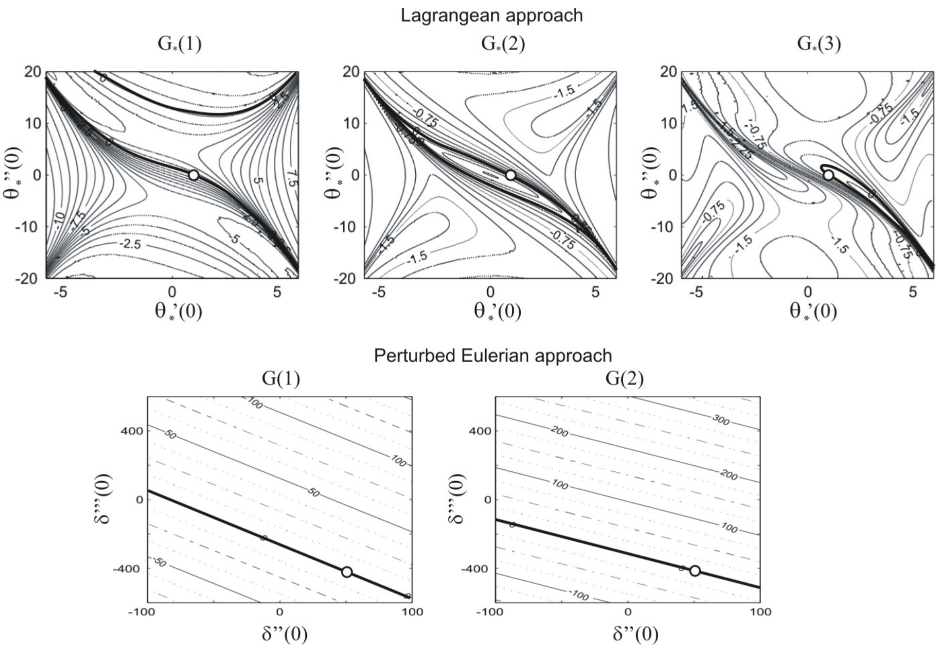

Figure 2: Level curves of $\mathbf{G}_*(\boldsymbol{\Upsilon}_*)$ for $\Upsilon_*(1) = \ell = 1.5697$ (the solution) and level curves of $\mathbf{G}(\boldsymbol{\Upsilon})$ for $\pi_1 = 1$, $\varepsilon = 1$ and $\alpha = 0.001$. Thick lines represent zero level curves. They intersect at the white dots, the solution of the shooting method.

to calculate successive equilibrium configurations of the drillstring. Indeed, the position of any contact becomes stationary in reference to the borehole with increasing distance between this contact and the bit, whilst it continues to slide along the moving drillstring.

The motivation to analyze this particular problem is multifold. First, there is the question of determining the transmission of forces between the rig and the bit (known as the *torque-and-drag* problem in the Petroleum Industry [3]), which is essentially controlled by the contacts between the drillstring and the borehole. Second, modeling the evolution of the borehole during drilling requires determination of the forces acting on the bit, which themselves depend on the deformed configuration of the drillstring. Finally, any analysis of the surface vibrations of the drillstring would benefit from *a priori* knowledge of the positions of the contacts along the string.

References

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