Numeration systems: a link between number theory and formal language theory

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Verification

Cryptography

Dynamical systems

Number theory

Symbolic dynamics

Automatic structure

Discrete geometry

Numeration systems

Combinatorics on words

Fractal

Combinatorial games

Formal language theory

Logic

Design of algorithms

Computability

Ergodic theory

Algorithm analysis

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	
$\mathbb{Q}$	numeration basis	
$\mathbb{R}$	substitutive	
Gaussian int.	abstract	
$\mathbb{C}$	Ostrowski system	
$\mathbb{F}_q[X]$	factorial system	
	β-expansions	
vectors	continued fractions	
of these	canonical number sys.	
$\vdots$	$\vdots$	

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	$A_2 = \{0, 1\}$
$\mathbb{Q}$	numeration basis	
$\mathbb{R}$	substitutive	$\mathrm{rep}_2(n)$, $n \in \mathbb{N}$, is a
Gaussian int.	abstract	finite word over A_2
$\mathbb{C}$	Ostrowski system	
$\mathbb{F}_q[X]$	factorial system	with $X \subseteq \mathbb{N}$,
	β-expansions	$\mathrm{rep}_2(X)$ is a
vectors	continued fractions	language over A_2
of these	canonical number sys.	
$\vdots$	$\vdots$	

Integer base, e.g., $k = 2$

$$\mathrm{rep}_2 : \mathbb{N} \to \{0, 1\}^*,\ n = \sum_{i=0}^{\ell} d_i\, 2^i,\ \mathrm{rep}_2(n) = d_\ell \cdots d_0$$

$$\mathrm{rep}_2(37) = 100101 \quad \text{and} \quad \mathrm{val}_2(100101) = 37$$

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	$A_2 = \{0,1\}$
$\mathbb{Q}$	numeration basis	
$\mathbb{R}$	substitutive	$\mathrm{rep}_2(r)$, $r \in \mathbb{R}$, is an
Gaussian int.	abstract	infinite word over A_2
$\mathbb{C}$	Ostrowski system	
$\mathbb{F}_q[X]$	factorial system	with $X \subseteq \mathbb{R}$,
	β-expansions	$\mathrm{rep}_2(X)$ is an
vectors	continued fractions	ω-language over A_2
of these	canonical number sys.	
⋮	⋮	maybe several rep.

Integer base, e.g., $k = 2$ (base-complement for neg. numbers)

$$\mathrm{rep}_2 : \mathbb{R} \to \{0,1\}^* \star \{0,1\}^\omega,\ \{r\} = \sum_{i=1}^{+\infty} d_i\, 2^{-i}.$$

The set of representations of $3/2$ is $0^+1 \star 10^\omega \cup 0^+1 \star 01^\omega$.

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	$A_F = \{0,1\}$
$\mathbb{Z}$	linear recurrence	
$\mathbb{Q}$	numeration basis	greedy choice
$\mathbb{R}$	substitutive	$\text{rep}_F(n)$, $n \in \mathbb{N}$, is a
Gaussian int.	abstract	finite word over A_F
$\mathbb{C}$	Ostrowski system	
$\mathbb{F}_q[X]$	factorial system	with $X \subseteq \mathbb{N}$,
	β-expansions	$\text{rep}_2(X)$ is a
vectors	continued fractions	language over A_F
of these	canonical number sys.	
⋮	⋮	maybe several rep.

Fibonacci numeration system (Zeckendorf 1972)

$\ldots, 34, 21, 13, 8, 5, 3, 2, 1 = (F_n)_{n\geq 0}$ and $\text{rep}_F(11) = 10100$

but $\text{val}_F(10100) = \text{val}_F(10011) = \text{val}_F(1111)$

$$U_{n+2} = U_{n+1} + U_n.$$

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	$A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$
$\mathbb{Q}$	numeration basis	
$\mathbb{R}$	substitutive	β-expansions are
Gaussian int.	abstract	infinite words over A_β
$\mathbb{C}$	Ostrowski system	
$\mathbb{F}_q[X]$	factorial system	maybe several rep.
	β-expansions	
vectors	continued fractions	β-development is
of these	canonical number sys.	the lexico. largest
$\vdots$	$\vdots$	

β-expansions (Rényi 1957, Parry 1960), e.g., $\beta = (1+\sqrt{5})/2$

$$r \in (0,1),\ r = \sum_{i=1}^{+\infty} c_i\, \beta^{-i} \qquad \beta^2 = \beta + 1$$

$$d_\beta(\pi - 3) = 000010101001000101010 \cdots .$$

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	$A = \mathbb{N}$
$\mathbb{Q}$	numeration basis	
$\mathbb{R}$	substitutive	$\mathrm{rep}(n)$, $n \in \mathbb{N}$, is a
Gaussian int.	abstract	finite word
$\mathbb{C}$	Ostrowski system	over an
$\mathbb{F}_q[X]$	factorial system	infinite alphabet
	β-expansions	
vectors	continued fractions	
of these	canonical number sys.	
$\vdots$	$\vdots$	

Factorial numeration system

$$\ldots,\ 720,\ 120,\ 24,\ 6,\ 2,\ 1 = (j!)_{j\geq 0}, \qquad n = \sum_{i=0}^{\ell} d_i\, i!,$$
$$\mathrm{rep}(719) = 54321.$$

H. Lenstra, Profinite Fibonacci numbers, **EMS Newsletter**'06

Sets of numbers	Numeration system	finite/infinite words or sequences
$\mathbb{N}$	integer base	
$\mathbb{Z}$	linear recurrence	$A = \{0, 1, X, X+1\}$
$\mathbb{Q}$	numeration basis	finite alphabet
$\mathbb{R}$	substitutive	
Gaussian int.	abstract	$\text{rep}_B(P)$, $P \in \mathbb{F}_2[X]$ is
$\mathbb{C}$	Ostrowski system	a finite word
$\mathbb{F}_q[X]$	factorial system	
	β-expansions	with $\mathcal{T} \subseteq \mathbb{F}_2[X]$
vectors	continued fractions	$\text{rep}_B(\mathcal{T})$ is a
of these	canonical number sys.	language over A
$\vdots$	$\vdots$	

“Polynomial base”, e.g., $B = X^2 + 1$, $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$

$$P = \sum_{i=0}^{\ell} C_i\, B^i \text{ with } \deg C_i < \deg B,$$

$$X^6 + X^5 + 1 = 1.B^3 + (X+1).B^2 + 1.B + X.B^0$$

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of these	canonical number sys.	
$\vdots$	$\vdots$	
numbers		formal language
arithmetic/	$\Leftrightarrow$	theory
algebraic		syntactical
properties		properties

The Chomsky's hierarchy :

- Recursively enumerable languages (Turing Machine)
- Context-sensitive languages (linear bounded T.M.)
- Context-free languages (pushdown automaton)
- Regular (or rational) languages (finite automaton)

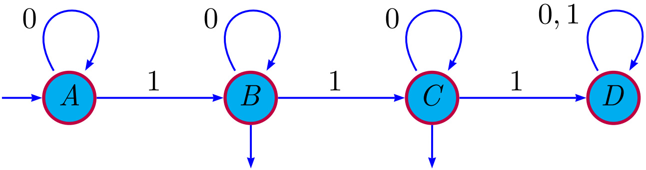
Deterministic Finite Automaton (DFA)

$\mathcal{A} = (Q, q_0, A, \delta, F)$

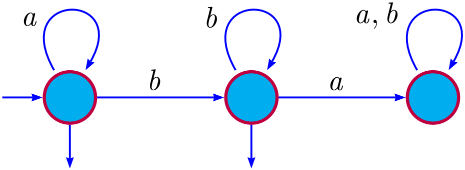
- A is a finite alphabet,
- Q finite set of states, $q_0 \in Q$ initial state
- $\delta : Q \times A \to Q$ transition function
- $F \subseteq Q$ set of final (or accepting) states

DFA form the simplest model of computation.

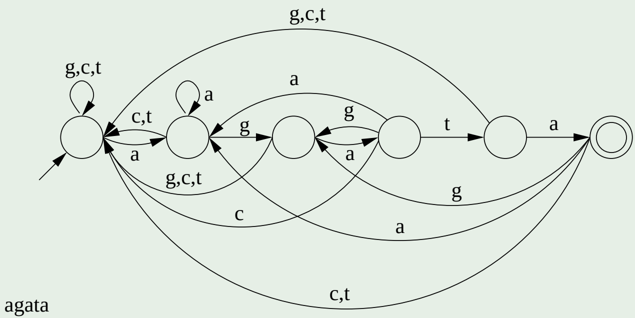
$$0^*10^* + 0^*10^*10^*$$



$$a^*b^*$$



Example (Use in bio-informatics, DNA: a,c,g,t)



Example (Use in computer science)

Model checking, program verification, “stringology”, . . .

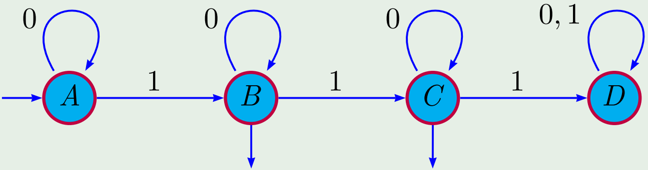
Sets of integers having a somehow simple description

Definition

A set $X \subset \mathbb{N}$ is *k-recognizable*, if $\mathrm{rep}_k(X)$ is a regular language.

A 2-recognizable set

$X = \{n \in \mathbb{N} \mid \exists i, j \geq 0 : n = 2^i + 2^j\} \cup \{1\}$

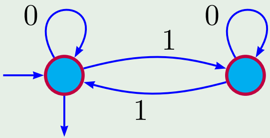


$$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, \ldots$$

$$1, 10, 11, 100, 101, 110, 1000, 1001, 1010, 1100, 10000, 10001, \ldots$$

Prouhet (1851) – Thue (1906) – Morse (1921)

$$\{n \in \mathbb{N} \mid s_2(n) \equiv 0 \bmod 2\}$$



$$0, 3, 5, 6, 9, 10, 12, 15, 17, 18, \ldots$$

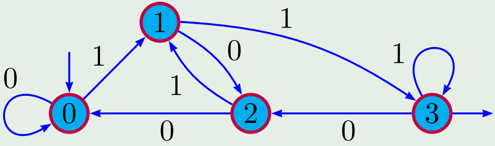
$$\varepsilon, 11, 101, 110, 1001, 1010, 1100, 1111, 10001, 10010, \ldots$$

The set of powers of 2

$$\mathrm{rep}_2(\{2^i \mid i \geq 0\}) = 10^*$$

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

An ultimately periodic set, e.g., $4\mathbb{N}+3$



$$3, 7, 11, 15, 19, 23, 27, 31, \ldots$$

Exercise

Let $k \geq 2$. Show that any arithmetic progression $p\mathbb{N}+q$ is k-recognizable (and consequently any ultimately periodic set).

B. Alexeev, Minimal dfas for testing divisibility, **JCSS**'04

Question

Does recognizability depends on the choice of the base ?
Is a 2-recognizable set also 3-recognizable or 4-recognizable ?

Exercise

Let $k, t \geq 2$. Show that $X \subset \mathbb{N}$ is k-recognizable
IFF it is k^t-recognizable. $0 \mapsto 00, 1 \mapsto 01, 2 \mapsto 10, 3 \mapsto 11$

Powers of 2 in base 3 :

$$2, 11, 22, 121, 1012, 2101, 11202, 100111, 200222, 1101221,$$

$$2210212, 12121201, 102020102, 211110211, 1122221122, 10022220021,$$

$$20122210112, 111022121001, 222122012002, 1222021101011,$$

$$10221112202022, 21220002111121, 120210012000012, \ldots$$

Two integers $k, \ell \geq 2$ are *multiplicatively independent* if $k^m = \ell^n \Rightarrow m = n = 0$, i.e., if $\log k / \log \ell$ is irrational.

Cobham's theorem (1969)

Let $k, \ell \geq 2$ be two multiplicatively independent integers.
A set $X \subseteq \mathbb{N}$ is k-rec. AND ℓ-rec. IFF X is ultimately periodic.

V. Bruyère, G. Hansel, C. Michaux, R. Villemaire, Logic and p-recognizable sets of integers, **BBMS**'94.

Some consequences of Cobham's theorem from 1969:

- k-recognizable sets are easy to describe but non-trivial,
- motivates characterizations of k-recognizability,
- motivates the study of "exotic" numeration systems,
- generalizations of Cobham's result to various contexts:

  multidimensional setting, logical framework, extension to Pisot systems, substitutive systems, fractals and tilings, simpler proofs, ...

B. Adamczewski, J. Bell, G. Hansel, D. Perrin, F. Durand, V. Bruyère, F. Point, C. Michaux, R. Villemaire, A. Bès, J. Honkala, S. Fabre, C. Reutenauer, A.L. Semenov, L. Waxweiler, M.-I. Cortez, ...

A possible application

The set of powers of 2 or the Thue–Morse set are 2-recognizable but NOT 3-recognizable.

$X = \{x_0 < x_1 < x_2 < \cdots\} \subseteq \mathbb{N}$

$$\mathbf{R}_X := \limsup_{i\to\infty} \frac{x_{i+1}}{x_i} \text{ and } \mathbf{D}_X := \limsup_{i\to\infty} (x_{i+1} - x_i).$$

Following G. Hansel, first part of the proof of Cobham's theorem is to show that X is *syndetic*, i.e., $\mathbf{D}_X < +\infty$.

Gap theorem (Cobham'72)

Let $k \geq 2$. If $X \subseteq \mathbb{N}$ is a k-recognizable infinite subset of $\mathbb{N}$, then either $\mathbf{R}_X > 1$ or $\mathbf{D}_X < +\infty$.

For instance, $\{n^t \mid n \geq 0\}$ is k-recognizable for no $k \geq 2$.

S. Eilenberg, Automata, Languages, and Machines, 1974.

- Logical characterization of k-recognizable sets

Büchi (1960) – Bruyère Theorem

A set $X \subset \mathbb{N}^d$ is k-recognizable IFF it is definable by a first order formula in the extended Presburger arithmetic $\langle \mathbb{N}, +, V_k \rangle$.

$V_k(n)$ is the largest power of k dividing $n \geq 1$, $V_k(0) = 1$.

$$\varphi_1(x) \equiv V_2(x) = x$$

$$\varphi_2(x) \equiv (\exists y)(V_2(y) = y) \wedge (\exists z)(V_2(z) = z) \wedge x = y + z$$

$$\varphi_3(x) \equiv (\exists y)(x = y + y + y + y + 3)$$

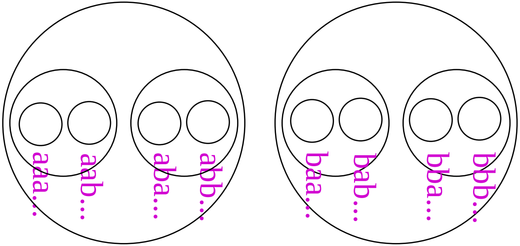
Restatement of Cobham's thm.

Let $k, \ell \geq 2$ be two multiplicatively independent integers.
A set $X \subseteq \mathbb{N}$ is k-rec. AND ℓ-rec. IFF X is definable in $\langle \mathbb{N}, + \rangle$.

Let $A^{\mathbb{N}} = A^{\omega}$ be the set of “infinite words over A”.
It is a metric space endowed with an ultra-metric distance given by

$$d(x, y) = 2^{-|x \wedge y|}$$

where $x \wedge y$ is the longest common prefix of x and y.



So we can speak of *convergent* sequences of infinite words or of a sequence of finite words converging to an infinite word.

- Automatic characterization of k-recognizable sets

THEOREM (COBHAM 1972) – UNIFORM TAG SEQUENCES

A set X is k-recognizable / k-automatic IFF its characteristic sequence is generated through an iterated k-uniform morphism + a coding.

$$g : \left\{ \begin{array}{lll} A & \mapsto & AB \\ B & \mapsto & BC \\ C & \mapsto & CD \\ D & \mapsto & DD \end{array} \right. \qquad f : \left\{ \begin{array}{lll} A & \mapsto & 0 \\ B & \mapsto & 1 \\ C & \mapsto & 1 \\ D & \mapsto & 0 \end{array} \right.$$

$$g(A) = AB,\ g^2(A) = ABBC,\ g^3(A) = ABBCBCCD, \ldots$$

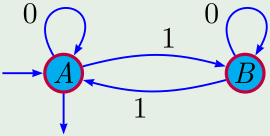
$$g^\omega(A) = ABBCBCCDBCCDCDDDBCCDCDDDCDDDDDDD\cdots$$

$$w = f(g^\omega(A)) = 01111110111010001110100010000000\cdots$$

feed a DFAO with k-ary rep. $, \forall n \geq 0,\ w_n = \tau(q_0 \cdot \mathrm{rep}_k(n))$

The Thue–Morse sequence is 2-automatic

$$T = \{n \in \mathbb{N} \mid s_2(n) \equiv 0 \bmod 2\}$$

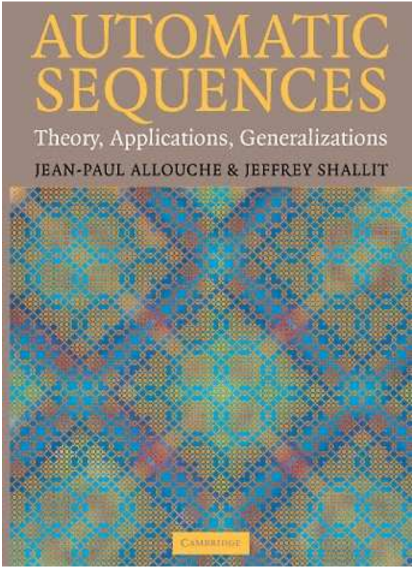


$$g : A \mapsto AB,\ B \mapsto BA, \quad f : A \mapsto 1,\ B \mapsto 0$$

$$f(g^\omega(A)) = 10010110011010010110100110010110\cdots$$

J.-P. Allouche, J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence. *Sequences and their applications*, 1999.

Axel Thue (1863–1922) Marston Morse (1892-1977)



J.-P. Allouche, J. Shallit, Cambridge Univ. Press, 2003.

Multidimensional setting, e.g., $k = 2,\ d = 2$

$$\mathrm{rep}_2 \begin{pmatrix} 5 \\ 35 \end{pmatrix} = \begin{pmatrix} 000101 \\ 100011 \end{pmatrix}, \quad \text{Alphabet } \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

One can easily define k-recognizable subsets of $\mathbb{N}^d$.

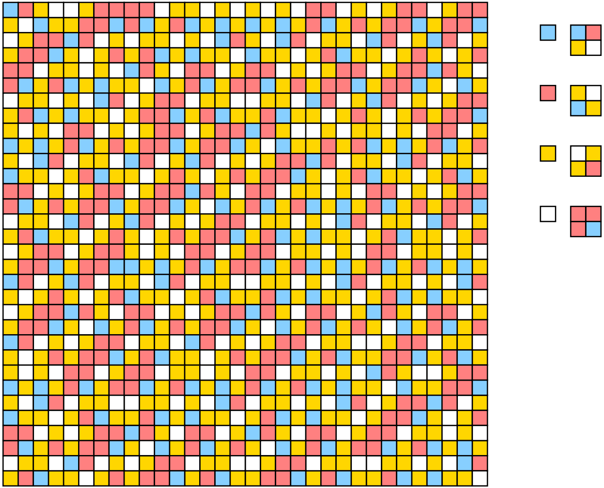
Cobham–Semenov' Theorem (1977)

Let $k, \ell \geq 2$ be two multiplicatively independent integers.
A set $X \subseteq \mathbb{N}^d$ is k-rec. AND ℓ-rec. IFF X is definable in $\langle \mathbb{N}, + \rangle$

Natural extension of ultimate periodicity :

- definability in $\langle \mathbb{N}, + \rangle$,
- semi-linear sets,
- Muchnik's local periodicity (TCS'03)

A 2-recognizable/2-automatic set in $\mathbb{N}^2$



O. Salon, Suites automatiques à multi-indices, **Sém TN Bord.**, 1986–1987.

Theorem (S. Eilenberg)

A sequence $(x_n)_{n\geq 0}$ is k-automatic IFF its k-kernel is finite,

$$\mathcal{K} = \{(x_{k^e n+r})_{n\geq 0} \mid \forall e \geq 0, r < k^e\}$$

For the Thue–Morse sequence
$(t_n)_{n\geq 0} = 10010110011010010110100110010110\cdots$,
the 2-kernel contains exactly the two sequences

$$10010110011010010110100110010110\cdots$$

$$01101001100101101001011001101001\cdots$$

because

$$t_n = 1 \Leftrightarrow (t_{2n} = 1 \wedge t_{2n+1} = 0),$$

$$t_n = 0 \Leftrightarrow (t_{2n} = 0 \wedge t_{2n+1} = 1).$$

A bit of combinatorics on words

A square : `bonbon`, `Shil`<u>`lala`</u>`hs`, <u>`coco`</u>`nut`

The easiest result (to attract students)

A (finite) word of length ≥ 4 over a 2-letter alphabet contains a *square*.

- Over a 2-letter alphabet, what pattern can be avoided ? e.g., cubes ?
- Over a larger alphabet, can we avoid squares ?

An overlap is a "bit more than a square": $auaua$

`Balalaika`, `rococo`, `alfalfa`

Theorem

The Thue-Morse sequence is overlap-free.

Corollary

Over a three-letter alphabet, there exists an infinite word avoiding squares.

Since the Thue–Morse word has no overlap, it never contains 000. So it can be uniquely factorized using factors in $\{1, 10, 100\}$. Just look if a 1 is followed by another 1, by one 0 or by two 0's.

$$100|10|1|100|1|10|100|10|1|10|100|1|100|10|1|\ 1\cdots$$

$$g : \begin{cases} 1 \mapsto 100 \\ 2 \mapsto 10 \\ 3 \mapsto 1 \end{cases}$$

The infinite word $1231321232131 23\cdots$ has no square, otherwise the Thue–Morse word would contain an overlap!

- Automatic words form a large and well-studied family of infinite words
- Many constructions in combinatorics on words rely on iterated morphisms

Dejean's conjecture 1972–2009

One can try to avoid rational powers, $\texttt{entente} = \texttt{ent}^{2+1/3}$ is a $7/3$-power. The *repetitive threshold* is the largest possible exponent e such that there exists no infinite word over a k-letter alphabet avoiding powers of exponent e or greater.

$$RT(2) = 2,\ RT(3) = 7/4,\ RT(4) = 7/5,$$

$$RT(k) = k/(k-1) \quad \forall k \geq 5$$

F. Dejean, N. Rampersad, M. Rao, J. Currie, M. Mohammad-Noori, J. Moulin Ollagnier, J.-J. Pansiot, A. Carpi

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Generalizing numeration systems to...

- Have new "recognizable" sets of integers
- Obtain a larger family of infinite words,
  a generalization of k-automatic sequences, e.g., morphic words

Take a sequence $(U_n)_{n\geq 0}$ of integers such that

- $U_{i+1} > U_i$, *non-ambiguity*
- $U_0 = 1$, *any integer can be represented*
- $\frac{U_{i+1}}{U_i}$ is bounded, *finite alphabet of digits* A_U

$$n = \sum_{i=0}^{\ell} c_i\, U_i, \quad \text{with} \quad c_\ell \neq 0 \quad \text{greedy expansion}$$

Any integer n corresponds to a word $\mathrm{rep}_U(n) = c_\ell \cdots c_0$.
A set $X \subset \mathbb{N}$ is *U-recognizable*, if $\mathrm{rep}_U(X)$ is a regular language.

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A first natural question

Let $U = (U_i)_{i \geq 0}$ be a strictly increasing sequence of integers,

is the whole set $\mathbb{N}$ U-recognizable ?

Theorem (Shallit '94)

If $\mathcal{L}_U = \mathrm{rep}_U(\mathbb{N})$ is regular, i.e., if $\mathbb{N}$ is U-recognizable, then $(U_i)_{i \geq 0}$ satisfies a linear recurrent equation.

Theorem (N. Loraud '95, M. Hollander '98)

They give (technical) sufficient conditions for $\mathcal{L}_U$ to be regular: "the characteristic polynomial of the recurrence has a special form".

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Many works on this topic has been done and the “best setting” are related to Pisot numbers.

- V. Bruyère, G. Hansel, Bertrand numeration systems and recognizability, TCS'97.
- Ch. Frougny, Numeration systems, Chap. 7 in M. Lothaire, Algebraic Combinatorics on Words, CUP 2002.
- F. Durand, M. Rigo, On Cobham's theorem, to appear Handbook of Automata (AutoMathA project), EMS Pub. House.
- Ch. Frougny, J. Sakarovitch, Chap. 2 in Combinatorics, Automata and Number Theory, CUP 2010.

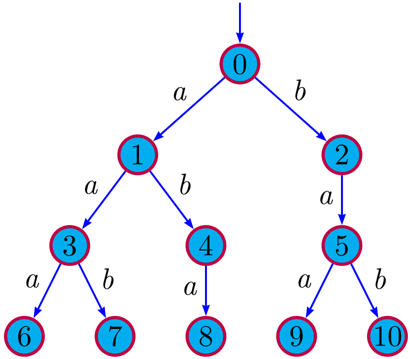
An abstract numeration system S is a regular language $L \subset A^*$ genealogically ordered where the alphabet A is totally ordered.

Words are ordered first by increasing lengths and then using the lexicographical ordering induced by the ordering of A.

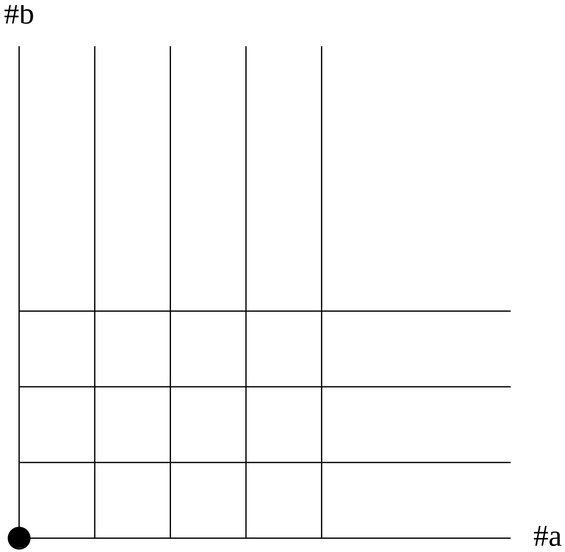
This ordering is a one-to-one correspondence between $\mathbb{N}$ and L.

The $(n+1)$th word in L is the S-representation of the integer n.

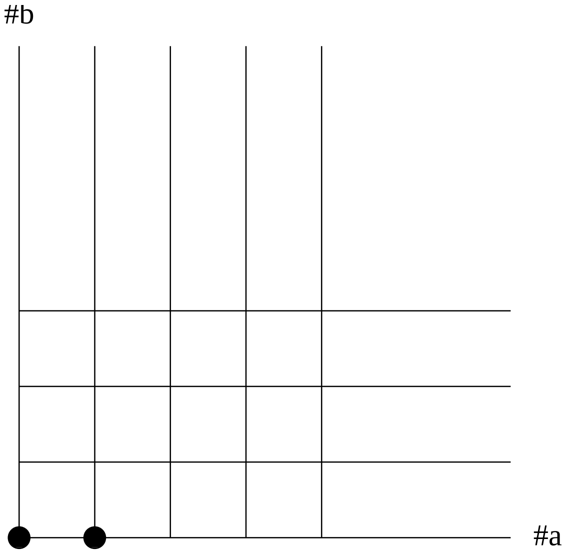
Example of a prefix-closed language $L = \{b, \varepsilon\}\{a, ab\}^*$



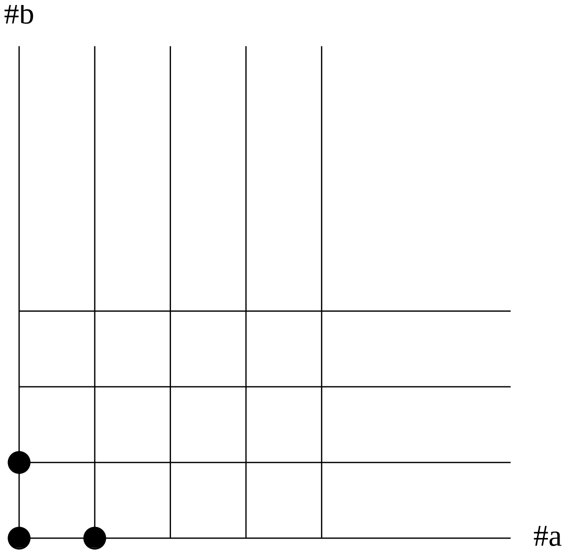
A non-positional numeration system $L = a^* b^*$



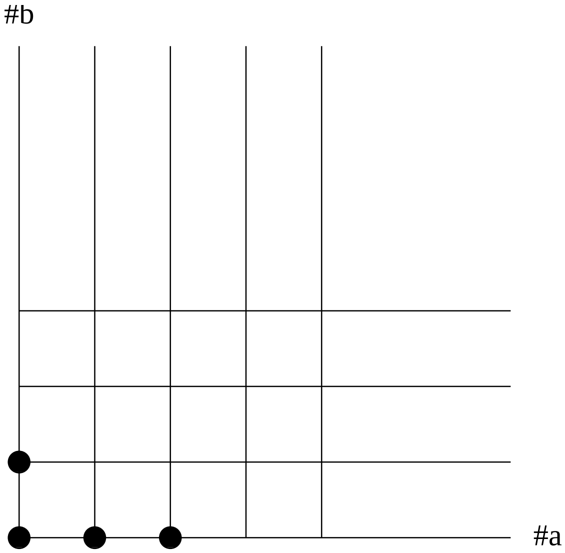
A non-positional numeration system $L = a^* b^*$



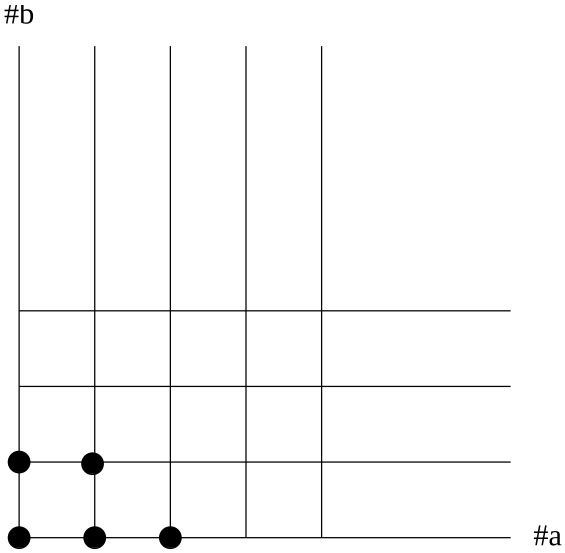
A non-positional numeration system $L = a^* b^*$



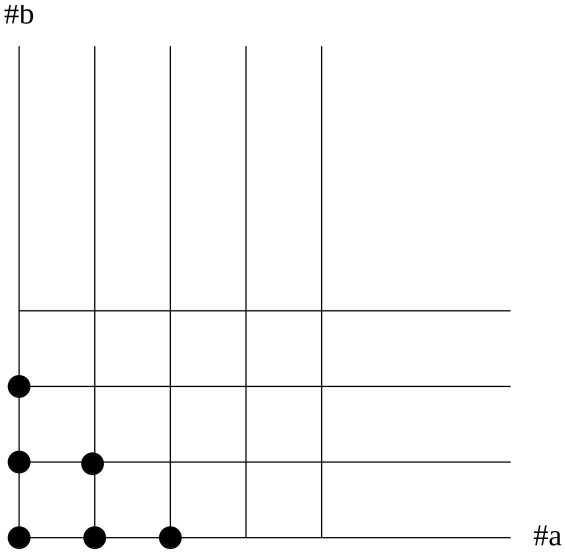
A non-positional numeration system $L = a^* b^*$



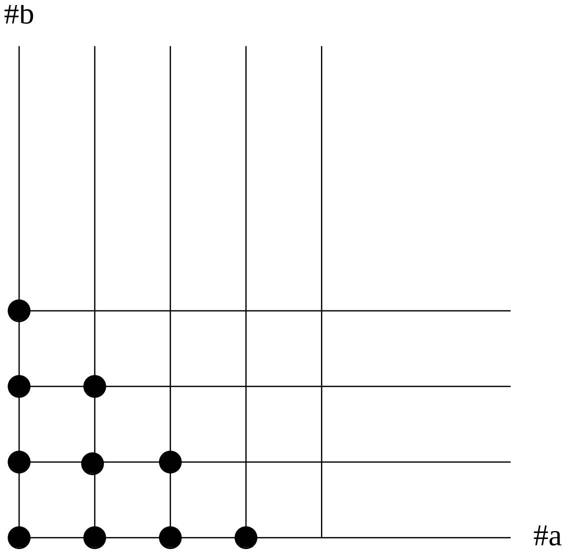
A non-positional numeration system $L = a^* b^*$



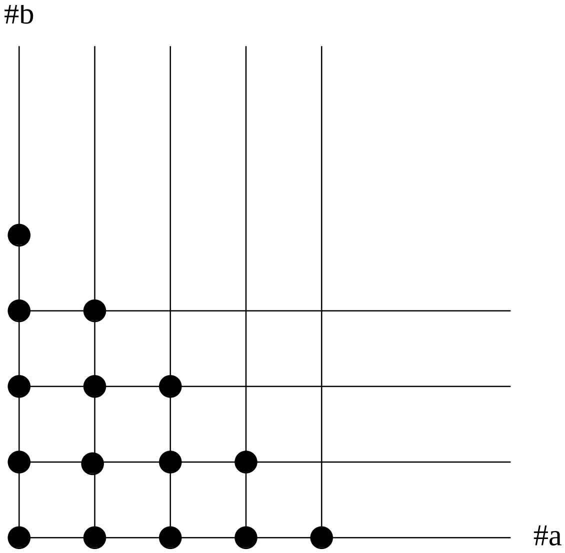
A non-positional numeration system $L = a^* b^*$



A non-positional numeration system $L = a^* b^*$

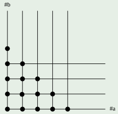


A non-positional numeration system $L = a^* b^*$



$L = a^*b^*,\ a < b$

ε	a	b	aa	ab	bb	aaa	aab	abb	$\cdots$
0	1	2	3	4	5	6	7	8	$\cdots$



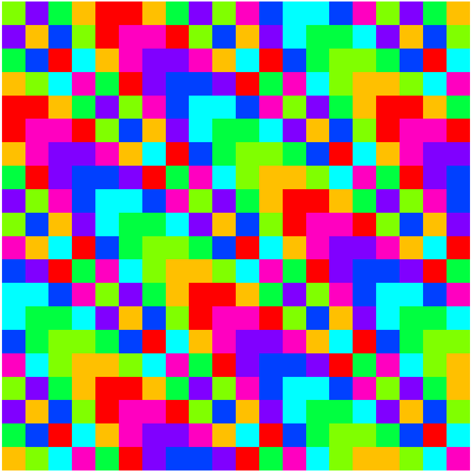
$$\mathrm{val}_{\mathcal{S}}(a^p b^q) = \frac{1}{2}(p+q)(p+q+1) + q = \binom{p+q+1}{2} + \binom{q}{1}$$

Katona, Lehmer, Fraenkel, Charlier, R., Steiner, Lew, Morales, ...

$$\text{Generalization}:\ \mathrm{val}_\ell(a_1^{n_1}\cdots a_\ell^{n_\ell}) = \sum_{i=1}^{\ell} \binom{n_i + \cdots + n_\ell + \ell - i}{\ell - i + 1}.$$

$$\forall n \in \mathbb{N}, \exists z_1, \ldots, z_\ell : n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell-1} + \cdots + \binom{z_1}{1}$$

with the condition $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0$



val$(a^p b^q)$ modulo 8

Theorem (P. Lecomte, M.R.)

Let S be an abstract numeration system.
Any ultimately periodic set is S-recognizable.

Theorem (D. Krieger *et al.* TCS'09)

Let L be a genealogically ordered regular language.
Any *periodic decimation* in L gives a regular language.
This result does not hold anymore if regular is replaced by context-free.

Something more general than k-automatic sequences ?

$$g : \begin{cases} A & \mapsto & ABCC \\ B & \mapsto & \varepsilon \\ C & \mapsto & BA \end{cases} \qquad f : \begin{cases} A & \mapsto & 010 \\ B & \mapsto & 1 \\ C & \mapsto & \varepsilon \end{cases}$$

$g(A) = ABCC,\ g^2(A) = ABCCBABA,$
$g^3(A) = ABCCBABAABCCABCC, \ldots$
$h(g^\omega(A)) = 0101101010100101010 1\cdots$

Remark

We can always assume that f is a coding (letter-to-letter) and g is a non-erasing morphism (in general non-uniform).

A. Cobham, On the Hartmanis-Stearns problem for a class of tag machines, '68
J.-P. Allouche, J. Shallit, CUP'03 J. Honkala, On the simplification of infinite morphic words, TCS'09

From k-automatic words to ... morphic/substitutive words

From k-recognizable subsets of $\mathbb{N}$ to ... substitutive sets

$$f(g^\omega(A)) = 0101101010100101010 1\cdots$$

Easy to generate the characteristic sequence of the substitutive set $\{1, 3, 4, 6, 8, 10, 13, \ldots\}$

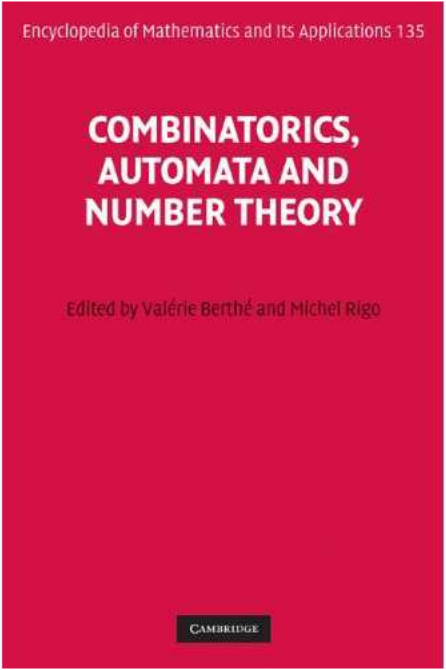
We still have a notion of "automaticity":

Maes–R. (JALC 2002)

An infinite word w is morphic IFF there exists an abstract numeration system S such that w is S-automatic.

P. Lecomte, R., Numeration systems on a regular language, **TOCS**'01.

P. Lecomte, R., Abstract numeration systems, Chap. 3 in Combinatorics, Automata and Number Theory, CUP 2010.


Encyclopedia of Mathematics and its Applications 135
COMBINATORICS, AUTOMATA AND NUMBER THEORY
Edited by Valérie Berthé and Michel Rigo
CAMBRIDGE

Transcendence of real numbers

$r \in (0,1)$, $k \in \mathbb{N} \setminus \{0,1\}$

$$r = \sum_{i=1}^{+\infty} c_i \, k^{-i} \qquad c_1 c_2 c_3 \cdots$$

Factor (or subword) complexity function : $p_w(n)$ is the number of distinct factors of length n occurring in w.

$$1 \leq p_w(n) \leq (\#A)^n \quad \text{and} \quad p_w(n) \leq p_w(n+1)$$

Morse–Hedlund theorem

The following conditions are equivalent:

- The word w is ultimately periodic, i.e., $w = xy^\omega$.
- The complexity function p_w is bounded by a constant,
- There exists $m \in \mathbb{N}$ such that $p_w(m) = p_w(m+1)$.

Transcendence of real numbers

Cobham 1972

If w is k-automatic, then p_w is $\mathcal{O}(n)$.

Pansiot (LNCS 172, 1984)

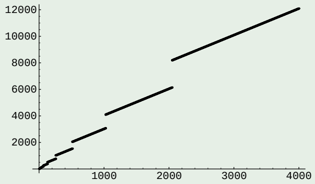
If w is pure morphic (no coding) and not ultimately periodic, then there exist constants C_1, C_2 such that $C_1 f(n) \leq p_w(n) \leq C_2 f(n)$ where $f(n) \in \{n, n\log n, n\log\log n, n^2\}$.

J.-P. Allouche, Sur la complexité des suites infinies, **BBMS**'94,
J. Cassaigne, F. Nicolas, Factor complexity, Chap. 4 in Combinatorics, Automata and Number Theory, CUP 2010.

Thue–Morse word

$$t = 10010110011010010110100110010110\cdots$$

$$p_t(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 4n - 2\cdot 2^m - 4 & \text{if } 2\cdot 2^m < n \le 3\cdot 2^m \\ 2n + 4\cdot 2^m - 2 & \text{if } 3\cdot 2^m < n \le 4\cdot 2^m \end{cases}$$



S. Brlek, Enumeration of factors in the Thue-Morse word, **DAM**'89
A. de Luca, S. Varricchio, On the factors of the Thue-Morse word on three symbols, **IPL**'88

Cobham's conjecture

Let α be an algebraic irrational real number. Then the k-ary expansion of α cannot be generated by a finite automaton.

Following this question :

Hartmanis–Stearns (Trans. AMS'65)

Does it exist an algebraic irrational real number computable in linear time by a (multi-tape) Turing machine? i.e., the first n digits of the representation computable in $\mathcal{O}(n)$ operations.

Transcendence of real numbers

J. P. Bell, B. Adamczewski, Automata in Number Theory, to appear Handbook (AutoMathA project).

Adamczewski–Bugeaud'07

Let $k \in \mathbb{N} \setminus \{0, 1\}$. The factor complexity of the k-ary expansion w of an algebraic irrational real number satisfies

$$\lim_{n \to +\infty} \frac{p_w(n)}{n} = +\infty.$$

Let $k \geq 2$ be an integer.
If α is an irrational real number whose k-ary expansion w has factor complexity in $\mathcal{O}(n)$, then α is transcendental.
So in particular, if w is k-automatic.

Bugeaud–Evertse'08

Let $k \geq 2$ be an integer and ξ be an algebraic irrational real number with $0 < \xi < 1$. Then for any real number $\eta < 1/11$, the factor complexity $p(n)$ of the k-ary expansion of ξ satisfies

$$\lim_{n \to +\infty} \frac{p(n)}{n(\log n)^{\eta}} = +\infty.$$

Positive view on k-recognizable sets 1/5

Let $\mathbb{K}$ be a field, $a(n) \in \mathbb{K}^{\mathbb{N}}$ be a $\mathbb{K}$-valued sequence and $k_1, \ldots, k_d \in \mathbb{K}$. The sequence $a(n)$ satisfies a *linear recurrence* over $\mathbb{K}$ if

$$a(n) = k_1 a(n-1) + \cdots + k_d a(n-d), \quad \forall n >>$$

Skolem–Mahler–Lech theorem

Let $a(n)$ be a linear recurrence over a field of characteristic 0. Then the zero set

$$\mathcal{Z}(a) = \{n \in \mathbb{N} \mid a(n) = 0\} \text{ is ultimately periodic.}$$

Remark

If $\mathbb{K}$ is a finite field, $a(n)$ (and so $\mathcal{Z}(a)$) is trivially ultimately periodic.

T. Tao, Effective Skolem–Mahler–Lech theorem in Structure and Randomness, AMS'08.

Positive view on k-recognizable sets 2/5

If $\mathbb{K}$ is an infinite field of positive characteristic...

Lech's example

$$a(n) := (1+t)^n - t^n - 1 \in \mathbb{F}_p(t).$$

The sequence a satisfies a linear recurrence, for $n > 3$

$$a(n) = (2+2t)\, a(n-1) + (1+3t+t^2)\, a(n-2) - (t+t^2)\, a(n-3).$$

We have

$$a(p^j) = (1+t)^{p^j} - t^{p^j} - 1 = 0$$

while $a(n) \neq 0$ if n is not a power of p, and so we obtain that

$$\mathcal{Z}(a) = \{1, p, p^2, p^3, \ldots\}.$$

Positive view on k-recognizable sets 3/5

Derksen's example

Consider the sequence $a(n)$ in $\mathbb{F}_p(x,y,z)$ defined by

$$a(n) := (x+y+z)^n - (x+y)^n - (x+z)^n - (y+z)^n + x^n + y^n + z^n.$$

It can be proved that :

- The sequence $a(n)$ satisfies a linear recurrence.
- The zero set is given by

$$\mathcal{Z}(a) = \{p^n \mid n \in \mathbb{N}\} \cup \{p^n + p^m \mid n, m \in \mathbb{N}\}.$$

$\mathcal{Z}(a)$ can be *more pathological* than in characteristic zero but. . . think about p-recognizable sets !

Positive view on k-recognizable sets 4/5

Theorem (H. Derksen'07)

Let $a(n)$ be a linear recurrence over a field of characteristic p. Then the set $\mathcal{Z}(a)$ is a p-recognizable set.

Derksen gave a further refinement of this result:
not all p-recognizable sets are zero sets of linear recurrences defined over fields of characteristic p.

Positive view on k-recognizable sets 5/5

Theorem (Adamczewski–Bell'2010)

Let $\mathbb{K}$ be a field and Γ be a finitely generated subgroup of $\mathbb{K}^*$. Consider the linear equations

$$a_1 X_1 + \cdots + a_d X_d = 1$$

where $a_1, \ldots, a_d \in \mathbb{K}$ and look for solutions in Γ^d.
The set of solutions is a "p-automatic subset of Γ^d" (not defined here).

If $\mathbb{K}$ is a field of characteristic 0, many contributions due to Beukers, Evertse, Lang, Mahler, van der Poorten, Schlickewei and Schmidt.

J.-H. Evertse, H.P. Schlickewei, W.M. Schmidt, Linear equations in variables which lie in a multiplicative group, Annals of Math. 2002.