Constant 2-labelling of a graph

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Abstract

We introduce the concept of constant 2-labelling of a graph and show how it can be used to obtain periodic sphere packing. Roughly speaking, a constant 2-labelling of a weighted graph is a 2-coloring $\{\bullet, \circ\}$ of its vertex set which preserves the sum of the weight of black vertices under some automorphisms. In this manuscript, we study this problem on complete graphs and on cycles. Our result on cycles allows us to determine (r, a, b)-codes in $\mathbb{Z}^2$ whenever $|a - b| > 4$.

Introduction

Given a graph $G = (V, E)$, a vertex v of G, a map $w : V \to \mathbb{R}$ and a subset A of the set $Aut(G)$ of all automorphisms of G, a *constant 2-labelling* of G is a mapping $\varphi : V \to \{\bullet, \circ\}$ such that

$$\sum_{\{u\in V|\varphi\circ\xi(u)=\bullet\}} w(u) = \sum_{\{u\in V|\varphi\circ\xi'(u)=\bullet\}} w(u) \quad \forall \xi, \xi' \in A_\bullet \text{ (respectively } A_\circ)$$

where $A_\bullet = \{\xi \in A \mid \varphi \circ \xi(v) = \bullet\}$ (resp. $A_\circ = \{\xi \in A \mid \varphi \circ \xi(v) = \circ\}$).

Constant 2-labellings are linked with distinguished colorings. A coloring is *distinguished* if it is not preserved by any non trivial automorphism of G. Introduced by Albertson and al [1], the *distinguishing number* of a graph is the smallest integer k such that there exist a distinguished coloring using k colors. This notion has already been studied in [4]. For a graph G, let φ be a non distinguished coloring of G. Then there exists a non trivial automorphism that preserves φ. If A denotes the set of automorphisms that preserve φ, then the coloring φ is a constant 2-labelling of G.

We can make some other straightforward observations about constant 2-labellings. Let a and b denote the following constants of a constant 2-labelling φ

$$a := \sum_{\{u\in V|\varphi\circ\xi(u)=\bullet\}} w(u) \text{ and } b := \sum_{\{u\in V|\varphi\circ\xi'(u)=\bullet\}} w(u) \text{ for } \xi \in A_\bullet, \xi' \in A_\circ.$$

Recall that a coloring of the vertex set V is *monochromatic* if all vertices have the same color.

Proposition 1. *Let $G = (V, E)$ be a weighted graph, $v \in V$, $w : V \to \mathbb{R}$ be the weight map and $A \subseteq Aut(G)$. If φ is a monochromatic coloring of V, then φ is a constant 2-labelling.*

In this case, the constant 2-labelling is said *trivial* and the corresponding constants are such that

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- $a = \sum_{u \in V} w(u)$ and b is not defined if φ is monochromatic black,
- a is not defined and $b = 0$ if φ is monochromatic white.

The following proposition allows us to consider either a coloring φ or the coloring obtained by switching the colors of φ. Let $\sigma : \{\bullet, \circ\} \to \{\bullet, \circ\}$ be a map such that $\sigma(\bullet) = \circ$ and $\sigma(\circ) = \bullet$. The *complementary* coloring of φ is the map $\sigma \circ \varphi$ and is denoted by $\overline{\varphi}$.

Proposition 2 (Complementary property)**.** *Let $G = (V, E)$ be a weighted graph, $w : V \to \mathbb{R}$ be the weight map, $v \in V$ and $A \subseteq Aut(G)$. Set $\omega := \sum_{u \in V} w(u)$. A coloring φ is a constant 2-labelling of G with respective constants a and b if and only if the coloring $\overline{\varphi}$ is a constant 2-labelling with respective constants $\omega - b$ and $\omega - a$.*

An interesting example is the weighted complete graph K_n.

Proposition 3. *Let $w : V(K_n) \to \mathbb{R}$, $v \in V(K_n)$ and $A = Aut(K_n)$. There is a non trivial constant 2-labelling of K_n if and only if $w(v_1) = w(v_2)$ for all $v_1, v_2 \in V \setminus \{v\}$.*

In this talk, we consider the following problem. Given a cycle of p weighted vertices, can we find a non trivial constant 2-labelling ? In particular, we consider eight different types of weighted cycles. Theorem 4 gives all the possible values of the constants a and b of constant 2-labellings of these cycles.

Next, we show how Theorem 4 can be useful to solve covering problems. Let $G = (V, E)$ be a graph and r a positive integer. A set $S \subseteq V$ of vertices is an (r, a, b)*-code* if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S. Such codes are also known as (r, a, b)-covering codes or (r, a, b)-isotropic colorings [2] or as perfect colorings [5]. When $r = 1$, an $(1, a, b)$-code is exactly a perfect weighted covering of radius one with weight $(\frac{b-a+1}{b}, \frac{1}{b})$. This particular case has been much studied. See [3] for existence and non-existence results in this case. Finally, thanks to Theorem 4, we describe all (r, a, b)-codes of $\mathbb{Z}^2$ with $|a - b| > 4$ and $r \geq 2$.

1 Constant 2-labelling of cycles

In this section, we consider weighted cycles with p vertices denoted by $\mathcal{C}_p$. These vertices $0, \ldots, p-1$ have respectively weights $w(0), \ldots, w(p-1)$. We will represent such a cycle by the word $w(0) \ldots w(p-1)$. Let $\mathcal{R}_k$ denote a k-rotation of $\mathcal{C}_p$, i.e.,

$$\mathcal{R}_k : \{0, \ldots, p-1\} \to \{0, \ldots, p-1\} : i \mapsto i + k \bmod p.$$

In the sequel, we always take $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. A coloring $\varphi : \{0, \ldots, p-1\} \to \{\bullet, \circ\}$ of a cycle $\mathcal{C}_p$ is a constant 2-labelling if, for every k-rotation of the coloring, the weighted sum of black vertices is a constant a (respectively b) whenever the vertex 0 is black (resp. white).

We consider eight particular types of weighted cycles $\mathcal{C}_p$ with at most 4 different weights namely z, x, y and t. The following words represent respectively cycles of Type 1–8 (see Figure 1) :

$$zx^{p-1},\ zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}},\ z(xy)^{\frac{p-1}{2}},\ z(xy)^{\frac{p-2}{2}}x,\ z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}},$$
$$z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}},\ z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}},\ z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$$

with $x \neq y$ and $p \geq 2$. Note that the exponents appearing in the representation of cycles must be integers. This implies extra conditions on p depending on the type of $\mathcal{C}_p$. We describe all constant 2-labellings of these cycles.

Theorem 4. *Let φ be a non trivial constant 2-labelling of a cycle $\mathcal{C}_p$ of type 1–8 with $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. Let $a = \sum_{\{u \in V \mid \varphi\circ\xi(u)=\bullet\}} w(u)$ and $b = \sum_{\{u \in V \mid \varphi\circ\xi'(u)=\bullet\}} w(u)$ for $\xi \in A_\bullet, \xi' \in A_\circ$. We have the following possible values of the constants a and b depending of the type of $\mathcal{C}_p$:*

Type	*Value of a*	*Value of b*	*Condition on parameters*
1	$\alpha x + z$	$(\alpha+1)x$	$\alpha \in \{0, \dots, p-2\}$
2	$2\alpha x + t + z$	$2(\alpha+1)x$	$\alpha \in \{0, \dots, \frac{p-4}{2}\}$
	$(\frac{p}{2}-1)x + z$	$(\frac{p}{2}-1)x + t$	
3	$\nexists$	$\nexists$	
4	$(\alpha+1)x + \alpha y + z$	$(\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p-4}{2}\}$
	$(\frac{p}{2}-1)y + z$	$\frac{p}{2}x$	
5	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}-1)x + (\frac{p}{3}+1)y$	$p \equiv 0 \pmod 3$
6	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}+1)x + (\frac{p}{3}-1)y$	$p \equiv 0 \pmod 3$
7	$a = (\frac{p}{2}-1)y + z$	$b = (\frac{p}{2}-1)x + t$	
	$a = \alpha(x+y) + t + z$	$b = (\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{2}-1\}$
8	$a = (\frac{p}{2}-2)y + z + t$	$b = \frac{p}{2}x$	
	$a = (2\alpha+2)x + 2\alpha y + z + t$	$b = (2\alpha+2)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{4}-1\}$
	$a = \frac{p}{4}x + (\frac{p}{4}-1)y + z$	$b = \frac{p}{4}x + (\frac{p}{4}-1)y + t$	
	$a = \frac{p}{2}x + (\frac{p}{4}-1)y + z$	$b = (p - \frac{p}{4})x$	$t = (\frac{p}{2} - \frac{p}{4})x + (\frac{p}{4} - \frac{p}{2} + 1)y$

Type 2 : $zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}}$ Type 5 : $z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}}$ Type 8: $z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$

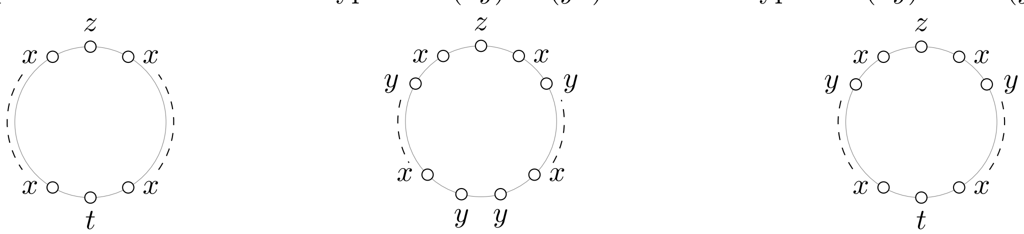

Figure 1: Types of weighted cycles $\mathcal{C}_p$.

2 Covering problems

In this section, we consider the graph of the infinite grid $\mathbb{Z}^2$. The vertices are all pairs of integers and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if $|x_1 - y_1| + |x_2 - y_2| = 1$. The infinite grid is a 4-regular graph, i.e., every vertex has 4 neighbours. Let the sets $L_e = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 = 0 \pmod 2\}$ and $L_o = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 = 1 \pmod 2\}$ denote the *even* and *odd* sub-lattices of $\mathbb{Z}^2$. Sets $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ and $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$ with $c \in \mathbb{Z}$ are called *diagonals* of $\mathbb{Z}^2$.

Recall that for a graph $G = (V, E)$ and a positive integer r, a set $S \subseteq V$ of vertices is an (r, a, b)*-code* if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S. For the infinite grid $\mathbb{Z}^2$, we consider balls defined relative to the Manhattan metric. The distance between two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ of $\mathbb{Z}^2$ is $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$. We can view an (r, a, b)-code of $\mathbb{Z}^2$ as a particular coloring φ with two colors black and white where the black vertices are the elements of the code. In other words, the coloring φ is such that a ball of radius r centered on a black (respectively white) vertex contains exactly a (resp. b) black vertices.

Firstly, we present the *projection and folding* method. Note that to apply this method, the coloring of the grid must satisfy some specific properties. Thanks to M. A. Axenovich's result about (r, a, b)-codes with $|a - b| > 4$ and $r \geq 2$ (see [2]), we can use the projection and folding method to prove Theorem 6. Secondly, we show how the method can be used in the case of (r, a, b)-codes of $\mathbb{Z}^2$.

2.1 Projection and folding

Let $r, t \in \mathbb{N}$ and $\varphi : \mathbb{Z}^2 \to \{\circ, \bullet\}$ be a coloring of $\mathbb{Z}^2$ such that the coloring of a line is obtained by doing a translation $\mathbf{t} = (t, 1)$ (respectively -$\mathbf{t} = (-t, -1)$) of the coloring of the line below (resp. above). In this case, if we know the coloring of one line and the translation $\mathbf{t}$, then the coloring of $\mathbb{Z}^2$ is known. In particular, for any vertex $\mathbf{x} \in \mathbb{Z}^2$, we have $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + \mathbf{t})$. Assume moreover that φ is such that $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0))$ for some $p \in \mathbb{Z}$ and all $\mathbf{x} \in \mathbb{Z}^2$. Suppose that p is the smallest integer satisfying this property.

Projection

Let $\mathbf{y} \in \mathbb{Z}^2$. Using the translation $\mathbf{t} = (t, 1)$, we can project the ball $B_r(\mathbf{y})$ on the line L containing $\mathbf{y}$. For easier notation, assume $\mathbf{y} = (0, 0)$. Let $Trans$ denote the set of all translated of $B_r(\mathbf{y})$ by a multiple of $\mathbf{t}$. Let $h : L \to \mathbb{N}$ be a map defined by

$$h((i, 0)) = \#\{T \in Trans \mid (i, 0) \in T\}.$$

The image of the line L by the mapping h, denoted by $h(L)$, is the *projection* of $B_r(\mathbf{y})$ with translation $\mathbf{t} = (t, 1)$. Note that $h((i, 0)) < \infty$ and h has a non zero value only finitely many times (see Figure 2 for example). This map is introduced to count the number of occurences in the ball $B_r(\mathbf{y})$ of vertices of L, up to translation $\mathbf{t}$. Observe that $\sum_{i \in \mathbb{Z}} h((i, 0)) = 2r^2 + 2r + 1$ since a ball of radius r contains exactly $2r^2 + 2r + 1$ vertices.

Folding

Using the translation $(p, 0)$, *i.e.*, $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0))$ for all $\mathbf{x} \in \mathbb{Z}^2$, we can fold a projection on a cycle of p weighted vertices. Let L be the line containing $\mathbf{y} = (0, 0)$ and $\{0, \ldots, p - 1\}$ be the set of vertices of the cycle $\mathcal{C}_p$. We define a map $w : \{0, \ldots, p - 1\} \to \mathbb{N}$ such that, for $i \in \{0, \ldots, p - 1\}$,

$$w(i) := \sum_{k \in \mathbb{Z}} h((i + kp, 0)).$$

The *folding* of the projection $h(L)$ is the cycle $\mathcal{C}_p$ with vertices $0, \ldots, p - 1$ of respective weights $w(0), \ldots, w(p_1)$.

Example 1. Consider balls of radius 3 of the infinite grid. Assume that the coloring φ of $\mathbb{Z}^2$ satisfies the translations $\mathbf{t} = (2, 1)$ and $(p, 0) = (5, 0)$, i.e.,

$$\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (2, 1)) \text{ and } \varphi(\mathbf{x}) = \varphi(\mathbf{x} + (5, 0)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2.$$

For $\mathbf{y} \in \mathbb{Z}^2$, we can compute the projection of $B_r(\mathbf{y})$ with translation $(2, 1)$ (see Figure 2) and its folding on a cycle $\mathcal{C}_5$ (see Figure 3).

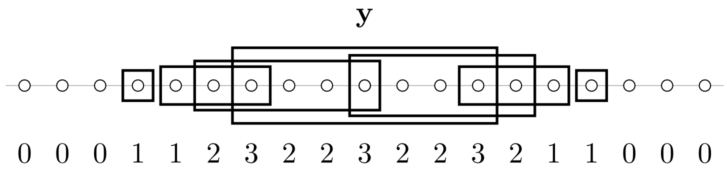

Figure 2: Projection on a line of $\mathbb{Z}^2$ of a ball $B_3(\mathbf{y})$ centered on $\mathbf{y}$ and of radius 3 with the translation $\mathbf{t} = (2, 1)$. The rectangles indicate the intersection of the line and elements of the set $Trans$. Under the line is the image of the line by the mapping h.

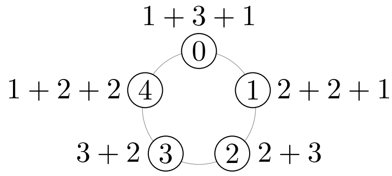

Figure 3: The folding of a ball $B_3(\mathbf{y})$ with translation $\mathbf{t} = (2, 1)$ on the cycle $\mathcal{C}_5$.

2.2 Application to (r, a, b)-codes

In order to use our projection and folding method for (r, a, b)-codes, we recall the notions of diagonal colorings and periodic colorings in the graph of the infinite grid $\mathbb{Z}^2$. A coloring φ of $\mathbb{Z}^2$ is *diagonal* if φ is such that the even and odd sublattices are the disjoint unions of monochromatic diagonals. Note that a diagonal coloring φ of $\mathbb{Z}^2$ is called *q-periodic* (respectively *q-antiperiodic*) if horizontal lines are colored q-periodically (resp. q-antiperiodically), i.e.,

$$\varphi((x_1, x_2)) = \varphi((x_1 + q, x_2)) \text{ (resp. } \varphi((x_1, x_2)) \neq \varphi((x_1 + q, x_2))) \qquad \forall (x_1, x_2) \in \mathbb{Z}^2.$$

For $r \geq 2$ and $|a - b| > 4$, M. A. Axenovich described all possible (r, a, b)-codes (see [2]).

Theorem 5 (M. A. Axenovich [2]). *If a coloring is an (r, a, b)-code with $r \geq 2$ and $|a - b| > 4$, then it is one of the following diagonal Colorings 1–5 :*

1. *q-periodic coloring where $q \in \{r, r+1\}$ is odd and the monochromatics diagonal are parallel.*
2. *q-antiperiodic coloring where $q \in \{r, r + 1\}$ is even.*
3. *q-periodic coloring where $q \in \{r, r + 1\}$ is even and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
4. *$(2r+1)$-periodic coloring and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
5. *2-periodic or 3-periodic coloring.*

This theorem allows us to apply the projection and folding method in this case. Let $r \geq 2$ and $a, b \in \mathbb{N}$ such that $|a - b| > 4$. Let φ be an (r, a, b)-code of $\mathbb{Z}^2$. By Theorem 5, φ is a diagonal coloring. Hence, φ is determined by the coloring of any horizontal line, e.g. $\{(x_1, 0) \mid x_1 \in \mathbb{Z}\}$, and by the orientation of the monochromatic diagonals in the even and odd sublattices. Observe that, by symmetry of the grid and balls of radius r, the case with non parallel monochromatic diagonals is equivalent to the case with parallel monochromatic diagonals in terms of counting vertices of a particular color appearing in the ball.

Hence we can assume that the monochromatic diagonals are all parallel. Without loss of generality, we suppose that they are all of the type $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ with $c \in \mathbb{Z}$. Since the coloring is diagonal, we have $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + \mathbf{t})$ for $\mathbf{t} = (1, 1)$ and all $\mathbf{x} \in \mathbb{Z}^2$. So we can apply the projection method. Moreover, by Theorem 5, φ is such that $\varphi(\mathbf{x} + (q, 0)) = \varphi(\mathbf{x})$ for some $q \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{Z}^2$. Hence, it is possible to apply the folding method.

Therefore, for $r \geq 2$ and $|a - b| > 4$, there exists an (r, a, b)-code of the infinite grid $\mathbb{Z}^2$ if and only if there exists a constant 2-labelling of some cycle $\mathcal{C}_p$, with $v = 0$, $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and a mapping w defined as before, such that

$$a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ and } b = \sum_{\{u \in V \mid \varphi \circ \xi' = \bullet\}} w(u) \quad \forall \xi \in A_\bullet,\ \xi' \in A_\circ.$$

Let $\mathbf{y} = (0, 0)$. By Theorem 5, we fold the ball $B_r(\mathbf{y})$ with translation $\mathbf{t} = (1, 1)$ on cycles $\mathcal{C}_p$ with $p \in \{2, 3, r, r+1, 2r, 2r+1, 2r+2\}$ accordingly with Colorings 1–5. So we consider the projection of $B_r(\mathbf{y})$ on the line L with translation $\mathbf{t} = (1, 1)$. We obtain for an even (respectively odd) radius r,

$$h((i,0)) = \begin{cases} r & \text{if } i \leq r \text{ and } i \text{ is odd (resp. even)} \\ r+1 & \text{if } i \leq r \text{ and } i \text{ is even (resp. odd)} \\ 0 & \text{otherwise} \end{cases}$$

Consider now Colorings 1–5. For each kind of coloring, we can determine the projection and folding of $B_r(\mathbf{y})$ on the cycle $\mathcal{C}_p$ according to the parity of r. Then, Theorem 4 gives the possible values of the constants a and b. Hence, we obtain the following theorem characterizing all (r, a, b)-codes of $\mathbb{Z}^2$ with $|a - b| > 4$.

Theorem 6. *Let $r, a, b \in \mathbb{N}$ such that $|a - b| > 4$ and $r \geq 2$. For all (r, a, b)-codes of $\mathbb{Z}^2$, the values of a and b are given in the following table.*

	a	b	*Conditions on parameter*
Coloring 1			
r even	$r + 1 + \alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0, \ldots, r-1\}$
r odd	$3r + 2 + \alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0, \ldots, r-2\}$
Coloring 2			
r even	$\frac{r}{2}(2r+1)$	$\frac{r^2}{2} + (\frac{r}{2}+1)(r+1)$	
r odd	$\frac{r+1}{2}(2r+1)$	$\frac{(r+1)^2}{2} + (\frac{r+1}{2}-1)r$	
Coloring 3			
r even	$2(\alpha+1)r + (2\alpha+3)(r+1)$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0, \ldots, \frac{r-4}{2}\}$
r even	$(r+1)^2$	r^2	
r odd	$2(\alpha+1)(r+1) + (2\alpha+1)r$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0, \ldots, \frac{r-3}{2}\}$
r odd	r^2	$(r+1)^2$	
Coloring 4			
	$\frac{(2r+1)^2}{3}$	$\frac{(2r+1)^2}{3} + 1$	$2r + 1 \equiv 0 \pmod 3$
Coloring 5			
r even	$(r+1)^2$	r^2	
r odd	r^2	$(r+1)^2$	
$r = 3k+1$	$\frac{2r^2+2r-1}{3} + \alpha\frac{2r^2+2r+2}{3}$	$(\alpha+1)\frac{2r^2+2r+2}{3}$	$\alpha \in \{0, 1\}$
$r = 3k-1$	$\frac{2r^2+2r}{3} - 2k + 1 + \alpha\frac{2r^2+2r}{3} + k$	$(\alpha+1)\frac{2r^2+2r}{3} + k$	$\alpha \in \{0, 1\}$
$r = 3k$	$\frac{2r^2+2r}{3} + 2k - 1 + \alpha\frac{2r^2+2r}{3} - k$	$(\alpha+1)\frac{2r^2+2r}{3} - k$	$\alpha \in \{0, 1\}$

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