Implicit Differentiation with Microscopes

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Implicit functions are an important tool in the fundamental mathematical analysis ; this theory has a lot of concrete applications in various domains : for example, many economical reasonings work with implicit functions like in the cases of a consumer and his indifference curves or of a firm and her isoquants.

The aim of this note is to give a systematic and suggestive approach to compute the successive derivatives of an implicit function which depends on one real variable. This method is intuitive : indeed, we use (virtual) microscopes but, as cost price, we work with hyperreal numbers.

Although Leibniz and Newton, for instance, already worked with "infinitesimals", the hyperreal numbers were rigorously introduced, in 1961, by A. Robinson [4], who developed the "nonstandard analysis". Here, we work with a didactical and simple presentation of the nonstandard analysis [3] : we shall essentially adopt Keisler's definitions and notations.

We recall that the hyperreal numbers extend the real ones with the same algebraic rules ; technically, the set ${}^*\mathbb{R}$ of the hyperreal numbers is a non-archimedean ordered field in which the real line $\mathbb{R}$ is embedded. Moreover, ${}^*\mathbb{R}$ contains at least (in fact infinitely many) one *infinitesimal* (i.e. a number ε such that its absolute value is less than every real number) which is unequal to 0 ; its reciproqual $\frac{1}{\varepsilon}$ is *infinite* (i.e. is a number such that its absolute value is greater than every real number) ; clearly, non-zero infinitesimals and infinite numbers are not real. A hyperreal number x which is not infinite is of course said to be *finite* ; in this case, there exists one and only one real number r which is *infinitely close* to x, i.e, such that the difference $x - r$ is an infinitesimal : r is called the *standard part* of x and is denoted by $r = \mathrm{st}(x)$; formally, st is a ring homomorphism from the set of finite hyperreal numbers to $\mathbb{R}$ and its kernel is the set of infinitesimals. Moreover, every function of one or several real variables has a *natural extension* for hyperreal numbers, with the same definition and the same properties as these of the original one : indeed, if a real-valued function is defined by a system of formulas, its extension can be obtained by applying the same formulas to the hyperreal system ; in this article, we adopt the same notation for a real function and for its natural extension.

The concept of (virtual) microscope is well-known (see, for example, [1], [2], [5]). For a point $P(a, b)$ in the hyperreal plane ${}^*\mathbb{R}^2$ and a positive infinite hyperreal number H, a microscope *pointed* on P and with H as *power* magnifies the distances from P by a factor H ; technically, it is a map, denoted by $\mathcal{M}_H^P$, defined on ${}^*\mathbb{R}^2$ as follows

$$\mathcal{M}_H^P : (x, y) \mapsto (X, Y) \text{ with } X = H\ (x - a) \text{ and } Y = H\ (y - b)$$

Then, we also have

$$x = a + \frac{X}{H} \text{ and } y = b + \frac{Y}{H}$$

For a real function f of two real variables x and y, we first consider, in the classical euclidean plane $\mathbb{R}^2$, the curve $\mathcal{C}$ defined by

$$f(x, y) = 0$$

We suppose that a point $P\,(r,s)$ belongs to $\mathcal{C}$ and that f is of class C^p, for p sufficiently large, in a neighbourhood of P. For more simplicity, we denote by d_1, d_2, d_{11}, d_{12}, ... , the corresponding partial derivatives of f at P, i.e. $d_1 = f_x\,(r,s)$, $d_2 = f_y\,(r,s)$, $d_{11} = f_{xx}\,(r,s)$, $d_{12} = f_{xy}\,(r,s)$, ... Moreover, we assume that $d_2 \neq 0$.

When we look at $\mathcal{C}$ through a microscope $\mathcal{M}_H^P$, where H is an arbitrary positive infinite hyperreal number, we see a curve characterized by the equation

$$f\left(r + \frac{X}{H}, s + \frac{Y}{H}\right) = 0$$

Taylor's Formula for f at P leads to

$$f\,(r,s) + d_1\,\frac{X}{H} + d_2\,\frac{Y}{H} + o\left[\frac{1}{H^2}\right] = 0$$

where a term like $o\left[\frac{1}{H^k}\right]$, for every integer k, is such that its product with H^{k-1} is infinitesimal. Because $f\,(r,s) = 0$, we also have

$$d_1 X + d_2 Y + o\left[\frac{1}{H}\right] = 0 \tag{1}$$

As we are only interested by the finite hyperreal numbers X and Y (otherwise they could not be really observed), we suppress all the infinitesimal details by taking the standard parts of the two members in (1) ; then we get

$$d_1\ \mathrm{st}\,(X) + d_2\ \mathrm{st}\,(Y) = 0 \Longleftrightarrow \mathrm{st}\,(Y) = -\frac{d_1}{d_2}\ \mathrm{st}\,(X) \tag{2}$$

It is well-known that the coefficient $m = -\frac{d_1}{d_2}$ is the slope of the tangent line $\mathcal{T}$ to $\mathcal{C}$ at P ; $\mathcal{T}$ is thus defined by the equation

$$y - s = m\ (x - r)$$

This non-vertical line $\mathcal{T}$ suggests intuitively that $\mathcal{C}$ is, close to P, the graph of a real function g of the variable x ; more precisely, the implicit function theorem ([3], p. 708) ensures the existence of a function g such that $g(r) = s$, the domain of g is an open interval I containing r and the graph of g is a subset of $\mathcal{C}$.

Although g is unknown, it is possible to compute its derivatives at r. For that, we first look at the graph of g through the microscope $\mathcal{M}_H^P$; that leads to this equality

$$s + \frac{Y}{H} = g\left(r + \frac{X}{H}\right)$$

so, by Taylor's Formula for g

$$s + \frac{Y}{H} = g(r) + g'(r)\ \frac{X}{H} + o\left[\frac{1}{H^2}\right]$$

and thus, as above,

$$\mathrm{st}\,(Y) = g'(r)\ \mathrm{st}\,(X) \tag{3}$$

A comparison between formulas (2) and (3) leads to

$$g'(r) = -\frac{d_1}{d_2}$$

In order to compute $g''(r)$, we must distinguish between the curve $\mathcal{C}$ and its tangent $\mathcal{T}$. For that, we use a stronger microscope, for example with a power H^2, and direct it to another point which is infinitely close ro P and belongs to $\mathcal{T}$ (otherwise we see again $\mathcal{C}$ and $\mathcal{T}$ as equal). We can choose, for example, the point $P_1\left(r + \frac{1}{H}, s + \frac{m}{H}\right)$ (note that point $P_2\left(r - \frac{1}{H}, s - \frac{m}{H}\right)$ could also be convenient).

On the one hand, the use of the microscope $\mathcal{M}_{H^2}^{P_1}$ to the graph of g leads to

$$\begin{aligned} s + \frac{m}{H} + \frac{Y}{H^2} &= g\left(r + \frac{1}{H} + \frac{X}{H^2}\right) \\ &= g(r) + g'(r)\ \left(\frac{1}{H} + \frac{X}{H^2}\right) + \frac{g''(r)}{2}\ \left(\frac{1}{H} + \frac{X}{H^2}\right)^2 + o\left[\frac{1}{H^3}\right] \end{aligned}$$

and thus, as previously,

$$\mathrm{st}\,(Y) = g'(r)\ \mathrm{st}\,(X) + \frac{1}{2}\ g''(r) \tag{4}$$

On the other hand, the application of $\mathcal{M}_{H^2}^{P_1}$ to the curve $\mathcal{C}$ gives

$$\begin{aligned} 0 &= f\left(r + \frac{1}{H} + \frac{X}{H^2}, s + \frac{m}{H} + \frac{Y}{H^2}\right) \\ &= f\,(r, s) + d_1\left(\frac{1}{H} + \frac{X}{H^2}\right) + d_2\left(\frac{m}{H} + \frac{Y}{H^2}\right) + \\ &\frac{1}{2}\left[d_{11}\ \left(\frac{1}{H} + \frac{X}{H^2}\right)^2 + 2d_{12}\left(\frac{1}{H} + \frac{X}{H^2}\right)\ \left(\frac{m}{H} + \frac{Y}{H^2}\right) + d_{22}\ \left(\frac{m}{H} + \frac{Y}{H^2}\right)^2\right] + 0\left[\frac{1}{H^3}\right] \end{aligned}$$

We easily get

$$d_1\ \mathrm{st}\,(X) + d_2\ \mathrm{st}\,(Y) + \frac{1}{2}\left[d_{11} + 2\ m\ d_{12} + m^2\ d_{22}\right] = 0 \tag{5}$$

The comparison between formulas (4) and (5) gives

$$g''(r) = -\frac{1}{d_2}\left(d_{11} + 2\ m\ d_{12} + m^2\ d_{22}\right) = \frac{|\overline{Hf}|}{(d_2)^3}$$

where $|\overline{Hf}|$ denotes the bordered hessian associated with f at P, i.e. the determinant of the matrix

$$\begin{pmatrix} 0 & d_1 & d_2 \\ d_1 & d_{11} & d_{12} \\ d_2 & d_{21} & d_{22} \end{pmatrix}$$

More generally, by the principle of induction, we can compute all the derivatives of the implicit function g at r. Indeed, let k be an integer greater than 1 and suppose that the numbers $g'(r)$, $g''(r)$, $\ldots$, $g^{(k-1)}(r)$ are well-known. Then we consider the point $P_{k-1}\left(r+\frac{1}{H}, s+\sum_{j=1}^{k-1}\frac{1}{j!}\frac{g^{(j)}(r)}{H^j}\right)$ and we apply the microscope $\mathcal{M}_{H^k}^{P_{k-1}}$ both on the graph of g and on the curve $\mathcal{C}$. So, after some elementary computations, we respectively obtain these two equalities

$$\mathrm{st}\,(Y) = m\ \mathrm{st}\,(X) + \frac{g^{(k)}(r)}{k!}$$

and

$$d_1\ \mathrm{st}\,(X) + d_2\ \mathrm{st}\,(Y) + p_k = 0$$

Then, we can deduce from these two last formulas

$$g^{(k)}(r) = -\frac{k!\ p_k}{d_2}$$

where p_k can be calculated in terms of the partial derivatives of the given function f at P.

For instance, we can so compute

$$g'''(r) = \frac{3\ \bar{H}f}{(d_2)^4}\left(\frac{d_1}{d_2}d_{22} - d_{12}\right) - \frac{1}{d_2}\left(d_{111} - 3\frac{d_1}{d_2}\ d_{112} + 3\frac{(d_1)^2}{(d_2)^2}\ d_{122} - \frac{(d_1)^3}{(d_2)^3}\ d_{222}\right)$$

In conclusion, we think that the use of microscopes gives a refreshing and new way to systematically compute the derivatives of infinite functions.

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