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Classification of finite reparametrization symmetry groups in the three-Higgs-doublet model

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Abstract Symmetries play a crucial role in electroweak symmetry breaking models with non-minimal Higgs content. Within each class of these models, it is desirable to know which symmetry groups can be implemented via the scalar sector. In N-Higgs-doublet models, this classification problem was solved only for $N = 2$ doublets. Very recently, we suggested a method to classify all realizable finite symmetry groups of Higgs-family transformations in the three-Higgs-doublet model (3HDM). Here, we present this classification in all detail together with an introduction to the theory of solvable groups, which play the key role in our derivation. We also consider generalized-CP symmetries, and discuss the interplay between Higgs-family symmetries and CP-conservation. In particular, we prove that presence of the $\mathbb{Z}_4$ symmetry guarantees the explicit CP-conservation of the potential. This work completes classification of finite reparametrization symmetry groups in 3HDM.

1 Introduction

The nature of the electroweak symmetry breaking is one of the main puzzles in high-energy physics. Very recently, the CMS and ATLAS collaborations at the LHC announced the discovery of the Higgs-like resonance at 126 GeV [1, 2], and their first measurements indicate intriguing deviations from the Standard Model (SM) expectations. Whether these data signal that a non-minimal Higgs mechanism is indeed at work and if so what it is, are among the hottest questions in particle physics these days.

In the past decades, many non-minimal Higgs sectors have been considered [3]. One conceptually simple and phenomenologically attractive class of models involves several Higgs doublets with identical quantum numbers (N-Higgs-doublet models, NHDM). Its simplest version with only two

doublets, 2HDM, was proposed decades ago [4], but it is still actively studied, see [5] for a recent review, and it has now become a standard reference model of the physics beyond the Standard Model (bSM). Constructions with more than two doublets are also extensively investigated [6–21].

Many bSM models aim at providing a natural explanation for the numerical values of (some of) the SM parameters. Often, it is done by invoking additional symmetries in the model. These are not related with the gauge symmetries of the SM but rather reflect extra symmetry structures in the "horizontal space" of the model. One of the main phenomenological motivations in working with several doublets is the ease with which one can introduce various symmetry groups. Indeed, Higgs fields with identical quantum numbers can mix, and it is possible that some of these Higgs-family mixing transformations leave the scalar sector invariant. Even in 2HDM, presence of such a symmetry in the Lagrangian and its possible spontaneous violation can lead to a number of remarkable phenomena such as various forms of CP-violation [4, 22–24], non-standard thermal phase transitions which may be relevant for the early Universe [25–27], natural scalar dark matter candidates [28–30]. For models with three or more doublets, an extra motivation is the possibility to incorporate into the Higgs sector non-abelian finite symmetry groups, which can then lead to interesting patterns in the fermionic mass matrices (for a general introduction into discrete symmetry groups relevant for particle physics, see [31]). In this respect, the very popular symmetry group has been A_4 [11–15], the smallest finite group with a three-dimensional irreducible representation, but larger symmetry groups also received some attention [6, 16, 17].

Given the importance of symmetries for the NHDM phenomenology, it is natural to ask: *which symmetry groups can be implemented in the scalar sector of NHDM for a given N?*

In the two-Higgs-doublet model (2HDM), this question has been answered several years ago [32–36], see also [5]

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for a review. Focusing on discrete symmetries, the only realizable Higgs-family symmetry groups are $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$. The $\mathbb{Z}_2$ group can be generated, for example, by the sign flip of one of the doublets (and it does not matter which, because once we focus on the scalar sector only, the simultaneous sign flip of both doublets does not change the Lagrangian), while the $(\mathbb{Z}_2)^2$ group is generated by sign flips and the exchange $\phi_1 \leftrightarrow \phi_2$. If generalized-*CP* transformations are also included, then $(\mathbb{Z}_2)^3$ becomes realizable as well, the additional generator being simply the *CP* conjugation.

With more than two doublets, the problem remains open. Although several attempts have been made in past to classify at least some symmetries in NHDM, [18, 19, 21], they led only to very partial results. The main obstacle here was the lack of the completeness criterion. Although many obvious symmetry groups could be immediately guessed, it was not clear how to prove that the given potential does not have other symmetries. An even more difficult problem is to prove that no other symmetry group can be implemented for a given N.

In the recent paper [37] we found such a criterion for abelian symmetry groups in NHDM for arbitrary N. Since abelian subgroups are the basic building blocks of any group, classification of realizable abelian symmetry groups in NHDM was an important milestone. We stress that this task is different from just classifying all abelian subgroups of $SU(3)$, because invariance of the Higgs potential places strong and non-trivial restrictions on possible symmetry groups.

In this paper, we solve the classification problem for all finite symmetry groups in 3HDM, including non-abelian groups. We do this by using the abelian groups in 3HDM found in [37] and by applying certain results and methods from the theory of solvable groups. Some of these results were already briefly described in [38]. Here, we present a detailed derivation of this classification together with an introduction to the relevant methods from finite group theory. In addition, we extend the analysis to symmetry groups which include both Higgs-family and generalized-*CP* transformations. This work, therefore, solves the problem of classification of finite reparametrization symmetry groups in 3HDM.

We would like to stress one important feature in which our method differs from more traditional approaches to symmetry classification problem, at least within the bSM physics. Usually, one starts by imposing invariance under certain transformations, and then one tries to recognize the symmetry group of the resulting potential. In this way it is very difficult to see whether all possible symmetries are exhausted. We approach the problem the other way around. We first restrict the list of finite groups which can appear as symmetry groups of 3HDM, and then we check one by one whether these groups can indeed be implemented.

The structure of this paper is the following. In Sect. 2 we describe different types of symmetry in the scalar sector of

NHDM and discuss the important concept of realizable symmetry groups. Section 3 contains an elementary introduction into the theory of (finite) solvable groups. Although it contains pure mathematics, we put it in the main text because it is a key part of the group-theoretic step of our classification, which is presented in Sect. 4. Then, in Sect. 5 we describe the methods which we will use to prove the absence of continuous symmetries. Sections 6 and 7 contain the main results of the paper: explicit constructions of the realizable symmetry groups and of the potentials symmetric under each group. Finally, in Sect. 8 we summarize and discuss our results. For the reader's convenience, we list in the Appendix potentials for each of the realizable non-abelian symmetry groups.

2 Symmetries of the scalar sector of multi-Higgs-doublet models

2.1 Reparametrization transformations

In NHDM we introduce N complex Higgs doublets with the electroweak isospin $Y = 1/2$, which interact with the gauge bosons and matter fields in the standard way, and also self-interact via a Higgs potential. The generic renormalizable Higgs potential can contain only quadratic and quartic gauge-invariant terms, and it can be compactly written as [39–41]:

$$V = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{abcd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d), \qquad (1)$$

where all indices run from 1 to N. Coefficients of the potential are grouped into components of tensors Y_{ab} and Z_{abcd}; there are N^2 independent components in Y and $N^2(N^2 + 1)/2$ independent components in Z.

In this work we focus only on the scalar sector of the NHDM. Therefore, once coefficients Y_{ab} and Z_{abcd} are given, the model is completely defined, and one should be able to express all its properties (the number and the positions of extrema, the spectrum and interactions of the physical Higgs bosons) via components of Y's and Z's. This explicit expression, however, cannot be written via elementary functions, and it remains unknown in the general case for any $N > 2$.

A very important feature of the most general potential is that any non-degenerate linear transformation in the space of Higgs doublets belonging to the group $GL(2, \mathbb{C})$ keeps the generic form of the potential, changing only the coefficients of Y and Z. We call such a transformation a *Higgs-basis change*. In addition, the *CP* transformation, which maps doublets to their hermitian conjugates $\phi_a \to \phi_a^\dagger$, also keeps the generic form of the potential, up to coefficient modification. Its combination with a Higgs-basis change represents a transformation which is usually

called a *generalized-CP transformation* [42–46]. The Higgs basis changes and generalized-*CP* transformations can be called together *reparametrization transformations* because they preserve the generic structure of the potential and lead only to its reparametrization.

A reparametrization transformation changes the basis in the space of Higgs doublets but does not modify the structural features of the model such as the number and the properties of minima, the symmetries of the potential and their spontaneous breaking at the minimum point. These properties must be the same for all the potentials linked by reparametrization transformations. Therefore, these properties must be expressible in terms of *reparametrization-invariant* combinations of Y's and Z's [39, 40, 47, 48].

If a reparametrization transformation maps a certain potential exactly to itself, that is, if it leaves certain Y's and Z's invariant, we say that the potential has a *reparametrization symmetry*. Usually, there is a close relation between the reparametrization symmetry group G of the potential and its phenomenological properties, both within the scalar and the fermion sectors. Therefore, understanding which groups can appear as reparametrization symmetry groups in NHDM with given N is of much importance for phenomenology of the model.

2.2 The group of kinetic-term-preserving reparametrization transformations

Often, one restricts the group of reparametrization transformations only to those transformations which keep the Higgs kinetic term invariant. In this case, a generic basis change becomes a unitary transformation $\phi_a \mapsto U_{ab}\phi_b$ with $U \in U(N)$. A kinetic-term-preserving generalized-*CP* transformation is an anti-unitary map $\phi_a \mapsto U_{ab}\phi_b^\dagger$, which can be written as $U_{CP} = U \cdot J$, with a unitary U and with J being the symbol for the *CP*-transformation.

The group $U(N)$ contains the group of overall phase rotations, which are already included in the gauge group $U(1)_Y$. Since we want to study structural symmetries of the NHDM potentials, we should disregard transformations which leave all the potentials invariant by construction. This leads to the group $U(N)/U(1) \simeq PSU(N)$. Note that $SU(N)$, which is often considered in these circumstances, still contains transformations which only amount to the overall phase shift of all doublets. They form the center of $SU(N)$, $Z(SU(N)) \simeq \mathbb{Z}_N$, and act trivially on all NHDM potentials. Being invariant under them does not represent any structural property of the Higgs potential, therefore, we are led again to the factor group $SU(N)/Z(SU(N)) = PSU(N)$. This allows us to write the group of kinetic-term-preserving reparametrization transformations as a semidirect product of the Higgs basis change group and the $\mathbb{Z}_2$ group generated by J (for a more detailed discussion, see [37]):

$$G_{\rm rep} = PSU(N) \rtimes \mathbb{Z}_2^*. \tag{2}$$

Here the asterisk indicates that the generator of the corresponding group is an anti-unitary transformation; we will use this notation throughout the paper.

Below, when discussing symmetry groups of the 3HDM potential, we will be either looking for subgroups of $PSU(3)$ (if only unitary transformations are allowed) or subgroups of this $G_{\rm rep}$ (when anti-unitary reparametrization transformations are also included). This should always be kept in mind when comparing our results with the groups which are discussed as symmetry groups in the 3HDM scalar sector. For example, in [16, 17] a 3HDM potential symmetric under $\Delta(27)$ or $\Delta(54)$ was considered, both groups being subgroups of $SU(3)$. However, they contains the center of $SU(3)$, which, we repeat, acts trivially on all Higgs potentials. Therefore, the structural properties of that model are defined by the factor groups $\Delta(27)/Z(SU(3)) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Delta(54)/Z(SU(3)) \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, which belong to $PSU(3)$.

2.3 Realizable symmetry groups

There is an important technical point which should be kept in mind when we classify symmetry groups of NHDM. When we impose a reparametrization symmetry group G on the potential, we restrict its coefficients in a certain way. It might happen then that the resulting potential becomes symmetric under a *larger* symmetry group $\widetilde{G}$ properly containing G.

One drawback of this situation is that we do not have control over the true symmetry properties of the potential: if we construct a G-symmetric potential, we do not know a priori what is its full symmetry group $\widetilde{G}$. This might be especially dangerous if G is finite while $\widetilde{G}$ turns out to be continuous, as it might lead to unwanted Goldstone bosons. Another undesirable feature is related with symmetry breaking. Suppose that we impose invariance of the potential under group G but we do not check what is the true symmetry group $\widetilde{G}$. After electroweak symmetry breaking, the symmetry group of the vacuum is $G_v \leq \widetilde{G}$, and it can happen that G_v is *not* a subgroup of G. This is not what we normally expect when we construct a G-symmetric model, and it is an indication of a higher symmetry.

Examples of these situations were encountered in literature before. For instance, the authors of [19] explicitly show that trying to impose a $\mathbb{Z}_p$, $p > 2$, group of rephasing transformations in 2HDM unavoidably leads to a potential with continuous Peccei–Quinn symmetry. For 3HDM they find an even worse example, when a cyclic group immediately leads to a $U(1) \times U(1)$-symmetric potential. Another well-known example is the A_4-symmetric 3HDM potential, which at certain values of parameters admits vacua with the S_3 symmetry, although S_3 is not a subgroup of A_4, see an explicit study in [15]. The explanation is that the potential

at these values of parameters becomes symmetric under S_4 which contains both A_4 and S_3.

In order to avoid such situations altogether, we must always check for each G whether the G-symmetric potentials are invariant under any larger group. We are interested only in those groups G, for which there exists a G-invariant potential with the property that no other reparametrization transformation leaves it invariant (either within $PSU(3)$ or within G_{rep}, depending on whether we include anti-unitary transformations). Following [21, 37], we call such groups *realizable*.

Using the terminology just introduced we can precisely formulate the two main questions which we address in this paper:

1. considering only non-trivial kinetic-term-preserving Higgs-basis transformations (i.e. group $PSU(3)$), what are the realizable finite symmetry groups in 3HDM?
2. more generally, considering non-trivial kinetic-term-preserving reparametrization transformations, which can now include generalized-CP transformations (i.e. group G_{rep}), what are the realizable finite symmetry groups in 3HDM?

For abelian groups, these questions were answered in [37] for general N. Here we focus on non-abelian finite realizable groups for $N = 3$.

3 Solvable groups: an elementary introduction

Our classification of realizable groups of Higgs-family symmetries in 3HDM contains two essential parts: the group-theoretic and the calculational ones. The group-theoretic part will make use of some methods of pure finite group theory, which are not very familiar to the physics community (although they are quite elementary for a mathematician with expertise in group theory). To equip the reader with all the methods needed to understand the group-theoretic part of our analysis, we begin by giving a concise introduction to the theory of solvable groups. In doing so, we mention only methods and results which are relevant for the particular problem of this paper. For a deeper introduction to solvable groups and finite group theory in general, see e.g. [49].

3.1 Basics

We assume that the reader is familiar with the basic definitions from group theory. We only stress here that we will work with finite groups, therefore the *order* of the group G (the number of elements in G) denoted as $|G|$ is always finite, and so is the order of any element g (the smallest positive integer n such that $g^n = e$, the identity element of the group).

A group G is called abelian if all its elements commute. An alternative way to formulate it is to say that all *commutators* in the group are trivial: $[x, y] = xyx^{-1}y^{-1} = e$ for all $x, y \in G$. Working with commutators is sometimes easier than checking the commutativity explicitly. For example, it is easy to prove that if every non-trivial element of the group has order two, $g^2 = e$, then the group is abelian. Indeed, for any $x, y \in G$ we have

$$[x, y] = xyx^{-1}y^{-1} = xyxy = (xy)^2 = e, \tag{3}$$

which means that x and y commute.

A group G can have proper subgroups $H < G$ (whenever we do not require that the subgroup H is proper, we write $H \leq G$), whose order must, by Lagrange's theorem, divide the order of the group: $|H|$ divides $|G|$. If proper subgroups exist, some of them must be abelian. A simple way to obtain an abelian subgroup is to pick up an element $g \in G$ and consider its powers: if order of the element g is n, we will get the cyclic group $\mathbb{Z}_n < G$.

The inverse of Lagrange's theorem is not, generally speaking, true: namely, if p is a divisor of $|G|$, the group G does not necessarily have a subgroup of order p. However, if p is a prime which enters the prime decomposition of $|G|$, then according to Cauchy's theorem such a subgroup must exist (this group is $\mathbb{Z}_p$ because there are no other groups of prime order). It immediately follows that if we have the list of all abelian subgroups of a given finite group G, then the prime decomposition of $|G|$ can only contain primes which are present in the orders of these abelian subgroups.

In fact, there is an existence criterion stronger than Cauchy's theorem. Namely, if p^a is the highest power of the prime p that enters the prime decomposition of $|G|$, then G contains a subgroup of this order, which is called the Sylow p-subgroup of the group G. This theorem (known as the Sylow-E theorem) is the starting point of the theory of Sylow subgroups, see Chap. 1 in [49].

There are several ways to *present* a finite group. One possibility is to list all its elements and write down the $|G| \times |G|$ multiplication table. Clearly, this presentation becomes impractical for a sufficiently large group. A more compact and powerful way is known as *presentation by generators and relations*. We call a subset $M = \{g_1, g_2, \dots\}$ of the elements of G a *generating set* (and its elements are called *generators*) if every $g \in G$ can be written as a product of elements of M or their inverses. The fact that G is generated by the set M is denoted as $G = \langle M \rangle$. Finding a minimal generating set for a given group and listing equalities which these generators satisfy is precisely presentation of the group by generators and relations. For example, the symmetry group of the regular n-sided polygon has the following presentation by generators and relations:

$$D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n = e \rangle. \tag{4}$$

This group is known as the *dihedral group* and has order $|D_{2n}| = 2n$ (note that there exists an alternative convention for denoting dihedral groups: D_n; the one which we use has its order in the subscript).

3.2 Normal subgroups and extensions

Consider two groups G and H. Suppose we have a map f from G to H, $f : G \to H$, which sends every $g \in G$ into its image $f(g) \in H$. If this map preserves the group operation, $f(g_1)f(g_2) = f(g_1 g_2)$, then it is called a *homomorphism*. If this map is surjective (i.e. it covers the entire H) and injective (distinct elements from G have distinct images in H), then f is invertible and is called an isomorphism.

In the case when $H = G$, we deal with an isomorphism of the group onto itself, which is called an *automorphism*. One can note that composition of two automorphisms is also an automorphism, and define the group structure on the set of all automorphisms of G. This *automorphism group* is denoted as $Aut(G)$. The trivial automorphism which fixes every element of G is the identity element of $Aut(G)$.

Let us now consider a special class of automorphisms called *inner* automorphisms, or *conjugations*. Fix an element $g \in G$ and define $f : x \mapsto g^{-1}xg$ for every $x \in G$. It can be immediately checked that f is an automorphism, and that it sends a subgroup of G into a (possibly another) subgroup of G. It can, however, happen that certain subgroups will be mapped onto themselves: $g^{-1}Hg = H$. Subgroups which satisfy this invariance criterion for every possible $g \in G$ are called *normal*, or invariant subgroups. The fact that H is a normal subgroup of G is denoted as $H \lhd G$.

Even when a subgroup H is not normal in G, one can pick up some elements $g \in G$ such that $g^{-1}Hg = H$. The set of elements of G with the property $g^{-1}Hg = H$ forms a group, which is called the *normalizer* of H in G and denoted as $N_G(H)$. We then have $H \lhd N_G(H) \leq G$. Working with normalizers is a useful intermediate step in situations when it is not known whether the subgroup H is normal in G.

Having a normal subgroup $H \lhd G$ gives some information about the structure of G. One can define the group structure on the set of (left) cosets of H, which is now called the *factor group* G/H. Thus, one breaks the group into two smaller groups, which often simplifies its study. Given a normal subgroup $H \lhd G$, one can define the *canonical homomorphism* $\phi : G \to G/H$ which sends every element $g \in G$ into its coset gH. Its kernel (all elements g which are mapped by ϕ into the identity element of G/H) is precisely H. Thus, every normal subgroup is the kernel of the corresponding canonical homomorphism. The reverse statement is also true: kernels of homomorphisms are always normal subgroups.

The group-constructing procedure inverse to factoring is called *extension*. Given two groups, N and H, a group G is called an extension of H by N (denoted as $N \, . \, H$), if there exists $N_0 \lhd G$ such that $N_0 \simeq N$ and $G/N_0 \simeq H$. In the case when, in addition, H is also isomorphic to a subgroup of G and $G = NH$, we deal with a *split extension*. The criterion for G to be a split extension can also be written as existence of $N \lhd G$ and $H \leq G$ such that $NH = G$ and $N \cap H = 1$, so that $G/N = H$. The group G is then called a *semidirect product* $G = N \rtimes H$.

Even if two groups N and H are fixed, they can support several extensions and split extensions. Therefore one faces the problem of classifying of all extensions of two given groups.

For the most elementary example, consider extensions of $H = \mathbb{Z}_2$ (generated by a) by $N = \mathbb{Z}_2$ (generated by b), which should produce a group of order 4. Then, for a split extension, we need a group G which has two distinct subgroups isomorphic to N and H. The only choice is $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which can be presented as $\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$. For a non-split extension, we require that only N is isomorphic to a subgroup of G. Thus, we still have $b^2 = e$, while a^2 must *not* be the unit element. Then we have to set $a^2 = b$ producing the group $\mathbb{Z}_4$. So, $\mathbb{Z}_4$ does not split over $\mathbb{Z}_2$, while $\mathbb{Z}_2 \times \mathbb{Z}_2$ does.

3.3 Characteristic subgroups

In what concerns embedding of groups, normality is a relatively weak property. Namely, if $K \lhd H$ and $H \lhd G$, then K is not necessarily normal in G (it is instead called subnormal in G). Indeed, recall that a normal subgroup $K \lhd H$ stays invariant under all inner automorphisms on H. Here "inner" is meant with respect to the group H, namely, $h^{-1}Kh = K$ for all $h \in H$. However, since $H \lhd G$, one can fix $g \in G$ but $g \notin H$ and consider an automorphism on H defined by $H \to g^{-1}Hg$. This is indeed an automorphism on H because it induces a permutation of elements of H preserving its group property, but it is not inner, because g does not belong to H. Therefore K does not have to be invariant under it: $g^{-1}Kg \neq K$.

However, there is a stronger property which guarantees normality for embedded groups. Let us call a subgroup K *characteristic* in H if it is invariant under all (not only inner) automorphisms of H. Then, repeating the above arguments, we see that if K is characteristic in H, and H is normal in G, then K is also normal in G. Also, if K is characteristic in H and H is characteristic in G, then K is also characteristic in G. Thus, knowing that some subgroups are characteristic gives even more information than their normality.

There is one simple rule which guarantees that certain subgroups are characteristic. If we have a rule defined in terms of the group G which identifies its subgroup H uniquely, then H is characteristic in G. Two important examples are:

- the *center* of the group G denoted as $Z(G)$, which is the set of all elements $z \in G$ such that they commute with all elements of G:

$$Z(G) = \{z \in G \mid [z, g] = e \; \forall g \in G\}. \tag{5}$$

The center of an abelian group coincides with the group itself.

- the *commutator subgroup* (or *derived subgroup*) of G denoted as G' and defined as the subgroup generated by all commutators:

$$G' = \langle [x, y] \rangle, \quad x, y \in G. \tag{6}$$

Note that the word "generated" is needed because the set of commutators is generally speaking not closed under the group multiplication. Clearly, the commutator subgroup of an abelian group is trivial, therefore the size of G' can be used to qualitatively characterize how far G is from being abelian.

3.4 Consequences of existence of a normal maximal abelian subgroup

Let us now prove a rather simple group-theoretic result, which, however, will be important for our classification of symmetries in 3HDM. This result, loosely speaking, is the observation that a mere existence of a subgroup of G with some special properties can strongly restrict the structure of the group G.

First, an abelian subgroup $A < G$ is called a *maximal abelian subgroup* if there is no other abelian subgroup B with property $A < B \leq G$. Note that the word "maximal" refers not to the size but to containment. This definition does not specify a unique subgroup; in fact a group can have several maximal abelian subgroups. They correspond to terminal points in the partially ordered tree of abelian subgroups of G.

Suppose that A is an abelian subgroup of a finite group G. Elements of A, of course, commute among themselves. But it can also happen that there exist other elements $g \in G$, $g \notin A$, which also commute with all elements of A. The set of all such elements is called the *centralizer* of A in G:

$$C_G(A) = \{g \in G \mid [g, a] = e \; \forall a \in A\}. \tag{7}$$

It is easy to check that $C_G(A)$ is a subgroup of G, and it can be non-abelian. The name "centralizer" refers to the fact that although A is not the center in G, it is the center in $C_G(A)$.

Clearly, $A \leq C_G(A)$. If A is a proper subgroup of $C_G(A)$, then it means that A is not a maximal abelian subgroup. Indeed, we take an element $g \in C_G(A)$, $g \notin A$, and consider another subgroup $B = \langle A, g \rangle$. This subgroup is abelian and is strictly larger than A: $A < B \leq G$. On the other hand, an element $x \in G$ which commutes with all elements of B

will certainly commute with all elements of A, while the converse is not necessarily true. Therefore, we get the following chain: $A < B \leq C_G(B) \leq C_G(A)$. Next, we check whether B is a proper subgroup of $C_G(B)$. If so, we can enlarge it again in the same way by considering $C = \langle B, g' \rangle$, where $g' \in C_G(B)$, $g' \notin B$. We can continue this procedure until it terminates with an abelian subgroup K which is *self-centralizing*:

$$A < B < \cdots < K = C_G(K) \leq \cdots \leq C_G(B) \leq C_G(A). \tag{8}$$

Since there exists no other element in G which would commute with all elements of K, we conclude that K is a maximal abelian subgroup in G.

Let us now see what changes if the abelian subgroup A is normal. Any element $g \in G$ acting on A by conjugation induces an automorphism of A. Thus, we have a map from G to the group of automorphisms of A, $f : G \to Aut(A)$. The kernel of f consists of such g's which induce the trivial automorphism of A, that is, which leave every $a \in A$ unchanged: $g^{-1}ag = a \; \forall a \in A$. But this coincides with the definition of centralizer. Therefore we conclude that $\ker f = C_G(A)$.

The fact that $C_G(A)$ is the kernel of the homomorphism f implies that $C_G(A)$ is a normal subgroup of G. Note that it is essential that the abelian subgroup in question, A, is normal; if it were not, $C_G(A)$ would not have to be normal.

Now, if A is a normal maximal abelian subgroup of G, then $\ker f = C_G(A) = A$. In other words, the kernel of $G/A \to Aut(A)$ is trivial, and therefore, G/A is isomorphic to a subgroup of $Aut(A)$. Summarizing our discussion, if A is a normal maximal abelian subgroup of G, then G can be constructed as an extension of A by a subgroup of $Aut(A)$:

$$G \simeq A \,.\, K, \quad \text{where } K \leq Aut(A). \tag{9}$$

This is a powerful structural implication for the group G of existence of a normal maximal abelian subgroup.

3.5 Automorphism groups

For future reference, we give some details on the automorphism groups $Aut(A)$ of certain abelian groups A. In this subsection we will use the additive notation for the group operation.

Suppose $A = \mathbb{Z}_n$ is the cyclic group of order n with generator e: $ne = \underbrace{e + \cdots + e}_{n \text{ times}} = 0$. An automorphism σ acting on A is a group-structure-preserving permutation of elements of A. Since A is generated by e, this automorphism is completely and uniquely defined once we assign the value of $\sigma(e) = k$ and make sure that $m\sigma(e) \neq 0$ for all $0 < m < n$. This holds when k and n are coprime ($k = 1$ is coprime to any n). The number of integers less than n and

coprime to n is called the Euler function $\varphi(n)$. Thus, we have $|Aut(\mathbb{Z}_n)| = \varphi(n)$. For a prime p, the Euler function is obviously $\varphi(p) = p - 1$. In general, if $p_1^{k_1} \cdots p_s^{k_s}$ is the prime decomposition for n, then

$$\varphi\big(p_1^{k_1} \cdots p_s^{k_s}\big) = \varphi\big(p_1^{k_1}\big) \cdots \varphi\big(p_s^{k_s}\big)$$
$$= \big(p_1^{k_1} - p_1^{k_1-1}\big) \cdots \big(p_s^{k_s} - p_s^{k_s-1}\big).$$

Suppose now that p is prime and

$$A = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ times}} = (\mathbb{Z}_p)^n.$$

Then G can be considered as an n-dimensional vector space over a finite field $\mathbb{F}_p$ of order p. Vectors in this space can be written as

$$x = k_1 e_1 + \cdots + k_n e_n,$$

where numbers $k_i \in \mathbb{F}_p$ and "basis vectors" e_i are certain non-zero elements of the ith group $\mathbb{Z}_p$. The group of all automorphisms on $(\mathbb{Z}_p)^n$ is then the general linear group in this space $GL_n(p)$.

Again, in order to define an automorphism σ acting on A, it is sufficient to assign where the basis vectors e_i are sent by σ and to make sure that they stay linearly independent: that is, if $m_1 \sigma(e_1) + \cdots + m_n \sigma(e_n) = 0$, with $m_i \in \mathbb{F}_p$, then all $m_i = 0$. In order to calculate $|GL_n(p)|$, we just need to find to how many different bases the initial basis $\{e_1, \ldots, e_n\}$ can be mapped to. The first vector, e_1, can be sent to $p^n - 1$ vectors, the second vector, e_2, can then be sent to $p^n - p$ vectors linearly independent with $\sigma(e_1)$, and so forth. The result is

$$\big|GL_n(p)\big| = \big(p^n - 1\big)\big(p^n - p\big) \cdots \big(p^n - p^{n-1}\big)$$
$$= p^{\frac{n(n-1)}{2}}(p - 1)\big(p^2 - 1\big) \cdots \big(p^n - 1\big). \quad (10)$$

In particular, $|Aut(\mathbb{Z}_p \times \mathbb{Z}_p)| = |GL_2(p)| = p(p-1)(p^2 - 1)$, and the p-subgroup of $Aut(\mathbb{Z}_p \times \mathbb{Z}_p)$ can only be $\mathbb{Z}_p$.

3.6 Nilpotent groups

In group theory, a powerful tool to investigate structure and properties of groups is to establish existence of subgroup series with certain properties. For example, a finite collection of normal subgroups $N_i \lhd G$ is called a *normal series* for G if

$$1 = N_0 \le N_1 \le N_2 \le \cdots \le N_r = G. \quad (11)$$

Restricting the properties of the factor groups N_i/N_{i-1} for all i, one can infer non-trivial consequences for the group G.

If all the factor groups in the normal series lie in the centers, $N_i/N_{i-1} \le Z(G/N_{i-1})$ for $1 \le i \le r$, then (11) becomes a *central series*, and the group G is then called *nilpotent*. The smallest number r for which the central series exists is called the nilpotency class of G.

Clearly, abelian groups are nilpotent groups of class 1 because for them $G \le Z(G)$. A non-abelian group G whose factor group by its center $G/Z(G)$ gives an abelian group is a nilpotent group of class 2, etc. So, nilpotent groups are often regarded as "close relatives" of abelian groups in the class of non-abelian ones. One important class of nilpotent groups is p-groups, i.e. finite groups whose order is a power of a prime p.

Nilpotent groups bear several remarkable features. We mention here only two of them which we will use below. First, a nilpotent group has a normal self-centralizing, and therefore maximal, abelian subgroup (Lemma 4.16 in [49]), whose implications were discussed above. Second, if H is a proper subgroup of a nilpotent group G, then H is also a proper subgroup of $N_G(H)$ (Theorem 1.22 in [49]). In other words, the only subgroup of a nilpotent group G which happens to be self-normalizing is the group G itself.

3.7 Solvable groups

A group G is called *solvable* if it has a normal series (11) in which all factor groups N_i/N_{i-1} are abelian. This is a broader definition than the one of nilpotent groups. Therefore we can expect that both criteria and properties of solvable groups will be weaker than for nilpotent groups.

One particular example is that unlike nilpotent groups, a solvable group does not have to possess a normal self-centralizing abelian subgroup. However, what it does possess is just a *normal abelian subgroup*. In order to prove this statement, let us first introduce another series of nested subgroups, called the *derived series*. We first find G', the derived subgroup of G, then we find its derived subgroup, $G'' = (G')'$, then the third derived subgroup, $G^{(3)} = (G'')'$, and so on. The derived series is simply

$$\cdots \le G^{(3)} \le G'' \le G' \le G. \quad (12)$$

The relation of the derived series with solvability is the following: G is solvable if and only if its derived series terminates, i.e. $G^{(m)} = 1$ for some integer $m \ge 0$ (Lemma 3.9 in [49]). The basic idea behind the proof of this statement is the observations that G' is the unique smallest normal subgroup of G with an abelian factor group. Indeed, if $N \lhd G$ and $\phi : G \to G/N$ is the canonical homomorphism, then $\phi(G') = (G/N)'$ (commutators are mapped into commutators). If we want G/N to be abelian, then $(G/N)' = 1$, and $G' \le \ker \phi = N$. Therefore, whatever N_{r-1} we choose in (11), it will contain G'. This argument can be continued

through the series, and since the normal series terminates, so does the derived series.

Now, since $G^{(m)} = 1$ for some finite m, we can consider $G^{(m-1)}$. It is an abelian group because its derived subgroup is trivial. Being a characteristic subgroup of $G^{(m-2)}$, it is definitely normal in G. Thus, we obtain the desired normal abelian subgroup.

A normal abelian subgroup is not guaranteed to be maximal. One can, of course, extend it to a maximal abelian subgroup, but then it is not guaranteed to be normal. Thus, in order to use the result (9), we need to prove the existence of an abelian subgroup which combines both properties. This situation is not generic: a solvable groups does not have to possess a normal maximal abelian subgroup. However, it can possess it in certain cases, and we will show below that in what concerns finite symmetry groups in 3HDM, they do contain such a subgroup.

4 Structure of the finite symmetry groups in 3HDM

4.1 Abelian subgroups and Burnside's theorem

Our goal is to understand which finite groups G can be realized as Higgs-family symmetry groups in the scalar sector of 3HDM. We stress that we look for realizable groups only, see discussion in Sect. 2.3.

Since finite groups have abelian subgroups, it is natural first to ask which abelian subgroups G can have. This can be immediately inferred from our paper [37] devoted to abelian symmetry groups in NHDM. In the particular case of 3HDM, only the following groups can appear as abelian subgroups of a finite realizable symmetry group G:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3. \tag{13}$$

The first four are the only realizable finite subgroups of maximal tori in $PSU(3)$. The last group, $\mathbb{Z}_3 \times \mathbb{Z}_3$, is on its own a maximal abelian subgroup of $PSU(3)$, but it is not realizable because a $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric potential is automatically symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, see explicit expressions below. However, since it appears as an abelian subgroup of a finite realizable group, it must be included into consideration. Trying to impose any other abelian Higgs-family symmetry group on the 3HDM potential unavoidably makes it symmetric under a continuous group.

Let us first see what order the finite (non-abelian) group G can have. We note that the orders of all abelian groups in (13) have only two prime divisors: 2 and 3. Thus, by Cauchy's theorem, the order of the group G can also have only these two prime divisors: $|G| = 2^a 3^b$. Then according to Burnside's $p^a q^b$-theorem the group G is solvable (Theorem 7.8 in [49]), and this means that G contains a normal abelian subgroup, which belongs, of course, to the list (13).

In order to proceed further, we need to prove that one can in fact find a normal maximal (that is, self-centralizing) abelian subgroup of G, a property which is not generic to solvable groups but which holds in our case.

4.2 Existence of a normal abelian self-centralizing subgroup

Suppose $A < G$ is a normal abelian subgroup, whose existence follows from the solvability of G. In this subsection we prove that even if it is not self-centralizing, i.e. $A < C_G(A)$, then there exists another abelian subgroup $B > A$, which is normal and self-centralizing in G.

Suppose that $A < C_G(A)$. Then for every $b \in C_G(A) \setminus A$, the group $A_b = \langle A, b \rangle$ is an abelian subgroup of G, which properly contains A. Figure 1 should help visualize embedding of various abelian subgroups of this kind in $C_G(A)$. Note that $C_G(A)$ can be non-abelian. There are two possibilities compatible with the list (13):

(i) $A = \mathbb{Z}_2$, and then A_b can be either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$,
(ii) $A = \mathbb{Z}_3$, and then $A_b = \mathbb{Z}_3 \times \mathbb{Z}_3$.

Thus $C_G(A)$ is either a 2-group or a 3-group. Below we assume that $p = 2$ if $C_G(A)$ is a 2-group, and $p = 3$ if $C_G(A)$ is a 3-group.

Since $C_G(A)$ is a p-group, it is nilpotent, and according to discussion in Sect. 3.6, it possesses a normal maximal abelian subgroup B (which of course can be represented as A_b for some b), while B properly includes $A = \mathbb{Z}_p$: $A < B \leq C_G(A)$. In particular, B is self-centralizing in $C_G(A)$, so according to our discussion in Sect. 3.4, the factor group $C_G(A)/B$ is a subgroup of $Aut(B)$. If $B = C_G(A)$, then $C_G(A)$ is abelian and, being a centralizer of a normal subgroup, it is normal in G. Clearly $B \leq C_G(B) \leq C_G(A) = B$, therefore $C_G(A)$ is the desired normal abelian self-centralizing subgroup of G.

Assume now that $B \neq C_G(A)$:

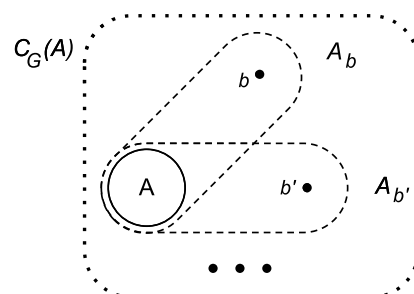$$A < B = \underbrace{C_{C_G(A)}(B)}_{=C_G(B)} < C_G(A) < G. \tag{14}$$

Fig. 1 Illustration of $C_G(A)$ and some of its subgroups

The illustration in Fig. 1 refers to this case. Since B is an abelian subgroup of G, it must be in list (13). So, either $B = \mathbb{Z}_p \times \mathbb{Z}_p$ or $B = \mathbb{Z}_{p^2}$ (the last case occurs only if $p = 2$), and in any of these cases we obtain $|B| = p^2$. Now, recall that $C_G(A)$ is a p-group, and so is $C_G(A)/B$. If $B = \mathbb{Z}_p \times \mathbb{Z}_p$, then $C_G(A)/B$ is a p-subgroup of $GL_2(p)$, in particular, $|C_G(A)/B| = p$. If $B = \mathbb{Z}_{p^2}$, then $C_G(A)/B$ is a p-subgroup of $Aut(\mathbb{Z}_{p^2})$. Since $\varphi(p^2) = p(p-1)$, it follows that $|C_G(A)/B| = p$. So in any case we have $|C_G(A)| = p^3$.

Now the arguments depend on p.

- In the case $p = 2$, we see that $C_G(A)$ is a non-abelian group of order 8. Thus $C_G(A)$ is either dihedral group D_8 or the quaternion group Q_8. If $C_G(A)$ is dihedral, then it possesses the unique (and hence characteristic) subgroup $H = \mathbb{Z}_4$, so H is the desired normal self-centralizing subgroup of G. If $G = Q_8$ is quaternion then, as we describe in Sect. 6.3.3, trying to impose a Q_8 symmetry group on the 3HDM potential will result in a potential symmetric under a continuous group. Thus, this situation cannot happen if we search for finite realizable groups G. Note that this feature is purely calculational and does not rely on the existence of a normal maximal abelian subgroup which we prove here.

- In the case $p = 3$, we find that $C_G(A)$ is a non-abelian group of order $p^3 = 27$ and exponent 3, i.e. for every $g \in C_G(A)$ we have $g^3 = 1$. It is non-abelian and cannot contain elements of order 9 because (13) does not contain abelian groups of orders 9 or 27.

In this case we do not yet know whether B is normal in G, but it is definitely normal in its own normalizer $B \triangleleft N_G(B) \leq G$. Moreover $C_G(A) \leq N_G(B)$, since B is normal in $C_G(A)$. These relations are visualized by the following relations:

$$B \triangleleft C_G(A) \leq N_G(B) \leq G < PSU(3). \qquad (15)$$

We can then consider the factor group $N_G(B)/B$. We know that $B = \mathbb{Z}_3 \times \mathbb{Z}_3$ is a maximal abelian group in $PSU(3)$ [37]; therefore it is self-centralizing in $PSU(3)$ and, consequently, in G and in its subgroup $N_G(B)$. Then, in particular, we see that $N_G(B)/B$ is a subgroup of $Aut(B) = GL_2(3)$. Moreover, the analysis which will be exposed in detail in Sect. 7 allows us to state that $N_{PSU(3)}(B)/B = SL_2(3)$, so $N_G(B)/B$ is a subgroup of $SL_2(3)$. We show in Sect. 7 that one cannot use elements of order 3 from $SL_2(3)$ because the potential will then become invariant under a continuous symmetry group. Therefore, $N_G(B)/B$ cannot have elements of order 3, which implies that B is a Sylow 3-subgroup of $N_G(B)$. The same statement holds for every group that lies "between" $N_G(B)$ and B, in particular, to $C_G(A)$. This contradicts the fact that $|C_G(A) : B| = 3$ and $C_G(A) \leq N_G(B)$. So this case is impossible.

Summarizing the group-theoretic part of our derivation, we proved that any finite group G which can be realized as a Higgs-family symmetry group in 3HDM is solvable, and in addition it contains a normal self-centralizing abelian subgroup A belonging to the list (13). Then, according to (9) the group G can be constructed as an extension of A by a subgroup of $Aut(A)$.

This marks the end of the group-theoretic part of our analysis. We now need to check all the five candidates for A, whose explicit realization were already given in [37], and by means of direct calculations see which extension can work in 3HDM.

5 Detecting continuous symmetries

Before we embark on analyzing each particular abelian group and its extensions, let us discuss an important issue. In this paper, we focus on discrete symmetries of the scalar sector in 3HDM. The symmetry groups we study must be realizable, that is, we need to prove that a potential symmetric under a finite group G is not symmetric under any larger group containing G. In particular, we must prove that a given G-symmetric potential does not have any continuous symmetry.

In principle, it would be desirable to derive a basis-invariant criterion for existence or absence of a continuous symmetry. Such condition is known for 2HDM [32–36], while for the more than two doublets a necessary and sufficient condition is still missing. However, in certain special but important cases it is possible to derive a *sufficient* condition for absence of any continuous symmetry. Since this method relies on the properties of the orbit space in 3HDM, we start by briefly describing it.

5.1 Orbit space in 3HDM

The formalism of representing the space of electroweak-gauge orbits of Higgs fields via bilinears was first developed for 2HDM [32–36, 50–52], and then generalized to N doublets in [20]. Below we focus on the 3HDM case.

The Higgs potential depends on the Higgs doublets via their gauge-invariant bilinear combinations $\phi_a^\dagger \phi_b$, $a, b = 1, 2, 3$. These bilinears can be organized into the following real scalar r_0 and real vector r_i, $i = 1, \ldots, 8$:

$$r_0 = \frac{(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)}{\sqrt{3}},$$

$$r_3 = \frac{(\phi_1^\dagger \phi_1) - (\phi_2^\dagger \phi_2)}{2},$$

$$r_8 = \frac{(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) - 2(\phi_3^\dagger \phi_3)}{2\sqrt{3}},$$

$$r_1 = \text{Re}(\phi_1^\dagger \phi_2), \qquad r_2 = \text{Im}(\phi_1^\dagger \phi_2), \qquad r_4 = \text{Re}(\phi_3^\dagger \phi_1),$$
$$r_5 = \text{Im}(\phi_3^\dagger \phi_1), \qquad r_6 = \text{Re}(\phi_2^\dagger \phi_3), \qquad r_7 = \text{Im}(\phi_2^\dagger \phi_3).$$
$$(16)$$

The last six components can be grouped into three "complex coordinates":

$$r_{12} = (\phi_1^\dagger \phi_2) = r_1 + i r_2,$$
$$r_{45} = (\phi_3^\dagger \phi_1) = r_4 + i r_5, \qquad (17)$$
$$r_{67} = (\phi_2^\dagger \phi_3) = r_6 + i r_7.$$

It is also convenient to define the normalized coordinates $n_i = r_i / r_0$. The orbit space of the 3HDM is then represented by an algebraic manifold lying in the $1+8$-dimensional Euclidean space of r_0 and r_i and is defined by the following (in)equalities [20]:

$$r_0 \geq 0, \qquad \vec{n}^2 \leq 1, \qquad \sqrt{3} d_{ijk} n_i n_j n_k = \frac{3\vec{n}^2 - 1}{2}, \quad (18)$$

where d_{ijk} is the fully symmetric $SU(3)$ tensor. It can also be derived that $|\vec{n}|$ is bounded from below:

$$\vec{n}^2 = \alpha, \qquad \frac{1}{4} \leq \alpha \leq 1. \qquad (19)$$

The value of α parametrizes $SU(3)$-orbits inside the orbit space. In particular, we will use this relation below when substituting $r_3^2 + r_8^2$ by $\alpha r_0^2 - |r_{12}|^2 - |r_{45}|^2 - |r_{67}|^2$.

Any $U(3)$ transformation in the space of doublets ϕ_1, ϕ_2, ϕ_3 leaves r_0 invariant and induces an $SO(8)$ rotation of the vector r_i. Note that this map is not surjective, namely not every $SO(8)$ rotation of r_i can be induced by a $U(3)$ transformation in the space of doublets. Therefore, unlike in 2HDM, we do not expect the orbit space of 3HDM to be $SO(8)$-symmetric, and the last condition in (18) stresses that.

Let us take a closer look at the (n_3, n_8)-subspace. It follows from (18) that the orbit space intersects this plane along the equilateral triangle shown in Fig. 2. Its vertices P, P', P'' lie on the "neutral" manifold, which satisfy the condition $\vec{n}^2 = 1$ and which would correspond to the neutral vacuum if the minimum of the potential were located there, while the line segments joining them correspond to the charge-breaking vacuum, see details in [20]. The orbit space in this plane clearly lacks the rotational symmetry and has only the symmetries of the equilateral triangle.

5.2 Absence of continuous symmetries

The convenience of the formalism of bilinears is that the most general Higgs potential becomes a quadratic form in this space:

$$V = -M_0 r_0 - M_i r_i + \frac{1}{2} \Lambda_{00} r_0^2 + \Lambda_{0i} r_0 r_i + \frac{1}{2} \Lambda_{ij} r_i r_j. \quad (20)$$
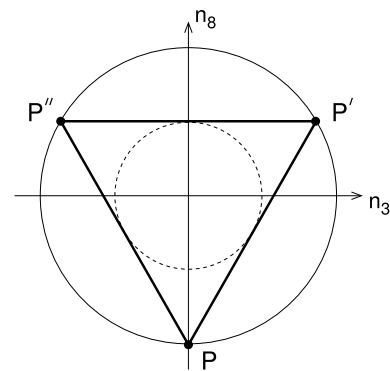
Fig. 2 The orbit space of 3HDM in the (n_3, n_8)-subspace (all other $n_i = 0$). The *outer* and *inner circles* correspond to $|\vec{n}| = 1$ and $|\vec{n}| = 1/2$, respectively

The real symmetric matrix Λ_{ij} has eight real eigenvalues (counted with multiplicity). In order for the potential to be symmetric under a continuous group of transformations, Λ_{ij} must have eigenvalues of multiplicities > 1. Note that any statement about eigenvalues of Λ_{ij} is basis-invariant and therefore it can be checked in any basis. Furthermore, if we find a basis in which Λ_{ij} has a block-diagonal form, and if eigenvalues from different blocks are distinct, then a continuous symmetry requires that each block is either invariant under this symmetry, or contains eigenvalues with multiplicity > 1.

Let us consider an important special case of this situation. Suppose that the potential has no terms of type $(\phi_a^\dagger \phi_a)(\phi_b^\dagger \phi_c)$, where a, b, c are all distinct. This implies the absence of terms $r_{0,3,8} r_{1,2,4,5,6,7}$, and the block-diagonal form of Λ_{ij}, in which two blocks correspond to the (r_3, r_8) subspace and to its orthogonal complement. Suppose also that the eigenvalues of Λ_{ij} in the (r_3, r_8) subspace are distinct from those in the orthogonal complement. It follows then that any possible continuous symmetry must act *trivially* in the (r_3, r_8) subspace, because the orbit space here lacks the rotational invariance. However, if r_0, r_3, and r_8 are fixed, then $\phi_1^\dagger \phi_1$, $\phi_2^\dagger \phi_2$, and $\phi_3^\dagger \phi_3$ are also fixed. So, the doublets do not mix, and the possible continuous symmetry group can only be a subgroup of the group of pure phase rotations, which were studied in [37].

If in addition it is known that a given potential is not symmetric under continuous phase rotations, then we conclude that it does not have any continuous symmetry from $PSU(3)$. It turns out that all the cases of various finite symmetry groups we consider below, except the last one, are of this type. Since the arguments of this section provide a sufficient condition for absence of continuous symmetries, they guarantee that the corresponding potentials can have only finite symmetry groups. Absence of a continuous symmetry in the very last case will be proved separately.

6 Possible extensions: the torus chain

We now check all the candidates for A from the list (13) and see which extension can work in 3HDM. In this section we will deal with the first four groups from the list, which arise as subgroups of the maximal torus; the last group will be considered later. For each group A, we use its explicit realization given in [37] as a group of rephasing transformations, and then we search for additional transformations from $PSU(3)$ with the desired multiplication properties.

6.1 Representing elements of $PSU(3)$

Before we start analysis of each case, let us make a general remark on how we describe the elements of $PSU(3)$. Using the bar notation for the canonical homomorphism $SU(3) \to PSU(3)$, we denote $\bar{H} < PSU(3)$ if its full preimage in $SU(3)$ is H. Denoting the center of $SU(3)$ as $Z = Z(SU(3)) \simeq \mathbb{Z}_3$, we have $Z = \{1, z, z^2\}$, where

$$z = \mathrm{diag}(\omega, \omega, \omega), \quad \omega = e^{2\pi i/3}. \tag{21}$$

The elements of the group H ($a, b, \ldots \in H$) will be written as 3×3 matrices from $SU(3)$. The elements of $\bar{H}$ ($\bar{a}, \bar{b}, \ldots \in \bar{H}$) are the corresponding cosets of Z in H. Explicit manipulation with these cosets is inconvenient, therefore in our calculation we represent an element $\bar{a} \in PSU(3)$ by any of the three representing elements from $SU(3)$: a, az, or az^2. We will usually choose a and then prove that this representation is faithful (does not depend on the choice of representing element).

6.2 Extending $\mathbb{Z}_2$ and $\mathbb{Z}_3$

The smallest group from the list is $A = \mathbb{Z}_2$, whose automorphism group is $Aut(\mathbb{Z}_2) = \{1\}$, so that $G = \mathbb{Z}_2$. This case was already considered in [37].

The next possibility is $A = \mathbb{Z}_3$, whose $Aut(\mathbb{Z}_3) = \mathbb{Z}_2$. The only non-trivial case to be considered is $G/A = \mathbb{Z}_2$, which implies that G can be either $\mathbb{Z}_6$ or $D_6 \simeq S_3$, the symmetry group of the equilateral triangle. The former can be disregarded because it does not appear in the list (13), thus we focus only on the D_6 case.

6.2.1 Constructing D_6

The group D_6 is generated by two elements a, b with the following relations: $a^3 = 1$, $b^2 = 1$, $ab = ba^2$. Following [37], we represent the $\mathbb{Z}_3$ group by phase rotations:

$$a = \mathrm{diag}(\omega, \omega^2, 1). \tag{22}$$

There are in fact three such groups which differ only by the choice of the doublet which is fixed. However, their generators, a, az, and az^2, differ only by a transformation from

the center, and therefore all of them correspond to the same generator $\bar{a}$ from $PSU(3)$. It is straightforward to check that selecting a to represent $\bar{a}$ is a faithful representation.

The explicit solution of the matrix equation $ab = ba^2$ shows that $b \in SU(3)$ must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{23}$$

with an arbitrary δ. The choice of the mixing pair of doublets (ϕ_1 and ϕ_2 in this case) is fixed by the choice of invariant doublet in a.

The fact that b is not uniquely defined means that there exists not a single D_6 group but a whole family of D_6 groups parametrized by the value of δ. Below, when checking whether a potential is D_6 symmetric, we will need to check its invariance under all possible D_6's from this family.

The generic $\mathbb{Z}_3$-symmetric potential contains the part invariant under any phase rotation

$$V_0 = -\sum_{1 \le i \le 3} m_i^2 (\phi_i^\dagger \phi_i) + \sum_{1 \le i \le j \le 3} \lambda_{ij} (\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j)$$
$$+ \sum_{1 \le i < j \le 3} \lambda'_{ij} (\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i),$$

and the following additional terms:

$$V_{\mathbb{Z}_3} = \lambda_1 (\phi_2^\dagger \phi_1)(\phi_3^\dagger \phi_1) + \lambda_2 (\phi_1^\dagger \phi_2)(\phi_3^\dagger \phi_2)$$
$$+ \lambda_3 (\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3) + h.c. \tag{24}$$

with complex λ_1, λ_2, λ_3. At least two of them must be non-zero, otherwise the potential will be symmetric under a continuous group of Higgs-family transformations [37]. Let us denote their phases by ψ_1, ψ_2, and ψ_3, respectively. If the parameters of V_0 satisfy

$$m_{11}^2 = m_{22}^2, \qquad \lambda_{11} = \lambda_{22},$$
$$\lambda_{13} = \lambda_{23}, \qquad \lambda'_{13} = \lambda'_{23}, \tag{25}$$

and if, in addition, $|\lambda_1| = |\lambda_2|$, then the whole potential becomes symmetric under one particular D_6 group constructed with b in (23) with the value of $\delta = (\psi_2 - \psi_1 + \pi)/3 + 2\pi k/3$. The extra freedom given by $2\pi k/3$ corresponds to three order-two elements of D_6: b, ab, a^2b. We opt to define b by setting $k = 0$. Alternatively, we can be compactly write the condition as

$$3\delta = \pi - \psi_1 + \psi_2. \tag{26}$$

To summarize, the criterion of the D_6 symmetry of the potential is that, after a possible doublet relabeling, conditions (25) and (26) are satisfied.

Let us also note that when constructing the group D_6 we could have searched for b satisfying not $ab = ba^2$ but $ab = ba^2 \cdot z^p$, with $p = 1, 2$. Solutions of this equation exist, but they do not lead to any new possibilities. Indeed, let us introduce $a' = az^p$. Then, we get $a'b = ba'^2$. Thus, we get the same equation for b as before, up to a cyclic permutation of doublets, the possibility which we already took into account.

6.2.2 Proving that D_6 is realizable

This construction allows us to write down an example of the D_6-symmetric potential: it is V_0 restricted by conditions (25) plus $V_{\mathbb{Z}_3}$ in (24) subject to $|\lambda_1| = |\lambda_2|$. In order to show that D_6 is realizable, we need to demonstrate that this potential is not symmetric under any larger Higgs-family transformation group.

This proof is short and contains two steps. First, we note that the conditions described in Sect. 5 are fulfilled: the (r_3, r_8)-subspace does not couple to its orthogonal complement via Λ_{ij}, and that the eigenvalues in these two subspaces are defined by different sets of free parameters. The extra terms (24) guarantee that there is only finite group of phase rotations, the group $\mathbb{Z}_3$. Therefore, the sufficient conditions described in Sect. 5 are satisfied, and the generic D_6-symmetric potential has no continuous symmetry.

Second, we need to show that the generic D_6-symmetric potential has no higher discrete symmetries. This is proved by the simple observation that all other finite groups to be discussed below which could possibly contain D_6 lead to *stronger* restrictions on the potential than (25) and $|\lambda_1| = |\lambda_2|$. Therefore, not satisfying those stronger restrictions will yield a potential symmetric only under D_6.

6.2.3 Including antiunitary transformations

Any generalized-*CP* (antiunitary) transformation acting on three doublets is of the form

$$J' = c \cdot J, \quad c \in PSU(3). \tag{27}$$

Here J is the operation of hermitian conjugation of the doublets. If G is the symmetry group of unitary transformations, then it is normal in $\langle G, J' \rangle$, and J' induces automorphisms in G. So, when we search for J', we require that

$$(J')^2 \in G, \qquad (J')^{-1} a J' \in G, \tag{28}$$

where a generically denotes the generators of G. If such a transformation is found, the group is extended from G to $G \rtimes \mathbb{Z}_2^*$, where the asterisk on the group indicates that its generator is antiunitary.

Note the crucial point of our method: when extending G by an antiunitary transformation, we require that the unitary transformation symmetry group remains G. The logic

is simple. If we start with a realizable group G of unitary transformations but do not impose condition (28), we will end up with a potential being symmetric under $\tilde{G} \rtimes \mathbb{Z}_2^*$, with $\tilde{G} > G$. But at the end of this paper we will have a complete list of all finite realizable symmetry groups of unitary transformations, and this list will contain $\tilde{G}$ anyway. So, this possibility is not overlooked but will be studied in its due time after construction of $\tilde{G}$.

Now, turning to extension of D_6 by an antiunitary symmetry, we first note that the resulting group $D_6 \rtimes \mathbb{Z}_2^*$ is a non-abelian group of order 12 containing a normal subgroup D_6. Among the three non-abelian groups of order 12, there exists only one group, namely $D_6 \times \mathbb{Z}_2^*$, with a subgroup D_6 (which is automatically normal because all subgroups of index 2 are normal). This fact can also be proved in a more general way without knowing the list of groups of order 12. Note that it contains, among other, the subgroup $\mathbb{Z}_6^*$; its presence does not contradict the list (13) because that list refers only to the groups of unitary transformations.

Next, let us denote the generator of $\mathbb{Z}_2^*$ by $J' = cJ$. Since J' centralizes the entire D_6, it follows that $(J')^{-1}aJ' = a$, $(J')^{-1}bJ' = b$, and $(J')^2 = cJcJ = cc^* = 1$. The matrix c satisfying these conditions must be of the form

$$c = \begin{pmatrix} 0 & e^{i\gamma} & 0 \\ e^{i\gamma} & 0 & 0 \\ 0 & 0 & -e^{-2i\gamma} \end{pmatrix}, \tag{29}$$

with arbitrary γ. Requiring the potential to stay invariant under J', we obtain the following conditions on γ: $6\gamma = -2(\psi_1 + \psi_2) = 2\psi_3$. Therefore, if the following *extra* condition is fulfilled:

$$2(\psi_1 + \psi_2 + \psi_3) = 0 \tag{30}$$

the D_6-invariant potential becomes symmetric under the group $D_6 \times \mathbb{Z}_2^*$. If this condition is not satisfied, the symmetry group remains D_6 even in the case when antiunitary transformations are allowed. We conclude that both D_6 and $D_6 \times \mathbb{Z}_2^*$ are realizable in 3HDM.

It is interesting to note that if we set $\lambda_3 = 0$, then the potential would still be invariant under D_6. However, in this case it becomes symmetric under J' with $6\gamma = -2(\psi_1 + \psi_2)$, without any extra condition on ψ_1 and ψ_2, and the potential becomes automatically invariant under $D_6 \times \mathbb{Z}_2^*$. So, we conclude that the fact that D_6 is still realizable even if antiunitary transformations are included is due to the special feature of the $\mathbb{Z}_3$-symmetry: we have three, not two terms in the $\mathbb{Z}_3$-symmetric potential, and it is the third term that prevents an automatic antiunitary symmetry.

6.3 Extending $\mathbb{Z}_4$

Let us now take $A = \mathbb{Z}_4$ generated by a. Then $Aut(\mathbb{Z}_4) = \mathbb{Z}_2$, so that $G = \mathbb{Z}_4 . \mathbb{Z}_2$ generated by a and some $b \notin \mathbb{Z}_4$. The

two non-abelian possibilities for G are the dihedral group D_8 representing symmetries of the square, and the quaternion group Q_8. In both cases $b^{-1}ab = a^3$, with the only difference that $b^2 = 1$ for D_8 while $b^2 = a^2$ for Q_8. Note that extension leading to the dihedral group is split, $D_8 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$, while Q_8 is not.

6.3.1 Constructing D_8

Representing a by phase rotations $a = \mathrm{diag}(i, -i, 1)$, we find that b satisfying these conditions is again of the form (23) with arbitrary δ. However, now we do not have the freedom to choose the pair of doublets which are mixed by b: this pair is fixed by a. Also, unlike the $\mathbb{Z}_3$ case, the matrix equation $ab = ba^3 \cdot z$ does not have solutions for $b \in SU(3)$.

The $\mathbb{Z}_4$-symmetric potential (for this choice of a) is $V_0 + V_{\mathbb{Z}_4}$, where

$$V_{\mathbb{Z}_4} = \lambda_1 (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) + \lambda_2 (\phi_1^\dagger \phi_2)^2 + h.c. \tag{31}$$

The phases of λ_1 and λ_2 are, as usual, denoted as ψ_1 and ψ_2, respectively. Upon b, the first term here remains invariant, while the second term transforms as

$$(\phi_1^\dagger \phi_2)^2 \mapsto e^{-4i\delta}(\phi_2^\dagger \phi_1)^2. \tag{32}$$

This means that the potential (31) is *always* symmetric under (23) provided that we choose

$$\delta = \psi_2/2. \tag{33}$$

Therefore, in order to get a D_8-symmetric potential we only require that V_0 satisfies conditions (25). The proof that D_8 is realizable (as long as only unitary transformations are concerned) follows along the same lines as in Sect. 6.2.2.

6.3.2 Including antiunitary transformations

In [37] we found that exactly the same conditions, namely (25) and (33), must be satisfied for existence of an antiunitary transformation commuting with the elements of $\mathbb{Z}_4$. This transformation is again $J' = cJ$, where c is given by (29) with $6\gamma = 2\psi_1$, and it commutes with all elements of D_8. Therefore, if we include antiunitary transformations, we automatically get the group $D_8 \times \mathbb{Z}_2^*$, while D_8 becomes non-realizable. Note that the resulting group does not contain $\mathbb{Z}_8^*$. Indeed, we showed in [37] that imposing $\mathbb{Z}_8^*$ symmetry group leads to a potential with continuous symmetry.

6.3.3 Attempting at Q_8

Solving matrix equations $ab = ba^3$ and $b^2 = a^2$, we get the following form of b:

$$b(Q_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

By checking how $V_{\mathbb{Z}_4}$ in (31) transforms under it, we find that the first term simply changes its sign. The only way to make the potential symmetric under Q_8 is to set $\lambda_1 = 0$. But then we know from [37] that the potential becomes invariant under a continuous group of phase rotations. Therefore, Q_8 is not realizable.

6.4 Extending $\mathbb{Z}_2 \times \mathbb{Z}_2$

If $A = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) = GL_2(2) = S_3$. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be realized as the group of independent sign flips of the three doublets with generators $a_1 = \mathrm{diag}(1, -1, -1)$ (equivalent to the sign flip of the first doublet) and $a_2 = \mathrm{diag}(-1, 1, -1)$ (equivalent to the sign flip of the second doublet), so that $a_1 a_2$ is equivalent to the sign flip of the third doublet. The potential symmetric under this group contains V_0 and additional terms

$$\begin{aligned} V_{\mathbb{Z}_2 \times \mathbb{Z}_2} \\ = \tilde{\lambda}_{12}(\phi_1^\dagger \phi_2)^2 + \tilde{\lambda}_{23}(\phi_2^\dagger \phi_3)^2 + \tilde{\lambda}_{31}(\phi_3^\dagger \phi_1)^2 + h.c. \end{aligned} \tag{35}$$

with at least two among coefficients $\tilde{\lambda}_{ij}$ being non-zero. The coefficients can be complex; as usual we denote their phases as ψ_{ij}. This model is also known as Weinberg's 3HDM [6].

The non-abelian finite group G can be constructed as extension of A by $\mathbb{Z}_2$, by $\mathbb{Z}_3$, or by S_3.

6.4.1 Extension $(\mathbb{Z}_2 \times \mathbb{Z}_2) . \mathbb{Z}_2$

Consider first the extension $(\mathbb{Z}_2 \times \mathbb{Z}_2) . \mathbb{Z}_2$. The only extension leading to a non-abelian group is $(\mathbb{Z}_2 \times \mathbb{Z}_2) . \mathbb{Z}_2 = D_8$, and we already proved that this group is realizable. Nevertheless, we prefer to explicitly work it out to see the reduction of free parameters.

The element b which we search for must act on $\{a_1, a_2, a_1 a_2\}$ as a transposition of any pair. In addition, $b^2 \in \mathbb{Z}_2 \times \mathbb{Z}_2$. It does not matter which pair of generators is transposed, as this choice can be changes by renumbering the doublets. So, we take b such that $b^{-1}a_1 b = a_2$ and $b^{-1}a_2 b = a_1$. Then, b^2 can be either 1 or $a_1 a_2$, because choices $b^2 = a_1$ or a_2 lead to inconsistent relations. Indeed, if we assume $b^2 = a_1$, then

$$a_2 = b^{-1}a_1 b = b^{-1}b^2 b = b^2 = a_1,$$

which is a contradiction. In both cases ($b^2 = 1$ and $b^2 = a_1 a_2$) we get the group D_8. Even more, we get the *same* D_8 group: if $b^2 = a_1 a_2$, then $b' = ba_1$ satisfies $b'^2 = 1$, while its action on a_1 and a_2 remains the same. So, it is sufficient to focus on the $b^2 = 1$ case only.

Again, explicitly solving the matrix equations, we get b of the form (23) with arbitrary δ. Then, we check how the potential (35) changes upon b and find that we need to set

$$4\delta = 2\psi_{12}, \qquad 2\delta = -(\psi_{23} + \psi_{31}), \qquad |\tilde{\lambda}_{23}| = |\tilde{\lambda}_{31}|. \tag{36}$$

Equations on the phase δ can be satisfied if

$$2(\psi_{12} + \psi_{23} + \psi_{31}) = 0 \quad \Leftrightarrow \quad \mathrm{Im}(\tilde{\lambda}_{12}\tilde{\lambda}_{23}\tilde{\lambda}_{31}) = 0. \tag{37}$$

So, if: (1) this condition is satisfied, (2) two among $|\tilde{\lambda}_{ij}|$ are equal, (3) condition on V_0 (25) is satisfied, then the potential is D_8-symmetric. Note also that if $\tilde{\lambda}_{12} = 0$ (which we are allowed to consider because (35) contains three rather than two terms), then condition on the phases is not needed.

It might seem that these conditions on the potential to make it D_8-symmetric are more restrictive than in the $\mathbb{Z}_4$ extension we studied above. However, note that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric potential (35) has six free parameters, and we placed two conditions to reduce the number of free parameters in the D_8 potential to four (apart from V_0). On the other hand, (31) had only four from the beginning, and without any restriction this number survives. Therefore we have the same number of degrees of freedom when constructing D_8 in either way.

6.4.2 Constructing $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes Z_3 = T$

The extension by $\mathbb{Z}_3$ is necessarily split, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, leading to the group $T \simeq A_4$, the symmetry group of the tetrahedron. To construct it, we need b such that $b^3 = 1$ with the property that b acts on $\{a_1, a_2, a_1 a_2\}$ by cyclic permutations. Fixing the order of permutations by $b^{-1} a_1 b = a_2$, we find that b must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta_1} & 0 \\ 0 & 0 & e^{i\delta_2} \\ e^{-i(\delta_1 + \delta_2)} & 0 & 0 \end{pmatrix}, \tag{38}$$

with arbitrary δ_1, δ_2. It then follows that if the coefficients in (35) satisfy

$$|\tilde{\lambda}_{12}| = |\tilde{\lambda}_{23}| = |\tilde{\lambda}_{31}|, \tag{39}$$

then $V_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is symmetric under one particular b with

$$\delta_1 = \frac{2\psi_{12} - \psi_{31} - \psi_{23}}{6}, \qquad \delta_2 = \frac{2\psi_{23} - \psi_{31} - \psi_{12}}{6}.$$

Then, by a rephasing transformation one also make the phases of all $\tilde{\lambda}_{ij}$ equal and bring (35) to the following form:

$$V_T = \tilde{\lambda}\big[(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2\big] + h.c. \tag{40}$$

with a complex $\tilde{\lambda}$. In this form, the parameters $\delta_1 = \delta_2 = 0$, and the matrix b is just the cyclic permutation of the doublets. In addition, the symmetry under b places stronger conditions on the parameters of V_0, so that the most general V_0 satisfying them is

$$
\begin{aligned}
V_0 = {}& -m^2\big[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)\big] \\
& + \lambda\big[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)\big]^2 \\
& + \lambda'\big[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) \\
& + (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)\big] \\
& + \lambda''\big(|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2\big). \tag{41}
\end{aligned}
$$

6.4.3 Constructing $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3 = O$

The last extension, $(\mathbb{Z}_2 \times \mathbb{Z}_2) . S_3$, is also split, otherwise we would obtain $\mathbb{Z}_6$. It leads to the group $O \simeq S_4$, the symmetry group of the octahedron and the cube. As it includes T as a subgroup, the most general O-symmetric potential is V_0 from (41) plus V_T from (40) with the additional condition that $\tilde{\lambda}$ is real (the extra symmetry with respect to the T-symmetric case is a transposition of any two doublets).

6.4.4 Including antiunitary transformations

The case of D_8 has been already considered in Sect. 6.3.2.

The tetrahedral potential $V_T + V_0$ from (40) and (41) is symmetric under the following antiunitary transformation:

$$J' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot J, \tag{42}$$

which generates a $\mathbb{Z}_2^*$ group. Therefore the symmetry group of this potential is the full achiral tetrahedral group $T_d \simeq T \rtimes \mathbb{Z}_2^*$, which is isomorphic to S_4.

The octahedral potential is a particular case of the tetrahedral one, therefore it is also invariant under an antiunitary transformation. The extra $\mathbb{Z}_2^*$ subgroup is generated by the complex conjugation, J, and this transformation commutes with the entire Higgs-family group O. Therefore, the symmetry group of the potential is the full achiral octahedral symmetry group $O_h \simeq O \times \mathbb{Z}_2^*$.

6.5 Extensions of abelian groups by an antiunitary transformation

The last type of extension we need to consider is of the type $A . \mathbb{Z}_2^*$, where A is one of the four abelian groups of Higgs-family transformations lying in a maximal torus, that is, the first four groups in the list (13), while the $\mathbb{Z}_2^*$ is as usual generated by an antiunitary transformation $J' = cJ$. This problem was partly solved in [37], where such extensions leading to *abelian* groups were analyzed. It was established that only the following four abelian groups of this type are realizable: $\mathbb{Z}_2^*$, $\mathbb{Z}_4^*$, $\mathbb{Z}_2 \times \mathbb{Z}_2^*$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^*$. Here, we consider *non-abelian* extensions of this type.

6.5.1 Anti-unitary extension of $\mathbb{Z}_3$

The smallest non-abelian group we can have is $\mathbb{Z}_3 \rtimes \mathbb{Z}_2^* \simeq D_6$. We stress that this D_6 group we search for is different from what we analyzed in Sect. 6.2, because there the D_6 group contained only unitary transformations, see a discussion in Sect. 8.3. Using the same notation for the generator a of the $\mathbb{Z}_3$ group, we find that the transformation c in the definition of J' must be diagonal: $c = \mathrm{diag}(e^{i\xi_1}, e^{i\xi_2}, e^{-i(\xi_1+\xi_2)})$. Then, studying how the $\mathbb{Z}_3$-symmetric potential $V_0 + V_{\mathbb{Z}_3}$ changes under $J' = cJ$, we find that the only condition to be satisfied is (30).

If this condition is satisfied, then the potential is invariant under $\mathbb{Z}_3 \rtimes \mathbb{Z}_2^* \simeq D_6$, if not, then the symmetry group remains $\mathbb{Z}_3$. This proves that both groups are realizable in 3HDM. Note that in contrast with the $D_6 \times \mathbb{Z}_2^*$ case, we do not place any extra condition such as (25).

6.5.2 Anti-unitary extension of $\mathbb{Z}_4$

A priori, the two non-abelian extensions here are again D_8 and Q_8. With the usual convention for a, the generator of $\mathbb{Z}_4$, we again find that c must be of the same diagonal form. This immediately excludes the Q_8 case because we have $(J')^2 = c^*c = 1$.

The case of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^* \simeq D_8$ is possible. Even more, it turns out that the $\mathbb{Z}_4$-symmetric potential $V_0 + V_{\mathbb{Z}_4}$ is *always* symmetric under some J' of this type. It means, therefore, that if anti-unitary transformations are included, $\mathbb{Z}_4$ is not realizable anymore: the true symmetry group of the potential is $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^* \simeq D_8$. In more physical terms, we conclude that the *presence of a $\mathbb{Z}_4$ group of Higgs-family transformations makes the potential explicitly CP-conserving*.

6.5.3 Anti-unitary extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$

The only non-abelian extension of the type $(\mathbb{Z}_2 \times \mathbb{Z}_2) . \mathbb{Z}_2^*$ can produce only D_8, which was already considered. We only remark here that c turns out to be of the type (29), which places extra constraints on V_0. Not satisfying these constraints will keep the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which means that it is realizable.

7 The $\mathbb{Z}_3 \times \mathbb{Z}_3$ chain

7.1 The group and its extensions

The last abelian group from the list (13), $\mathbb{Z}_3 \times \mathbb{Z}_3$, requires a special treatment due to a number of reasons. First, it does not belong to any maximal torus of $PSU(3)$ but is a maximal abelian subgroup of $PSU(3)$ on its own [37], and its full preimage in $SU(3)$ is the non-abelian group $\Delta(27)$ [53].

Second, its automorphism group $Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is sufficiently large and requires an accurate description.

Let us first remind the reader how this group is constructed. We first consider the subgroup of $SU(3)$ generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{43}$$

This group known as $\Delta(27)$ is non-abelian because a and b do not commute, but their commutator lies in the center of $SU(3)$:

$$[a, b] = aba^{-1}b^{-1} = z^2 \in Z(SU(3)). \tag{44}$$

Therefore, its image under the canonical homomorphism $SU(3) \rightarrow PSU(3)$ becomes the desired abelian group $\Delta(27)/\mathbb{Z}_3 = \mathbb{Z}_3 \times \mathbb{Z}_3$. The true generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are cosets $\bar{a} = aZ(SU(3))$ and $\bar{b} = bZ(SU(3))$ from $PSU(3)$, and they obviously commute: $[\bar{a}, \bar{b}] = 1$. Note that since $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a maximal abelian subgroup in $PSU(3)$, there is no other element in $PSU(3)$ commuting with all elements of this group, so $C_{PSU(3)}(\mathbb{Z}_3 \times \mathbb{Z}_3) = \mathbb{Z}_3 \times \mathbb{Z}_3$.

If the normal self-centralizing abelian subgroup of G, whose existence was proved in Sect. 4.2, is $A = \mathbb{Z}_3 \times \mathbb{Z}_3$, then G can be constructed as an extension of A by a subgroup of $Aut(\mathbb{Z}_3 \times \mathbb{Z}_3) = GL_2(3)$, the general linear group of transformations of two-dimensional vector space over the finite field $\mathbb{F}_3$. The order of this group is $|GL_2(3)| = 48$, and it will prove useful if we now digress and describe the structure of this group in some detail.

7.1.1 $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a vector space over $\mathbb{F}_3$

The finite field $\mathbb{F}_3$ is defined as the additive group of integers mod 3, in which the multiplication is also introduced. It is convenient to denote the elements of this field as $0, 1, -1$ with obvious addition and multiplication laws. Unlike the integers themselves, $\mathbb{F}_3$ is closed under division by a non-zero number, the property that makes $\mathbb{F}_3$ a field.

A vector space over a finite field is defined just as over any "usual" field. The group $\mathbb{Z}_3 \times \mathbb{Z}_3$ can be thought of as a 2D vector space over $\mathbb{F}_3$; its elements are (with the additive notation for the group operation) $\bar{x} = q_a\bar{a} + q_b\bar{b}$, where $q_a, q_b \in \mathbb{F}_3$, and $\bar{a}, \bar{b}$ are, as before, the generators of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$. In the multiplicative notation, we write $\bar{x} = \bar{a}^{q_a}\bar{b}^{q_b}$.

It is possible to define an antisymmetric scalar product in this space. For any $\bar{x} \in \mathbb{Z}_3 \times \mathbb{Z}_3$, take any element of its preimage, $x \in \Delta(27)$. Then, for any two elements $\bar{x}, \bar{y} \in \mathbb{Z}_3 \times \mathbb{Z}_3$, construct the number $(\bar{x}, \bar{y})$ as $[x, y] \in \mathbb{F}_3$. This map is faithful: although we can select different x for a given $\bar{x}$, all of them give the same $[x, y]$.

Clearly, $(\bar{x}, \bar{y}) = -(\bar{y}, \bar{x})$, in the additive notation. Besides, the so defined product is linear in both arguments:

$$(\bar{x}_1 + \bar{x}_2, \bar{y}) = (\bar{x}_1, \bar{y}) + (\bar{x}_2, \bar{y}),$$
$$(\bar{x}, \bar{y}_1 + \bar{y}_2) = (\bar{x}, \bar{y}_1) + (\bar{x}, \bar{y}_2). \tag{45}$$

Indeed, for any three elements of any group the following relation holds:

$$[xy, z] = xyzy^{-1}x^{-1}z^{-1} = xyzy^{-1} \cdot z^{-1}x^{-1}xz \cdot x^{-1}z^{-1}$$
$$= x[y, z]x^{-1}[x, z]. \tag{46}$$

If in addition all commutators take values in the center of the group $SU(N)$, then x and x^{-1} can be canceled, and we get $[xy, z] = [y, z][x, z]$. In our case we represent $x_1 = \bar{a}^{q_{a1}} \bar{b}^{q_{b1}} z^{r_1}$ and similarly for x_2 and y, and noting that all z^{r_i} are inessential, we recover the above linearity in the first argument. Thus, $\mathbb{Z}_3 \times \mathbb{Z}_3$ becomes a vector space over $\mathbb{F}_3$ equipped with an antisymmetric scalar product.

Note that all antisymmetric products in $\mathbb{Z}_3 \times \mathbb{Z}_3$ are proportional to $(\bar{a}, \bar{b})$. Indeed, if two elements $\bar{x}$ and $\bar{x}'$ are defined by their vectors $\vec{q} = (q_a, q_b)$ and $\vec{q}' = (q_a', q_b')$, then due to bilinearity we get

$$(\bar{x}, \bar{x}') = (q_a q_b' - q_b q_a')(\bar{a}, \bar{b}) = \epsilon_{ij} q_i q_j'(\bar{a}, \bar{b}), \tag{47}$$

where ϵ_{ij} is the standard antisymmetric tensor with $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$.

7.1.2 The automorphism group of $\mathbb{Z}_3 \times \mathbb{Z}_3$

The automorphism group of $\mathbb{Z}_3 \times \mathbb{Z}_3$ can then be viewed as the group of non-degenerate matrices with elements from $\mathbb{F}_3$ acting in this 2D space, which explains why $Aut(\mathbb{Z}_3 \times \mathbb{Z}_3) = GL_2(3)$. Each matrix q can be defined by its action on the generators $\bar{a}, \bar{b}$: $\bar{a} \mapsto q_{aa}\bar{a} + q_{ab}\bar{b}$, $\bar{b} \mapsto q_{ba}\bar{a} + q_{bb}\bar{b}$, and can therefore be written as

$$q = \begin{pmatrix} q_{aa} & q_{ab} \\ q_{ba} & q_{bb} \end{pmatrix}, \quad \det q \neq 0. \tag{48}$$

The group operation in $GL_2(3)$ is just the matrix product.

Recall now that the elements of both the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group and its automorphism group are represented in our case as unitary or antiunitary transformations of the three doublets (that is, we work not with the abstract groups but with their three-dimensional complex representations). Since $\mathbb{Z}_3 \times \mathbb{Z}_3$ is assumed to be normal in G, the elements $g \in Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$ act on the elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$ by conjugation: $\bar{x} \mapsto g^{-1}\bar{x}g$, which we denoted by $g(\bar{x})$. Then the antisymmetric product defined above changes upon this action in the following way:

$$(g(\bar{x}), g(\bar{y}))$$

$$= g^{-1}[x, y]g = g^{-1}z^r g$$
$$= \begin{cases} z^r = (\bar{x}, \bar{y}), & \text{if } g \text{ is unitary,} \\ (z^*)^r = (z^{-1})^r = -(\bar{x}, \bar{y}), & \text{if } g \text{ is anti-unitary.} \end{cases} \tag{49}$$

Here we used the fact the commutator of any two elements of $\Delta(27)$ lies in the center $Z(SU(3))$, and that the CP conjugation operator J acts on any $x \in SU(3)$ by $J^{-1}xJ = x^*$. So, unitary transformations preserve the antisymmetric product, while anti-unitary ones flip its sign.

Generically, the subgroup of a general linear group which conserves an antisymmetric bilinear product in a vector space is called symplectic. Here we have the group $Sp_2(3) < GL_2(3)$. It turns out that $Sp_2(3) = SL_2(3)$. Indeed, suppose $g \in GL_2(3)$ acts in the 2D space over $\mathbb{F}_3$ by mapping $q_i \mapsto g(q) = g_{ii'}q_{i'}$. Then, the product transforms as

$$(\bar{x}, \bar{y}) \mapsto (g(\bar{x}), g(\bar{y})) = \epsilon_{ij} g_{ii'} g_{jj'} q_{i'}^{(x)} q_{j'}^{(y)} (\bar{a}, \bar{b})$$
$$= \det g \cdot (\bar{x}, \bar{y}). \tag{50}$$

Since $\det g = \pm 1$, we get two kinds of transformations: those which conserve all products ($\det g = 1$, so that $g \in SL_2(3)$) and those which flip their signs ($\det g = -1$), hence the identification of $Sp_2(3)$ and $SL_2(3)$ follows.

We conclude that the finite symmetry group G of unitary transformations with the normal self-centralizing abelian subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3$ can be constructed as extension $(\mathbb{Z}_3 \times \mathbb{Z}_3) . K$, where $K \leq SL_2(3)$.

7.1.3 Explicit description of $SL_2(3)$

The structure of the group $SL_2(3)$ is well-known, but it will prove useful to have the explicit expressions for some of its elements.

The order of the group is $|SL_2(3)| = 24$. It contains elements of order 2, 3, 4, and 6, generating the corresponding cyclic subgroups. The subgroup $\mathbb{Z}_2$ is generated by the center of the group

$$c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{51}$$

which in the multiplicative notation means $\bar{a} \mapsto \bar{a}^2, \bar{b} \mapsto \bar{b}^2$. There are four distinct $\mathbb{Z}_3$ subgroups generated by

$$f_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$f_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \tag{52}$$

three $\mathbb{Z}_4$ subgroups generated by

$$d_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$
$$d_3 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{53}$$

and four $\mathbb{Z}_6$ subgroups, which we do not write explicitly because they are absent in the list (13).

Every element of $SL_2(3)$ can be represented by a unique (up to center) $SU(3)$ matrix, which can be found by explicitly solving the corresponding matrix equations defining the action of this element. For example, the transformation c is defined by

$$c(a) = c^{-1}ac = a^2, \qquad c(b) = c^{-1}bc = b^2. \qquad (54)$$

Rewriting these equations as 3×3 matrix equations $ac = ca^2$, $bc = cb^2$ and solving them explicitly, we find the matrix c:

$$c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \qquad (55)$$

7.2 Generic potential

A generic potential symmetric under $\mathbb{Z}_3 \times \mathbb{Z}_3$ is

$$\begin{aligned}
V = {} & -m^2 \big[\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\big] \\
& + \lambda_0 \big[\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\big]^2 \\
& + \frac{\lambda_1}{\sqrt{3}}\big[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2 \\
& - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) - (\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\big] \\
& + \lambda_2\big(|\phi_1^\dagger\phi_2|^2 + |\phi_2^\dagger\phi_3|^2 + |\phi_3^\dagger\phi_1|^2\big) \\
& + \big(\lambda_3\big[(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3) + (\phi_2^\dagger\phi_3)(\phi_2^\dagger\phi_1) \\
& + (\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2)\big] + h.c.\big)
\end{aligned} \qquad (56)$$

with real m^2, λ_0, λ_1, λ_2 and complex λ_3. All values here are generic. This potential can be found by taking the potential symmetric under the $\mathbb{Z}_3$ group of phase rotations described above and then requiring that it be invariant under the cyclic permutations on the doublets. Written in the space of bilinears, the potential has the form

$$\begin{aligned}
V = {} & -\sqrt{3}m^2 r_0 + 3\lambda_0 r_0^2 + \sqrt{3}\lambda_1\big(r_3^2 + r_8^2\big) \\
& + \lambda_2\big(|r_{12}|^2 + |r_{45}|^2 + |r_{67}|^2\big) \\
& + \lambda_3\big(r_{12}r_{45}^* + r_{67}r_{12}^* + r_{45}r_{67}^*\big) \\
& + \lambda_3^*\big(r_{12}^* r_{45} + r_{67}^* r_{12} + r_{45}^* r_{67}\big) \\
= {} & -\sqrt{3}m^2 r_0 + 3\lambda_0 r_0^2 + \Lambda_{ij} r_i r_j.
\end{aligned} \qquad (57)$$

It is important to prove that this potential has no continuous symmetry. Using the approach described in Sect. 5, we calculate the eigenvalues of Λ_{ij} and find that it has four distinct eigenvalues of multiplicity two:

$$\sqrt{3}\lambda_1, \qquad \lambda_2 + \lambda_3 + \lambda_3^*, \qquad \lambda_2 + \omega\lambda_3 + \omega^2\lambda_3^*,$$
$$\lambda_2 + \omega^2\lambda_3 + \omega\lambda_3^*. \qquad (58)$$

The first eigenvalue corresponds to the subspace (r_3, r_8), while the rest are three 2D subspaces within its orthogonal complement $(r_1, r_2, r_4, r_5, r_6, r_7)$. For generic values of the coefficients, they do not coincide. Then, according to our discussion in Sect. 5, a continuous symmetry group, if present, must consist only of phase rotations of the doublets. But the λ_3 term selects only the $\mathbb{Z}_3$ group of phase rotations, which proves that no continuous symmetry leaves this potential invariant.

7.3 Extension $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$

It turns out that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not realizable because the potential (56) is symmetric under a larger group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \Delta(54)/\mathbb{Z}_3$, which is generated by $\bar{a}, \bar{b}, \bar{c}$ with the following relations:

$$\bar{a}^3 = \bar{b}^3 = 1, \qquad \bar{c}^2 = 1, \qquad [\bar{a}, \bar{b}] = 1,$$
$$\bar{c}\bar{a}\bar{c} = \bar{a}^2, \qquad \bar{c}\bar{b}\bar{c} = \bar{b}^2.$$

In terms of explicit transformation laws, $\bar{c}$ is the coset $cZ(SU(3))$, with c being the exchange of any two doublets, for example (55). Note that $\langle \bar{a}, \bar{c} \rangle = S_3$ is the group of arbitrary permutations of the three doublets. Thus, if $G = (\mathbb{Z}_3 \times \mathbb{Z}_3).K$, then a G-symmetric potential must be a restriction of (56), and K must contain a $\mathbb{Z}_2$ subgroup.

There are three kinds of subgroups of $SL_2(3)$ containing $\mathbb{Z}_2$ but not containing $\mathbb{Z}_6$: $\mathbb{Z}_2$, $\mathbb{Z}_4$, and Q_8. In each case it would give a split extension, so G must contain a subgroup isomorphic to one of these groups. Since, as we argued above, the quaternion group Q_8 is not realizable in 3HDM, K can only be $\mathbb{Z}_2$ or $\mathbb{Z}_4$. Therefore, the only additional case to consider is $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, the group also known as $\Sigma(36)$ [53].

7.4 Extension $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$

There are three distinct $\mathbb{Z}_4$ subgroups in $SL_2(3)$ generated by d_1, d_2, and d_3, listed in (53). In principle, all of them are conjugate inside $SL_2(3)$, but for our purposes all of them need to be checked. Explicit solutions of the matrix equations give the following transformations:

$$d_1 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix},$$

$$d_2 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ \omega & \omega^2 & \omega \end{pmatrix}, \qquad (59)$$

$$d_3 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 1 & \omega^2 \\ \omega & 1 & \omega \\ 1 & \omega & \omega \end{pmatrix}.$$

Note that the prefactor $i/\sqrt{3}$ can also be written as $1/(\omega^2 - \omega)$.

Let us mention here that when searching for explicit $SU(3)$ realizations of the transformations d_1, we solve equations $d_1^{-1}ad_1 = b$, $d_1^{-1}bd_1 = a^2$. However, we could also use other representative matrices, a' and b', which differ from a and b by transformations from the center. For example, we can also ask for solutions of

$$d_1'^{-1}ad_1' = z^{n_1}b, \qquad d_1'^{-1}bd_1' = z^{n_2}a^2. \tag{60}$$

However, the solution of this equation can be written as

$$d_1' = d_1 a^{n_1} b^{n_2}. \tag{61}$$

Therefore the resulting group $\langle \bar{d}_1', \bar{a}, \bar{b} \rangle$ coincides with $\langle \bar{d}_1, \bar{a}, \bar{b} \rangle$. The similar results hold for d_2 and d_3.

7.4.1 Conditions for the $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ symmetry

We should now check how the potential (56) changes under these transformations and when it remains invariant. The calculation is simplified if we introduce the following combinations of bilinears (here i^*j stands for $\phi_i^\dagger \phi_j$):

$$A_0 = 1^*1 + 2^*2 + 3^*3, \qquad A_1 = 1^*1 + \omega 2^*2 + \omega^2 3^*3,$$

$$A_2 = A_1^*, \qquad B_0 = 1^*2 + 2^*3 + 3^*1,$$

$$B_1 = 1^*2 + \omega 2^*3 + \omega^2 3^*1,$$

$$B_2 = 1^*2 + \omega^2 2^*3 + \omega 3^*1,$$

$$B_0^* = 2^*1 + 3^*2 + 1^*3, \qquad B_1^* = 2^*1 + \omega^2 3^*2 + \omega 1^*3,$$

$$B_2^* = 2^*1 + \omega 3^*2 + \omega^2 1^*3.$$

Next, introducing

$$X = \frac{1}{\sqrt{3}}\Big[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2)$$

$$\qquad - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1) \Big],$$

$$\quad = \frac{1}{\sqrt{3}}|A_1|^2,$$

$$Y = \big|\phi_1^\dagger \phi_2\big|^2 + \big|\phi_2^\dagger \phi_3\big|^2 + \big|\phi_3^\dagger \phi_1\big|^2 \tag{62}$$

$$\quad = \frac{|B_0|^2 + |B_1|^2 + |B_2|^2}{3},$$

$$Z^* = (\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_1) + (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2)$$

$$\quad = \frac{|B_0|^2 + \omega^2|B_1|^2 + \omega|B_2|^2}{3},$$

we write the potential (56) as

$$V = -\sqrt{3}m^2 r_0 + 3\lambda_0 r_0^2 + \lambda_i^* X_i,$$

$$\text{where } \lambda_i^* X_i = \lambda_1 X + \lambda_2 Y + \lambda_3 Z^* + \lambda_3^* Z \tag{63}$$

is the scalar product of the vector of coefficients and the vector of coordinates. Now, it follows from explicit calculations that the action of d_i can be compactly represented by the following transformations:

$$d_1: \quad A_1 \to B_0, \qquad B_0 \to A_1^*,$$
$$\qquad B_1 \to \omega^2 B_2, \qquad B_2 \to B_1^*,$$
$$d_2: \quad A_1 \to B_1, \qquad B_1 \to \omega A_1^*,$$
$$\qquad B_0 \to \omega B_2^*, \qquad B_2 \to B_0,$$
$$d_3: \quad A_1 \to B_2, \qquad B_2 \to \omega A_1^*,$$
$$\qquad B_0 \to B_1, \qquad B_1 \to \omega B_0^*,$$

or even more compactly

$$d_1: \quad |A_1|^2 \leftrightarrow |B_0|^2, \qquad |B_1|^2 \leftrightarrow |B_2|^2,$$
$$d_2: \quad |A_1|^2 \leftrightarrow |B_1|^2, \qquad |B_0|^2 \leftrightarrow |B_2|^2, \tag{64}$$
$$d_3: \quad |A_1|^2 \leftrightarrow |B_2|^2, \qquad |B_0|^2 \leftrightarrow |B_1|^2.$$

Therefore, their action in the space of (X, Y, Z, Z^*) is given by the following hermitian and unitary matrices:

$$T(d_1) = \frac{1}{3} \begin{pmatrix} 0 & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & 2 & -1 & -1 \\ \sqrt{3} & -1 & -1 & 2 \\ \sqrt{3} & -1 & 2 & -1 \end{pmatrix},$$

$$T(d_2) = \frac{1}{3} \begin{pmatrix} 0 & \sqrt{3} & \omega^2\sqrt{3} & \omega\sqrt{3} \\ \sqrt{3} & 2 & -\omega^2 & -\omega \\ \omega\sqrt{3} & -\omega & -1 & 2\omega^2 \\ \omega^2\sqrt{3} & -\omega^2 & 2\omega & -1 \end{pmatrix},$$

and $T(d_3) = [T(d_2)]^*$. It can be also noted that $T(d_2)$ acts in the space of $(X, Y, \omega^2 Z, \omega Z^*)$ by the matrix $T(d_1)$. So, $T(d_1)$, $T(d_2)$ and $T(d_3)$ represent the same type of transformations acting in the spaces (X, Y, Z, Z^*), $(X, Y, \omega^2 Z, \omega Z^*)$, or $(X, Y, \omega Z, \omega^2 Z^*)$, respectively. That is, if (x, y, z, z^*) is an eigenvector of $T(d_1)$, then $(x, y, \omega z, \omega^2 z^*)$ is an eigenvector of $T(d_2)$ and $(x, y, \omega^2 z, \omega z^*)$ is an eigenvector of $T(d_3)$. This observation restores the expected symmetry among the three types of $\mathbb{Z}_4$ subgroup inside $SL_2(3)$.

Since these matrices are hermitian and unitary, they act by pure reflections, which implies that each of them is diagonalizable and has eigenvalues ± 1. If we want the potential to be symmetric under one of these d_i, it must induce the same transformations in the space of $\lambda_i = (\lambda_1, \lambda_2, \lambda_3^*, \lambda_3)$. Therefore, in order to find conditions that the potential is invariant under d_i, we need to find eigenvectors of $T(d_i)$ corresponding to the eigenvalue -1 and require that λ_i's projection on these eigenvectors is zero.

Consider first $T(d_1)$. It has two eigenvectors corresponding to the eigenvalue -1: $(-\sqrt{3}, 1, 1, 1)$ and $(0, 0, 1, -1)$.

Therefore, we obtain the following condition for the potential to be symmetric under d_1:

$$\lambda_3 \text{ is real and } \lambda_3 = \frac{\sqrt{3}\lambda_1 - \lambda_2}{2}. \tag{65}$$

Similarly, for d_2 we have

$$\omega\lambda_3 \text{ is real and } \omega\lambda_3 = \frac{\sqrt{3}\lambda_1 - \lambda_2}{2}. \tag{66}$$

For d_3 we have the complex conjugate condition. Therefore, the potential (56) is symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ if

$$\left(\frac{2\lambda_3}{\sqrt{3}\lambda_1 - \lambda_2}\right)^3 = 1, \tag{67}$$

which encompasses all these cases. Let us also mention that when these conditions are taken into account, the spectrum of the matrix Λ_{ij} given in (58) becomes even more degenerate: it contains two eigenvalues of multiplicity four (we refer to this spectrum as $4 + 4$).

7.4.2 Absence of a continuous symmetry

In order for the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ to be realizable, we need to show that the potential (56) with parameters satisfying (67) is not symmetric under any continuous group.

We first note that even if such a continuous symmetry group existed, it could only be $U(1)$. Indeed, the spectrum of Λ_{ij} in our case is $4+4$, while for $U(1) \times U(1)$ and $SU(2)$ it must be $6 + 2$, and for $SO(3)$ it must be $5 + 3$.

Let us now consider, for example, the d_1-symmetric potential. Using $\sum_{i=1}^{8} r_i^2 = \alpha r_0^2$, where $1/4 \leq \alpha \leq 1$ parametrizes $SU(3)$-orbits in the orbit space, we can rewrite it as

$$V = -\sqrt{3}m^2 r_0 + (3\lambda_0 + \alpha\sqrt{3}\lambda_1) r_0^2$$
$$- \frac{\sqrt{3}\lambda_1 - \lambda_2}{2}\left(|r_{12} - r_{45}|^2 + |r_{45} - r_{67}|^2\right.$$
$$\left. + |r_{67} - r_{12}|^2\right). \tag{68}$$

Suppose the potential (68) is invariant under a $U(1)$ group of transformations of doublets, generated by the generator t from the algebra $su(3)$. Since the potential (68) is invariant under the S_3 group of arbitrary permutations of the doublets, then the same potential must be also invariant under other $U(1)$ subgroups which are generated by various t^g, which are obtained by acting on t by $g \in S_3$. If $t \neq t^g$ (or to be more accurate, if their corresponding $U(1)$ groups are different), then the continuous symmetry group immediately becomes larger than $U(1)$, which is impossible. Therefore, t^g must be equal (up to sign) to t for all $g \in S_3$. In other words, S_3 must stabilize the $U(1)$ symmetry group.

There exist only two elements in the algebra $su(3)$ with this property:

$$t_1 = \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix} \quad \text{and} \quad t_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{69}$$

t_2 generates pure phase rotations. It is explicitly S_3-invariant, therefore the corresponding $U(1)$ group is also invariant. t_1 induces $SO(3)$ rotations of the doublets around the axis $(1, 1, 1)$. It is $\mathbb{Z}_3$-invariant, while reflections from S_3 flip the sign of t_1. However, the $U(1)$ group is still invariant. Since t_1 and t_2 realize different representations of S_3, one cannot take their linear combinations. So, the list of possibilities is restricted only to t_1 and t_2 themselves.

The eigenvalues and eigenvectors of t_1 are

$$\zeta = 0: \ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \zeta = \sqrt{3}: \ \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix},$$
$$\zeta = -\sqrt{3}: \ \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}. \tag{70}$$

The presence of the eigenvalue $\zeta = 0$ implies that the combination $\phi_1 + \phi_2 + \phi_3$ is invariant under the $U(1)$ group generated by t_1. Bilinear invariants are

$$|\phi_1 + \phi_2 + \phi_3|^2, \qquad |\phi_1 + \omega^2\phi_2 + \omega\phi_3|^2,$$
$$|\phi_1 + \omega\phi_2 + \omega^2\phi_3|^2, \tag{71}$$

which simply means that $r_1 + r_4 + r_6$ and $r_2 + r_5 + r_7$ are, separately, invariant. So, if the potential depends only on r_0 and these two combinations, then it is symmetric under the $U(1)$ generated by t_1. The point is that our potential (68) cannot be written via these combinations only, therefore it is not invariant under this group.

Consider now t_2. Its eigensystem is

$$\zeta = 2: \ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \zeta = -1: \ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}. \tag{72}$$

There is no zero eigenvalue, therefore no linear combination of ϕ's is invariant. The independent bilinear combinations are

$$|\phi_1 + \phi_2 + \phi_3|^2, \quad |\phi_2 - \phi_3|^2,$$
$$|2\phi_1 - \phi_2 - \phi_3|^2, \tag{73}$$
$$\left(\phi_2^\dagger - \phi_3^\dagger\right)(2\phi_1 - \phi_2 - \phi_3).$$

In addition, there exists a triple product of ϕ's which is also invariant but it is irrelevant for our analysis because our potential contains only two ϕ's and two $\phi^\dagger$'s. These invariants can also be rewritten as the following linearly independent invariants (here $\rho_i = \phi_i^\dagger \phi_i$):

$$\rho_1 + 2r_6, \qquad \rho_2 + 2r_4, \qquad \rho_3 + 2r_1, \qquad r_2 + r_5 + r_7. \tag{74}$$

Despite the fact that we now have more invariants than in the previous case, it is still impossible to express (68) via these combinations. This means that (68) is not symmetric under t_2.

This completes the proof that the potential (56) subject to conditions (67) is not invariant under any continuous group.

7.4.3 Absence of a larger finite symmetry group

Although the group-theoretic arguments guarantee that no other extension can be used, it is still instructive to check what happens if we try to impose invariance under other subgroups of $SL_2(3)$.

Let us first note that if we try to impose simultaneous invariance under two among d_i (trying to get Q_8), we must set $\lambda_3 = 0$. But then the potential has an obvious continuous symmetry, and our attempt fails.

Next, let us assume that the potential is invariant under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, where the last $\mathbb{Z}_3$ is generated by one of the generators f in (52), for example $f = f_1$. Its representative matrix in $SU(3)$ is

$$f = \frac{-i}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{pmatrix}, \quad f^3 = 1. \tag{75}$$

An analysis similar to what was described above allows us to find the corresponding transformation matrix in the space of X, Y, Z, Z^*:

$$T(f_1) = \frac{1}{3} \begin{pmatrix} 0 & \sqrt{3} & \sqrt{3}\omega^2 & \sqrt{3}\omega \\ \sqrt{3} & 2 & -\omega^2 & -\omega \\ \sqrt{3}\omega^2 & -\omega^2 & -\omega & 2 \\ \sqrt{3}\omega & -\omega & 2 & -\omega^2 \end{pmatrix}. \tag{76}$$

It leads to the following conditions for the potential to be symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$:

$$\lambda_3 = \lambda_3^* \quad \text{and} \quad \lambda_1 = \frac{\lambda_2 - \lambda_3}{\sqrt{3}}. \tag{77}$$

In the space of bilinears, the potential can then be compactly written as

$$V = -\sqrt{3}m^2 r_0 + \left(3\lambda_0 + \sqrt{3}\lambda_1 \alpha\right) r_0^2$$
$$+ \lambda_3 |r_{12} + r_{45} + r_{67}|^2. \tag{78}$$

The spectrum of Λ_{ij} becomes of the type $6+2$. This high symmetry hints at existence of a possible continuous symmetry of the potential, and it is indeed the case. For example, the following $SO(2)$ rotations among three doublets, $\phi_a \mapsto R_{ab}(\alpha)\phi_b$, leave $r_{12} + r_{45} + r_{67}$ invariant:

$$R(\alpha) = \frac{1}{3} \begin{pmatrix} 1+2\cos\alpha & 1+2\cos\alpha'' & 1+2\cos\alpha' \\ 1+2\cos\alpha' & 1+2\cos\alpha & 1+2\cos\alpha'' \\ 1+2\cos\alpha'' & 1+2\cos\alpha' & 1+2\cos\alpha \end{pmatrix}, \tag{79}$$

with $\alpha \in [0, 2\pi)$ and $\alpha' = \alpha + 2\pi/3$ and $\alpha'' = \alpha + 4\pi/3$. Note that at $\alpha = 0$, $2\pi/3$ and $4\pi/3$ we recover the $\mathbb{Z}_3$ group $\langle b \rangle$.

We conclude therefore that imposing invariance under $\mathbb{Z}_3 < SL_2(3)$ makes the potential symmetric under a continuous group. In this way, we completely exhausted possibilities offered by $SL_2(3)$.

7.5 Anti-unitary transformations

We showed in Sect. 7.1 that antiunitary transformations correspond to elements of $GL_2(3)$ not lying in $SL_2(3)$ as they have negative determinant and flip the sign of the antisymmetric scalar product in $A = \mathbb{Z}_3 \times \mathbb{Z}_3$. The complex conjugation operator, J, acts in A by sending a to a^2 and leaving b invariant. Therefore, the corresponding matrix is

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{80}$$

Since any antiunitary transformation can be written as $J' = qJ$, where q is unitary, it follows that q must belong to $SL_2(3)$.

Next, we need to find which q's can be used. Clearly, $(J')^2 = qJqJ = qq^* \in SL_2(3)$. If we are looking for an antiunitary symmetry of a $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$-symmetric potential, then qq^* must be either 1 or c, which generates the center of $SL_2(3)$.

Let us first consider the second possibility.

$$\text{If } q = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad \text{then } q^* = \begin{pmatrix} x & -y \\ -z & t \end{pmatrix}. \tag{81}$$

Using this to solve $qq^* = c$, we get six possible solutions, but all of them have $\det q = -1$, that is, they do not belong to $SL_2(3)$. Therefore, the only possibility is $qq^* = 1$.

But then we can apply the results of our search for antiunitary transformations for the D_6 case. Our group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ contains the D_6 subgroup with $\delta = \pi$. Therefore, we arrive at the conclusion: in order for our potential to be symmetric under an antiunitary transformation, we must require

$$6 \arg \lambda_3 = 0. \tag{82}$$

If this criterion is satisfied, the symmetry group becomes $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2^*)$; otherwise the group remains $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Therefore, both groups are realizable in 3HDM.

Now, consider the case of the extended symmetry group, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 \simeq \Sigma(36)$. In this case (82) is satisfied automatically due to (67). We then conclude that in this case the realizable symmetry is $\Sigma(36) \rtimes \mathbb{Z}_2^*$.

8 Summary and discussion

8.1 List of realizable finite symmetry groups in 3HDM

Bringing together the results of the search for abelian symmetry groups [37] and of the present work, we can finally give the list of finite groups which can appear as the symmetry groups of the scalar sector in 3HDM. If only Higgs-family transformations are concerned, the realizable finite groups are

$$
\begin{aligned}
&\mathbb{Z}_2, \qquad \mathbb{Z}_3, \qquad \mathbb{Z}_4, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2, \\
&D_6, \qquad D_8, \qquad T \simeq A_4, \qquad O \simeq S_4, \\
&(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \Delta(54)/\mathbb{Z}_3, \\
&(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 \simeq \Sigma(36).
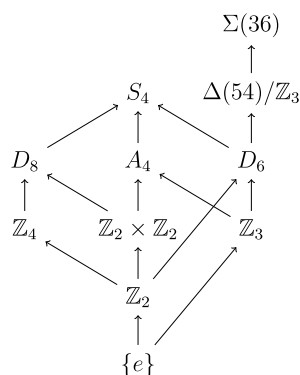\end{aligned}
\tag{83}
$$

This list is complete: trying to impose any other finite symmetry group of Higgs-family transformations leads to the potential with a continuous symmetry.

Figure 3 should help visualize relations among different groups from this list. Going up along a branch of this tree means that, starting with a potential symmetric under the lower group, one can restrict its free parameters in such a way that the potential becomes symmetric under the upper group.

If both unitary (Higgs-family) and antiunitary (generalized-*CP*) transformations are allowed, the list becomes

$$
\begin{aligned}
&\mathbb{Z}_2, \qquad \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \mathbb{Z}_2^*, \qquad \mathbb{Z}_4^*, \\
&\mathbb{Z}_2 \times \mathbb{Z}_2^*, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^*,
\end{aligned}
$$

Fig. 3 Tree of finite realizable groups of Higgs-family transformations in 3HDM

$$
\begin{aligned}
&\mathbb{Z}_3 \rtimes \mathbb{Z}_2^* \simeq D_6, \qquad \mathbb{Z}_4 \rtimes \mathbb{Z}_2^* \simeq D_8, \\
&D_6, \qquad D_6 \times \mathbb{Z}_2^*, \qquad D_8 \times \mathbb{Z}_2^*, \\
&A_4 \rtimes \mathbb{Z}_2^* \simeq T_d, \qquad S_4 \times \mathbb{Z}_2^* \simeq O_h, \\
&(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2, \qquad (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2^*), \\
&\Sigma(36) \rtimes \mathbb{Z}_2^*.
\end{aligned}
\tag{84}
$$

As usual, an asterisk here indicates that the generator of the corresponding group is an anti-unitary transformation. Note that Higgs-family transformation groups $\mathbb{Z}_4$, D_8, A_4, S_4, and $\Sigma(36)$ become non-realizable in this case, because potentials symmetric under them are automatically symmetric under an additional anti-unitary transformation. In all cases apart from A_4 this is a consequence of our finding in Sect. 6.5 that presence of the $\mathbb{Z}_4$ group of Higgs-family transformations *always* leads to an additional anti-unitary symmetry.

These lists complete the classification of realizable finite symmetry groups of the scalar sector of 3HDM. Conditions for the existence and examples of the potentials symmetric under each of these groups have been given in [37] and in the present work. For the reader's convenience, we collect examples with non-abelian groups in the appendix.

8.2 Interplay between Higgs-family symmetries and explicit *CP*-violation

In 2HDM, presence of any Higgs-family symmetry immediately leads to a generalized-*CP* symmetry. In other words, it is impossible to write down an explicitly *CP*-violating 2HDM potential with *any* Higgs-family symmetry. In this sense, generalized-*CP* symmetries can be viewed as the smallest building blocks of any symmetry group in 2HDM.

By comparing lists (83) and (84), we see that this conclusion is no longer true for 3HDM, namely there are some Higgs-family symmetry groups which are compatible with explicit *CP*-violation. However, we found another, quite remarkable feature in 3HDM: *the presence of a $\mathbb{Z}_4$ group of Higgs-family transformations guarantees that the potential is explicitly CP-conserving*. This is, of course, a sufficient but not necessary condition for explicit *CP*-violation. Put in other words, explicit *CP*-violation is incompatible with the Higgs-family symmetry group $\mathbb{Z}_4$.

8.3 Two different D_6 groups

It is interesting to note that the list (84) contains two different D_6 groups. One is $\mathbb{Z}_3 \rtimes \mathbb{Z}_2^*$, generated by a Higgs-family transformation of order 3 and a generalized-*CP* transformation. The other D_6 is a group of Higgs-family transformations only, and a potential invariant under it does not have any generalized-*CP* symmetry. Clearly, they lead to different phenomenological consequences, as the first case

is explicitly *CP*-conserving, while the latter is explicitly *CP*-violating.

Such a situation was absent in the two-Higgs-doublet model, where fixing the symmetry group uniquely defined the (tree-level) phenomenological consequences in the scalar sector. What makes it possible in 3HDM is a looser relation between Higgs-family and generalized-*CP* symmetries just discussed. In particular, it is possible to have a potential with the Higgs-family D_6 symmetry group without any generalized-*CP* symmetry. 2HDM does not offer this kind of freedom: any non-trivial Higgs-family symmetry group automatically leads to a generalized-*CP* symmetry.

8.4 Further directions of research

Certainly, our results do not provide answers to *all* symmetry-related questions which can be posed in 3HDM. Our paper should rather be regarded as the first step towards systematic exploration of all the possibilities offered by three Higgs doublets. Here are some further questions which deserve closer study:

- Continuous symmetry groups should also be included in the list. There exist only few Lie groups inside $PSU(3)$: $U(1)$, $U(1) \times U(1)$, $SU(2)$, $SU(2) \times U(1)$, $SO(3)$. The non-trivial question is which of these groups can be merged with some of the finite groups and with antiunitary transformations (the case of abelian groups was analyzed in [37]).

- It is well-known that the vacuum state does not have to respect all the symmetries of the Lagrangian, so the finite symmetry groups described here can be broken upon electroweak symmetry breaking. What are the symmetry breaking patterns for each of these groups? Clearly, if the symmetry group is very small, then the vacuum state can either conserve it or break it, either completely or partially. But when the finite group becomes sufficiently large, there are two important changes. First, some of the groups can never be conserved upon EWSB; the origin of this feature and some 3HDM examples were discussed in [54]. Second, a sufficiently large symmetry group cannot break down completely, as it would create too many degenerate vacua, which is not possible from the algebraic-geometric point of view. Indeed, in the geometric reformulation of the Higgs potential minimization problem [21], the points of the global minima in the (r_0, r_i)-space are precisely the contact points of two nine-dimensional algebraic manifolds: the orbit space and a certain quadric. Intersection of two algebraic manifolds of known degrees is also an algebraic manifold of a certain degree (the planar analog of this statement is Bezout's theorem). In the degenerate case when this manifold is reduced to a set of isolated points, there must exist an upper limit for the number of these points. Unfortunately, we have not yet found this number for 3HDM, but its existence is beyond any doubt.

- What are possible symmetries of the potential beyond the unitary and antiunitary transformations? For example, the full reparametrization group of the 2HDM potential is $GL(2,) \rtimes \mathbb{Z}_2^*$ rather than $SU(2) \rtimes Z_2^*$, [34–36]. It means that a potential can be left invariant by transformations which are neither unitary nor anti-unitary. Although these transformations played important role in the geometric constructions in the 2HDM orbit space, they did not produce new symmetry groups beyond what was already found from the unitary transformations. It would be interesting to check the situation in 3HDM. Unfortunately, the geometric method which worked well for 2HDM becomes much more intricate with more than two doublets [20, 21].

- It would also be interesting to see if the potential can have symmetries beyond reparametrization transformations. In the case of 2HDM, this problem was analyzed in [55, 56]. Although these additional symmetries cannot be extended to kinetic term, they could still provide useful information on the structure of the Higgs potential and properties of the physical Higgs bosons.

In summary, we found all finite groups which can be realized as symmetry groups of Higgs-family or generalized-*CP* transformations in the three-Higgs-doublet model. Our list (84) is complete: trying to impose any other discrete symmetry group on the 3HDM Higgs potential will make it symmetric under a continuous group.

Appendix: 3HDM potentials with non-abelian Higgs-family symmetry group

Here, for the reader's convenience, we list once again Higgs potentials with a given symmetry group. We focus here on cases with non-abelian groups from the list (84) because abelian ones were already discussed in detail in [37]. In each case we start from the most general potential compatible with the given realizable group presented in the main text and use the residual reparametrization freedom to simplify the coefficients of the potential (usually, it amounts to rephasing of doublets which makes some of the coefficients real). For each group G, the potential written below faithfully represents all possible Higgs potentials with realizable symmetry group G. In this sense, the symmetry group uniquely defines the phenomenology of the scalar sector of

3HDM, the only exception being D_6 with its two distinct realizations.

Group $D_6 \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2^*$ Consider the most general phase-independent part of the Higgs potential

$$V_0 = - \sum_{1 \le i \le 3} m_i^2 (\phi_i^\dagger \phi_i) + \sum_{1 \le i \le j \le 3} \lambda_{ij} (\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \sum_{1 \le i < j \le 3} \lambda'_{ij} (\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i),$$

and the additional terms

$$V_{\mathbb{Z}_3} = \lambda_1 (\phi_2^\dagger \phi_1)(\phi_3^\dagger \phi_1) + \lambda_2 (\phi_1^\dagger \phi_2)(\phi_3^\dagger \phi_2) + \lambda_3 (\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3) + h.c. \quad (A.1)$$

For generic λ_i, these terms are symmetric only under the group $\mathbb{Z}_3$ generated by

$$a_3 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = \exp\left(\frac{2\pi i}{3}\right). \quad (A.2)$$

If it happens that the product $\lambda_1 \lambda_2 \lambda_3$ is purely real, then by rephasing of doublets one can make all coefficients in (A.1) real. The resulting potential, $V_0 + V_{\mathbb{Z}_3}$, is symmetric under $D_6 \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2^*$ generated by a_3 and the *CP*-transformation.

Group $D_8 \simeq \mathbb{Z}_4 \rtimes \mathbb{Z}_2^*$ Consider now terms

$$V_{\mathbb{Z}_4} = \lambda_1 (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) + \lambda_2 (\phi_1^\dagger \phi_2)^2 + h.c., \quad (A.3)$$

which are symmetric under the group $\mathbb{Z}_4$ generated by

$$a_4 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (A.4)$$

It is always possible to compensate the phases of λ_1 and λ_2 by an appropriate rephasing of the doublets. Therefore, the potential $V_0 + V_{\mathbb{Z}_4}$ is symmetric under the group $D_8 \simeq \mathbb{Z}_4 \rtimes \mathbb{Z}_2^*$ generated by a_4 and the *CP*-transformation.

Group D_6 of unitary transformations Let us restrict the coefficients of V_0 in the way that guarantees the symmetry under $\phi_1 \leftrightarrow \phi_2$. Then, V_0 turns into

$$V_1 = -m_{11}^2 [(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2)] - m_{33}^2 (\phi_3^\dagger \phi_3) + \lambda_{11} [(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2] + \lambda_{33} (\phi_3^\dagger \phi_3)^2 + \lambda_{13} [(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2)](\phi_3^\dagger \phi_3) + \lambda_{12} (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda'_{13} [|\phi_1^\dagger \phi_3|^2 + |\phi_2^\dagger \phi_3|^2] + \lambda'_{12} |\phi_1^\dagger \phi_2|^2, \quad (A.5)$$

where all coefficients are real and generic. Imposing the same requirement on $V_{\mathbb{Z}_3}$ and performing rephasing, we obtain

$$V_{D_6} = \lambda_1 [(\phi_2^\dagger \phi_1)(\phi_3^\dagger \phi_1) - (\phi_1^\dagger \phi_2)(\phi_3^\dagger \phi_2)] + |\lambda_3| e^{i\psi_3} (\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3) + h.c., \quad (A.6)$$

where λ_1 is real and $\sin \psi_3 \ne 0$. The resulting potential, $V_1 + V_{D_6}$, is symmetric under D_6 generated by a_3 and

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (A.7)$$

There are no other Higgs-family or generalized-*CP* transformations which leave this potential invariant. Any explicitly *CP*-violating D_6-symmetric 3HDM potential can always be brought into this form.

Group $D_6 \times \mathbb{Z}_2^*$ If in the previous case we set $\sin \psi_3 = 0$ in (A.6), then the potential becomes symmetric under $D_6 \times \mathbb{Z}_2^*$ generated by a_3, b, and the generalized *CP*-transformation $b \cdot CP$.

Group $D_8 \times \mathbb{Z}_2^*$ The potential $V_1 + V_{\mathbb{Z}_4}$ is symmetric under the group $D_8 \times \mathbb{Z}_2^*$ generated by a_4, b, and $b \cdot CP$.

Group $A_4 \rtimes \mathbb{Z}_2^*$ A potential symmetric under $A_4 \rtimes \mathbb{Z}_2^*$ can be brought into the following form:

$$\begin{aligned} V_{A_4 \rtimes \mathbb{Z}_2^*} &= -m^2 [\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3] \\ &+ \lambda [\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3]^2 \\ &+ \lambda' [(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) \\ &+ (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)] \\ &+ \lambda'' (|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2) \\ &+ (\tilde{\lambda} [(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2] + h.c.) \quad (A.8) \end{aligned}$$

with complex $\tilde{\lambda}$. Its symmetry group is generated by independent sign flips of the individual doublets, by cyclic permutations of ϕ_1, ϕ_2, ϕ_3, and by the exchange of any pair of doublet together with the *CP*-transformation. An alternative form of this potential is

$$\begin{aligned} V_{A_4 \rtimes \mathbb{Z}_2^*} &= -m^2 [\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3] \\ &+ \lambda [\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3]^2 \\ &+ \lambda' [(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3)] \end{aligned}$$

$$+ (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)]$$
$$+ \lambda_{\mathrm{Re}}\big[(\mathrm{Re}\,\phi_1^\dagger \phi_2)^2 + (\mathrm{Re}\,\phi_2^\dagger \phi_3)^2 + (\mathrm{Re}\,\phi_3^\dagger \phi_1)^2\big]$$
$$+ \lambda_{\mathrm{Im}}\big[(\mathrm{Im}\,\phi_1^\dagger \phi_2)^2 + (\mathrm{Im}\,\phi_2^\dagger \phi_3)^2 + (\mathrm{Im}\,\phi_3^\dagger \phi_1)^2\big]$$
$$+ \lambda_{\mathrm{ReIm}}\big[\mathrm{Re}\,\phi_1^\dagger \phi_2\,\mathrm{Im}\,\phi_1^\dagger \phi_2 + \mathrm{Re}\,\phi_2^\dagger \phi_3\,\mathrm{Im}\,\phi_2^\dagger \phi_3$$
$$+ \mathrm{Re}\,\phi_3^\dagger \phi_1\,\mathrm{Im}\,\phi_3^\dagger \phi_1\big]. \tag{A.9}$$

Group $S_4 \times \mathbb{Z}_2^*$　If the parameter $\tilde{\lambda}$ in (A.8) is real or, equivalently, $\lambda_{\mathrm{ReIm}} = 0$ in (A.9), the potential becomes symmetric under $S_4 \times \mathbb{Z}_2^*$ generated by sign flips, all permutation of the three doublets, and the *CP*-transformation.

Group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \Delta(54)/\mathbb{Z}_3$　Consider the following potential:

$$V_{\Delta(54)/\mathbb{Z}_3}$$
$$= -m^2\big[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3\big]$$
$$+ \lambda_0\big[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3\big]^2$$
$$+ \lambda_1\big[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2)$$
$$- (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)\big]$$
$$+ \lambda_2\big[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2\big]$$
$$+ \lambda_3\big[(\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_1)$$
$$+ (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2)\big] + h.c. \tag{A.10}$$

with generic real m^2, λ_0, λ_1, λ_2 and complex λ_3. The symmetry group of this potential is $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \Delta(54)/\mathbb{Z}_3$. Here, $\Delta(54)$ is generated by the same a_3 and b as before and, in addition, by the cyclic permutation

$$c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A.11}$$

while the subgroup $\mathbb{Z}_3$ is the center of $SU(3)$.

Group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2^*)$　The potential (A.10) becomes symmetric under a generalized-*CP* transformation if $\lambda_3 = k \cdot \pi/3$ with any integer k. In this case, one can make λ_3 real by a rephasing transformation. The extra generator then is the *CP*-transformation.

Group $\Sigma(36) \rtimes \mathbb{Z}_2^*$　The same potential (A.10) becomes symmetric under the group $\Sigma(36) \rtimes \mathbb{Z}_2^*$ if, upon rephasing, $\lambda_3 = (3\lambda_1 - \lambda_2)/2$. The potential can then be rewritten as

$$V_{\Sigma(36) \rtimes \mathbb{Z}_2^*}$$
$$= -m^2 I_0 + \lambda_0 I_0^2 + 3\lambda_1 I_1$$
$$+ \frac{\lambda_2 - 3\lambda_1}{2}\big(|\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_3|^2 + |\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_1|^2$$
$$+ |\phi_3^\dagger \phi_1 - \phi_1^\dagger \phi_2|^2\big). \tag{A.12}$$

Here I_0 and I_1 are the $SU(3)$-invariants

$$I_0 = \frac{r_0}{\sqrt{3}} = \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3,$$
$$I_1 = \sum_i r_i^2 \tag{A.13}$$
$$= \frac{(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)}{3} + |\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2.$$

It is remarkable that this potential has only one "structural" free parameter, and the term containing it reduces the full $SU(3)$ symmetry group to a finite subgroup $\Sigma(36)$.

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