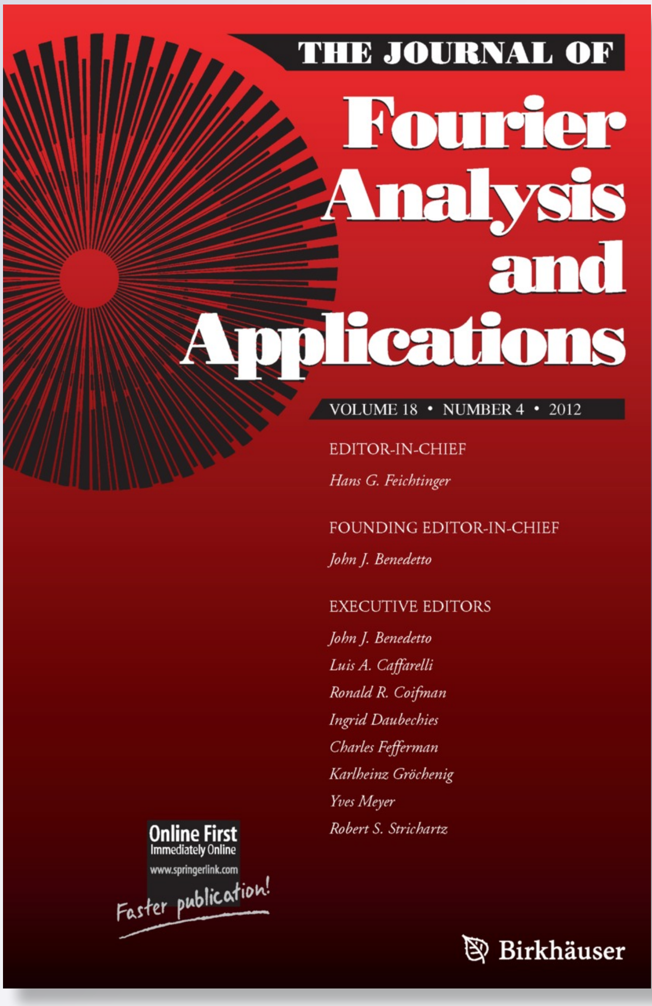
# A Wavelet Characterization for the Upper Global Hölder Index

**Marianne Clausel & Samuel Nicolay**

**Journal of Fourier Analysis and Applications**

ISSN 1069-5869
Volume 18
Number 4

J Fourier Anal Appl (2012) 18:750-769
DOI 10.1007/s00041-012-9220-y




Springer

J Fourier Anal Appl (2012) 18:750–769
DOI 10.1007/s00041-012-9220-y

# A Wavelet Characterization for the Upper Global Hölder Index

**Marianne Clausel · Samuel Nicolay**

Received: 11 July 2011 / Revised: 15 November 2011 / Published online: 1 February 2012


**Abstract** The upper Hölder index has been introduced to describe smoothness properties of a continuous function. It can be seen as the irregular counterpart of the usual Hölder index and has been used to investigate the behavior at the origin of the modulus of smoothness in many classical cases.

In this paper, we prove a characterization of the upper Hölder index in terms of wavelet coefficients. This result is a first step in the estimation of this exponent using wavelet methods.

**Keywords** Uniform Hölder regularity · Uniform Hölder irregularity · Discrete wavelet transform

**Mathematics Subject Classification (2000)** 26A16 · 42C40

## 1 Introduction

One of the most popular concept of uniform regularity is the uniform Hölder regularity, defined from the uniform Hölder spaces $C^\alpha(\mathbf{R}^d)$. For any $\alpha \in (0,1)$, a bounded function $f$ belongs to $C^\alpha(\mathbf{R}^d)$ if there exist $C, R > 0$ such that

$$\sup_{|x-y|\leq r} |f(x) - f(y)| \leq Cr^\alpha,$$

Communicated by Stéphane Jaffard.

M. Clausel
Laboratoire Jean Kuntzmann, Université de Grenoble, CNRS, 38041 Grenoble Cedex 9, France

S. Nicolay (✉)
Institut de Mathématique, Université de Liège, Grande Traverse, 12, Bâtiment B37, 4000 Liège (Sart-Tilman), Belgium
e-mail: S.Nicolay@ulg.ac.be

for any $r \in [0, R]$. This notion can be generalized for exponents greater than one (see Sect. 2.1). It has been widely used to study smoothness properties of classical models such as trigonometric series (see e.g. [22, 30]) and sample paths properties of processes (amongst these processes, let us cite the Brownian motion (see [24]) and the fractional Brownian motion (see e.g. [28]).

In many classical cases, the smoothness behavior of the investigated model is very simple. The studied function $f$ is both uniformly Hölder and uniformly anti-Hölder, i.e. there exist $C, R > 0$ such that

$$C^{-1}r^\alpha \le \inf_x \sup_{|y-x|\le r} |f(x) - f(y)| \le \sup_x \sup_{|x-y|\le r} |f(x) - f(y)| \le Cr^\alpha$$

for any $r \le R$ (see [9] and [10] for more details) and the smoothness properties of $f$ can be characterized using a single index,

$$\mathcal{H} = \lim_{r\to 0} \frac{\log \sup_{|x-y|\le r} |f(x) - f(y)|}{\log r}.$$

There are many well-known examples of such models (see [4, 5, 16, 17, 22, 30] for trigonometric series and [1–3, 6, 7, 31, 32] for sample paths of the FBM or some of its extensions).

Nevertheless, the smoothness properties of the model can be much more complex: in many cases, the uniform modulus of smoothness $\omega_f^1$ of $f$, that is the map

$$\omega_f^1 : r \mapsto \sup_{|x-y|\le r} |f(x) - f(y)|,$$

is quite general and can be "erratic". This is for example the case with the $\phi$–SNLD Gaussian models (see [31, 32]) or the lacunary fractional Brownian motion (see [8]), for which the uniform modulus of smoothness may be a general function that is not possible to estimate. It is then more convenient to describe the smoothness properties of the model using two indices:

$$\underline{\mathcal{H}} = \liminf_{r\to 0} \frac{\log \sup_{|x-y|\le r} |f(x) - f(y)|}{\log r} \tag{1}$$

and

$$\overline{\mathcal{H}} = \limsup_{r\to 0} \frac{\log \sup_{|x-y|\le r} |f(x) - f(y)|}{\log r}, \tag{2}$$

related to the behavior of the uniform modulus of smoothness of $f$ near 0.

Even in the case of Gaussian models, the estimation of these two indices is still an open problem. If the two indices $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are both equal to some $\mathcal{H} \in (0, 1)$, methods based on the wavelet decomposition or on discrete filtering (which has several similarities with the wavelet decomposition method) have proved to be often very efficient. The reader is referred to Flandrin (see [15]), Stoev et al. (see [29]) and the references therein for more informations on the wavelet-based methods and to Kent and Wood (see [23]), Istas and Lang (see [18]) and Coeurjolly (see [11, 12]) for more informations about quadratic variations-based methods.

The present work is a first step in the estimation of the two indices $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ in the general case. For this purpose, we investigate the relationship between these two Hölder indices and the wavelet decomposition of a function. The answer is well-known for the index $\underline{\mathcal{H}}$ (see [27] and Theorem 1 below). The main result of this paper is a characterization of the index $\overline{\mathcal{H}}$, called the upper Hölder exponent, by means of wavelets (see Theorem 3 and Corollary 4). Therefore, the results of the present paper should pave the way to the estimation of the indices $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ using wavelet methods.

This paper is organized as follows. In Sect. 2, we briefly recall the different concepts for uniform regularity and irregularity. Section 3 is devoted to the statement of our main results about the characterization of uniform irregularity by means of wavelets. Finally, Sect. 4 contains the proofs of the results stated in Sect. 3.

## 2 Upper and Lower Global Hölder Indices

In this section we first give the usual definition of global Hölder index, denoted here lower global Hölder index in order to make a distinction with the upper global Hölder index, which will be introduced afterward.

The definitions rely on the finite differences. For a function $f : \mathbf{R}^d \to \mathbf{R}$ and $x, h \in \mathbf{R}^d$, the first order difference of $f$ is

$$\Delta_h^1 f(x) = f(x+h) - f(x).$$

The difference of order $M$, where $M$ is an integer greater than 2, is iteratively defined by

$$\Delta_h^M f(x) = \Delta_h^{M-1} \Delta_h^1 f(x).$$

Let us introduce some notations. Given $\alpha > 0$, $[\alpha]$ will denote the greatest integer lower than $\alpha$. Throughout this paper, $M$ will designate the integer $M = [\alpha] + 1$ and we associate to a bounded function $f : \mathbf{R}^d \to \mathbf{R}$ its $M$-modulus of smoothness $\omega_f^M$:

$$\omega_f^M : r \mapsto \sup_{|h| \le r} \sup_{x \in \mathbf{R}^d} |\Delta_h^M f(x)|$$

### 2.1 The Lower Global Hölder Index

Let us recall the well-known notion of lower global Hölder index, usually called global Hölder index or uniform Hölder index.

**Definition 1** Let $\alpha > 0$ and $\beta \in \mathbf{R}$. The bounded function $f$ belongs to $C_\beta^\alpha(\mathbf{R}^d)$, if there exist $C, R > 0$ such that

$$\omega_f^M(r) \le C r^\alpha |\log r|^\beta, \tag{3}$$

for any $r \le R$. If $\beta = 0$, the space $C_0^\alpha(\mathbf{R}^d)$ is simply denoted $C^\alpha(\mathbf{R}^d)$.

A function $f$ is said to be uniformly Hölderian if for some $\alpha > 0$, $f \in C^\alpha(\mathbf{R}^d)$.

The above definition leads to a notion of global regularity.

**Definition 2** The lower global Hölder exponent of a uniformly Hölderian function $f$ is defined as

$$\underline{\mathcal{H}}_f = \sup\{\alpha > 0, f \in C^\alpha(\mathbf{R}^d)\}.$$

### 2.2 The Upper Global Hölder Index

The irregularity of a function can be studied through the notion of upper global Hölder index. The idea is to reverse inequality (3).

**Definition 3** Let $f : \mathbf{R}^d \to \mathbf{R}$ be a bounded function, $\alpha \geq 0$ and $\beta \in \mathbf{R}$; $f \in UI^\alpha_\beta(\mathbf{R}^d)$ if there exist $C, R > 0$ such that

$$\omega^M_f(r) \geq C r^\alpha |\log r|^\beta \tag{4}$$

for any $r \leq R$. If $\beta = 0$, the set $UI^\alpha_0(\mathbf{R}^d)$ is simply denoted $UI^\alpha(\mathbf{R}^d)$. A function belonging to $UI^\alpha(\mathbf{R}^d)$ is said to be uniformly irregular with exponent $\alpha$.

**Definition 4** The upper global Hölder exponent (or uniform irregularity exponent) of a bounded function $f$ is

$$\overline{\mathcal{H}}_f = \inf\{\alpha : f \in UI^\alpha(\mathbf{R}^d)\}.$$

Let us remark that the statement (4) is not a negation of the property $f \in C^\alpha(\mathbf{R}^d)$. Indeed $f$ does not belong to $C^\alpha(\mathbf{R}^d)$ if for any $C > 0$, there exists a decreasing sequence $(r_n)_n$ (depending on $C$) converging to 0 for which

$$\omega^M_f(r_n) \geq C r^\alpha_n.$$

We are thus naturally led to the following definition.

**Definition 5** Let $f : \mathbf{R}^d \to \mathbf{R}$ be a bounded function, $\alpha \geq 0$, $\beta \in \mathbf{R}$; $f \in C^\alpha_{w,\beta}(\mathbf{R}^d)$ if $f \notin UI^\alpha_\beta(\mathbf{R}^d)$, i.e. for any $C > 0$ there exists a decreasing sequence $(r_n)_n$ converging to 0 such that

$$\omega^M_f(r_n) \leq C r^\alpha_n |\log r_n|^\beta,$$

for any $n \in \mathbf{N}$. In the case where $\beta = 0$, the set $C^\alpha_{w,0}(\mathbf{R}^d)$ is denoted $C^\alpha_w(\mathbf{R}^d)$. A function belonging to $C^\alpha_w(\mathbf{R}^d)$ is said to be weakly uniformly Hölderian with exponent $\alpha$.

Roughly speaking, a function is weakly uniformly Hölderian with exponent $\alpha$ if for any $C > 0$, one can bound the $M$-modulus of smoothness $\omega^M_f$ of $f$ over $\mathbf{R}^d$ by $\theta(r_n) = C r^\alpha_n |\log r_n|^\beta$ for a remarkable decreasing sequence $(r_n)_n$ of scales, whereas for an Hölderian function, the $M$-modulus of smoothness of $f$ over $\mathbf{R}^d$ has to be bounded at each scale $r > 0$ by $\theta(r)$, for some $C > 0$.

## 3 A Wavelet Criterium for Uniform Irregularity

In this section we claim that both the lower and upper index of a bounded function can be characterized by means of wavelets.

### 3.1 The Discrete Wavelet Transform

Let us briefly recall some definitions and notations (for more precisions, see e.g. [13, 26, 27]). Under some general assumptions, there exists a function $\phi$ and $2^d - 1$ functions $(\psi^{(i)})_{1\le i<2^d}$, called wavelets, such that $\{\phi(x-k)\}_{k\in\mathbf{Z}^d} \cup \{\psi^{(i)}(2^j x - k) : 1 \le i < 2^d, k \in \mathbf{Z}^d, j \in \mathbf{Z}\}$ form an orthogonal basis of $L^2(\mathbf{R}^d)$. Any function $f \in L^2(\mathbf{R}^d)$ can be decomposed as follows,

$$f(x) = \sum_{k\in\mathbf{Z}^d} C_k \phi(x-k) + \sum_{j=1}^{+\infty} \sum_{k\in\mathbf{Z}^d} \sum_{1\le i<2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbf{R}^d} f(x)\psi^{(i)}(2^j x - k)\, dx,$$

and

$$C_k = \int_{\mathbf{R}^d} f(x)\phi(x-k)\, dx.$$

Let us remark that we do not choose the $L^2(\mathbf{R}^d)$ normalization for the wavelets, but rather an $L^\infty$ normalization, which is better fitted to the study of the Hölderian regularity. Hereafter, the wavelets are always supposed to belong to $C^\gamma(\mathbf{R}^d)$ with $\gamma$ sufficiently large (we require at least $\gamma > \alpha$) and the functions $\{\partial^s\phi\}_{|s|\le\gamma}$, $\{\partial^s\psi^{(i)}\}_{|s|\le\gamma}$ are assumed to have fast decay. Furthermore, in $\mathbf{R}^d$ we will use the tensor product wavelet basis (see [14, 27] and Sect. 4.2).

A dyadic cube of scale $j$ is a cube of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right),$$

where $k = (k_1, \ldots, k_d) \in \mathbf{Z}^d$. From now on, wavelets and wavelet coefficients will be indexed with dyadic cubes $\lambda$. Since $i$ takes $2^d - 1$ values, we can assume that it takes values in $\{0,1\}^d \setminus \{(0,\ldots,0)\}$; we will use the following notations:

- $\lambda = \lambda(i, j, k) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^{j+1}})^d$,
- $c_\lambda = c_{j,k}^{(i)}$,
- $\psi_\lambda = \psi_{j,k}^{(i)} = \psi^{(i)}(2^j \cdot - k)$.

To state our wavelet criteria, we will use the following notation: for any $j \ge 0$, we set

$$\|c_{j,\cdot}^{(\cdot)}\|_\infty = \sup_{i\in\{0,1\}^d\setminus\{(0,\ldots,0)\}} \sup_{k\in\mathbf{Z}^d} |c_{j,k}^{(i)}|.$$

### 3.2 Wavelets and Usual Uniform Regularity

The characterization of the lower global Hölder index in terms of wavelet coefficients is well-known.

The uniform Hölderian regularity of a function is closely related to the decay rate of its wavelet coefficients. Let us recall the following result (see [27]).

**Theorem 1** *Let $\alpha > 0$ such that $\alpha \notin \mathbf{N}$. We have $f \in C^\alpha(\mathbf{R}^d)$ if and only if there exists $C > 0$ such that*

$$\begin{cases} \forall k \in \mathbf{Z}^d, & |C_k| \leq C, \\ \forall j \geq 0, & \|c_{j,\cdot}^{(\cdot)}\|_\infty \leq C 2^{-j\alpha}. \end{cases} \tag{5}$$

This theorem yields a wavelet characterization of the lower Hölder index of a uniformly Hölderian function.

**Corollary 2** *Assume that $f$ is a uniformly Hölderian function*; *we have*

$$\underline{\mathcal{H}}_f = \liminf_{j \to \infty} \frac{\log_2 \|c_{j,\cdot}^{(\cdot)}\|_\infty}{-j}.$$

### 3.3 Wavelets and Uniform Irregularity

In this section, we aim at characterizing the uniform irregularity of a bounded function in terms of wavelets.

The main result if this paper is the following theorem.

**Theorem 3** *Let $\alpha > 0$ and $f$ be a bounded function on $\mathbf{R}^d$. If there exists $C > 0$ such that for any integer $j \geq 0$,*

$$\max\left( \sup_{\ell \geq j} \|c_{\ell,\cdot}^{(\cdot)}\|_\infty, 2^{-jM} \sup_{\ell \leq j} (2^{\ell M} \|c_{\ell,\cdot}^{(\cdot)}\|_\infty) \right) \geq C 2^{-j\alpha}, \tag{6}$$

*then $f \in UI^\alpha(\mathbf{R}^d)$.*

*Conversely, if $f$ is uniformly Hölderian and if for $\beta > 1$, $f$ belongs to $UI^\alpha_\beta(\mathbf{R}^d)$, then there exists $C > 0$ such that relation (6) holds for any $j \geq 0$.*

Let us make some remarks.

*Remark 1* Unlike the case of usual uniform Hölderian regularity, the case where $\alpha$ is a natural number is not a specific one.

*Remark 2* The assumptions of Theorem 3 are indeed optimal. See Appendix for more details.

*Remark 3* The condition

$$\|c_{j,\cdot}^{(\cdot)}\|_\infty \geq C 2^{-j\alpha},$$

for some $C > 0$ and any $j \geq 0$ is a sufficient (but not necessary) condition for uniform irregularity. In the general case,

$$\overline{\mathcal{H}}_f \neq \limsup_{j \to +\infty} \frac{\log_2 \|c_{j,\cdot}^{(\cdot)}\|_\infty}{-j}.$$

Following Theorem 1, a bounded function $f$ is not uniformly Hölderian with exponent $\alpha$, i.e. its $M$-modulus of smoothness is bounded from below by $\theta(r_n)$ for some specific decreasing sequence $(r_n)$ converging to 0, if and only if a similar property holds for its wavelet coefficients. The situation is completely different concerning uniform irregularity: the value of the $M$-modulus of smoothness at $r = 2^{-j}$ is influenced by the wavelet coefficients at scales below and above the scale $2^{-j}$. The $M$-modulus of smoothness of $f$ can be large at $r = 2^{-j}$ for any $j \in \mathbf{N}$ (even if for some scales $j$, the coefficients $(c_{j,k}^{(i)})$ are small or even vanish) provided that for any $j \in \mathbf{N}$, at a controlled distance of the scale $2^{-j}$, there exists some large wavelet coefficients. Such a behavior is met with the lacunary fractional Brownian motion, which admits some vanishing wavelet coefficients but that is almost surely locally uniformly irregular (see [8] for more details).

Theorem 3 leads to a wavelet characterization of the upper Hölder exponent.

**Corollary 4** *If $f$ is a uniformly Hölderian function, then*

$$\overline{\mathcal{H}}_f = \limsup_{j\to\infty} \frac{\log_2 \max(\sup_{\ell\geq j} \|c_{\ell,\cdot}^{(\cdot)}\|_\infty, 2^{-jM} \sup_{\ell\leq j}(2^{\ell M}\|c_{\ell,\cdot}^{(\cdot)}\|_\infty))}{-j}. \tag{7}$$

*Proof* One directly checks that if $\alpha$ is defined by the right-hand side of (7), Theorem 3 implies that $f \in UI^{\alpha+\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$ and $f \notin UI^{\alpha-\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$ such that $\alpha - \varepsilon \geq 0$. The conclusion is then straightforward. □

## 4 Proof of Theorem 3

Theorem 3 comes from a wavelet characterization (up to a logarithmic term) of the weak uniform Hölderian regularity. We first need to reformulate the property $f \in C_w^\alpha(\mathbf{R}^d)$ in terms of a modulus of continuity $\theta$ (defined by equality (12)), in order to obtain the following result.

**Proposition 5** *Let $\alpha > 0$;*

1. *If $f \in C_w^\alpha(\mathbf{R}^d)$ then, for any $C > 0$, there exists a strictly increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that for any $n \geq 0$ and any $j \in \{j_n, \dots, j_{n+1}-1\}$,*

$$\sup_{|\lambda|=2^{-j}} |c_\lambda| \leq C' C \inf(2^{-j_n\alpha}, 2^{(M-\alpha)j_{n+1}} 2^{-jM}), \tag{8}$$

*for some $C' > 0$ depending only on the chosen wavelet basis.*

2. *Conversely, if $f$ is uniformly Hölderian and if for any $C > 0$, there exists a strictly increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that (8) holds then $f \in C_{w,\beta}^\alpha(\mathbf{R}^d)$ for any $\beta > 1$.*

Next, we have to modify Proposition 5 by replacing the spaces $C_{w,\beta}^\alpha(\mathbf{R}^d)$ with $UI_\beta^\alpha(\mathbf{R}^d)$. This can be done thanks to the following lemma.

**Lemma 6** *The two following assertions are equivalent*:

1. *the wavelet coefficients of* $f$ *do not satisfy property* (8),
2. *there exists* $C' > 0$ *and an integer* $j_0$ *such that, for any* $j \geq j_0$,

$$\max\Big(\sup_{\ell\geq j}\sup_{|\lambda|=2^{-\ell}}|c_\lambda|, 2^{-jM}\sup_{\ell\leq j}\Big(2^{\ell M}\sup_{|\lambda|=2^{-\ell}}|c_\lambda|\Big)\Big)\geq C'\theta(2^{-j}), \tag{9}$$

*where* $\theta$ *is defined by equality* (12).

Once these results obtained, it is easy to show that if inequality (6) holds, then $f \notin C^\alpha_w(\mathbf{R}^d)$ and that if $f$ is uniformly Hölderian and satisfies $f \notin C^\alpha_{w,\beta}(\mathbf{R}^d)$ for $\beta > 1$, then relation (6) holds.

### 4.1 A Reformulation of the Property $f \in C^\alpha_w(\mathbf{R}^d)$

To prove Proposition 5, we first need to reformulate in a more appropriate way the property $f \in C^\alpha_w(\mathbf{R}^d)$.

Since modulus of smoothness $\omega^M_f$ is a non-decreasing function, $f \in C^\alpha_w(\mathbf{R}^d)$ if and only if, for any $C > 0$, there exists an increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that for any $r \in (2^{-j_{n+1}}, 2^{-j_n}]$ $(n \in \mathbf{N})$,

$$\omega^M_f(r) = \sup_{|h|\leq r}\sup_{x\in\mathbf{R}^d}|\Delta^M_h f(x)| \leq C2^{-j_n\alpha}. \tag{10}$$

Now, let $\Theta$ denote a piecewise constant function of the form

$$\Theta(r) = \sum_{n\in\mathbf{N}} 2^{-j_n\alpha}\chi_{(2^{-j_{n+1}},2^{-j_n}]}(r),$$

where $\chi_A$ is the characteristic function of the set $A$. The function $f$ belongs to $C^\alpha_w(\mathbf{R}^d)$ if and only if, for any $C > 0$, $C\Theta$ is an upper bound of the $M$-modulus of smoothness $\omega^M_f$ of $f$.

This characterization of the weak uniform regularity is not convenient to deal with, since

$$\limsup_{r\to 0}\frac{\Theta(2r)}{\Theta(r)}$$

may be infinite. To overcome this problem, in the next proposition we will reformulate the property $f \in C^\alpha_w(\mathbf{R}^d)$, giving a finer upper bound of $\omega^M_f$. To this end, let us remark that there is a link between the finite differences of $f$ at different scales.

**Proposition 7** *The bounded function* $f$ *belongs to* $C^\alpha_w(\mathbf{R}^d)$ *if and only if for any* $C > 0$, *there exists a strictly increasing sequence of integers* $(j_n)_{n\in\mathbf{N}}$ *such that for any* $j \in \{j_n, \dots, j_{n+1}-1\}$,

$$\sup_{|h|\leq 2^{-j}}\sup_{x\in\mathbf{R}^d}|\Delta^M_h f(x)| \leq C\inf(2^{-j_n\alpha}, 2^{M(j_{n+1}-j)}2^{-j_{n+1}\alpha}). \tag{11}$$

*Proof* Let us first assume that (10) holds. Since

$$\Delta^M_{2h} f(x) = \sum_{k=0}^{M}\binom{M}{k}\Delta^M_h f(x+kh),$$

we have

$$\omega_f^M(2r) = \sup_{|h|\le 2r}\sup_{x\in\mathbf{R}^d}|\Delta_h^M f(x)| \le 2^M \sup_{|h|\le r}\sup_{x\in\mathbf{R}^d}|\Delta_h^M f(x)| = 2^M\omega_f^M(r).$$

This inequality together with (10) imply that for any $j\in\{j_n,\dots,j_{n+1}-1\}$,

$$\begin{aligned}\omega_f^M(2^{-j}) &= \omega_f^M(2^{j_{n+1}-j}2^{-j_{n+1}})\\ &\le 2^{M(j_{n+1}-j)}\omega_f^M(2^{-j_{n+1}}) \le C2^{M(j_{n+1}-j)}2^{-j_{n+1}\alpha}.\end{aligned}$$

Hence, relation (11) holds. The converse assertion is obvious. □

Let us now remark that the piecewise function $\theta$ defined (on $(0,2^{-j_1}]$) as

$$\theta(r) = \sum_{n\in\mathbf{N}} \inf(2^{-j_n\alpha}, 2^{j_{n+1}(M-\alpha)}r^M)\chi_{(2^{-j_{n+1}},2^{-j_n}]}(r) \tag{12}$$

is a continuous function. Furthermore it satisfies additional interesting properties summed up in the following proposition.

**Proposition 8** *Let $\alpha>0$ and $(j_n)_{n\in\mathbf{N}}$ be an increasing sequence of integers. Let $\theta$ be defined by equality (12). The function $\theta$ obeys the following properties*:

1. *$\theta$ is a modulus of continuity, that is a non decreasing continuous function satisfying*

$$\limsup_{r\to 0}\frac{\theta(2r)}{\theta(r)} < \infty, \tag{13}$$

2. *for any $\beta>1$ and for any $J$ sufficiently large, the following relations are satisfied*:

$$\sum_{j=j_1}^{J} 2^{Mj}\theta(2^{-j}) \le CJ2^{MJ}\theta(2^{-J}), \tag{14}$$

$$\sum_{j\ge J}\frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} \le CJ^\beta\theta(2^{-J}), \tag{15}$$

$$2^{-Mj} = o(\theta(2^{-j})) \quad \text{as } j\to\infty. \tag{16}$$

*Proof* We first prove that $\theta$ is a modulus of continuity by showing that

$$\theta(2r) \le 2^M\theta(r). \tag{17}$$

Assume that there exists some $n\in\mathbf{N}$ such that

$$2^{-j_{n+1}} \le r \le 2^{-j_n-1}.$$

Since $2^{-j_{n+1}+1}\le 2r\le 2^{-j_n}$, one has

$$\theta(2r) = \inf(2^{-j_n\alpha}, 2^{j_{n+1}(M-\alpha)}(2r)^M) \le 2^M\theta(r).$$

On the other hand, if for some $n\in\mathbf{N}$, one has

$$2^{-j_n-1}\le r\le 2^{-j_n},$$

then $2^{-j_n} \leq 2r \leq 2^{-j_n+1}$ and thus

$$\begin{aligned}\theta(2r) &= \inf(2^{-j_{n-1}\alpha}, 2^{j_n(M-\alpha)}(2r)^M) \\ &\leq 2^M(2^{j_n}r)^M 2^{-j_n\alpha} = 2^M 2^{j_n(M-\alpha)} r^M.\end{aligned}$$

Since $M-\alpha>0$, one has

$$2^M 2^{j_n(M-\alpha)} r^M \leq 2^M 2^{j_{n+1}(M-\alpha)} r^M.$$

Moreover, since $r \leq 2^{-j_n}$,

$$2^M(2^{j_n}r)^M 2^{-j_n\alpha} \leq 2^M 2^{-j_n\alpha},$$

hence,

$$\theta(2r) \leq 2^M \inf(2^{-j_n\alpha}, 2^{j_{n+1}(M-\alpha)} r^M).$$

In any case, relation (17) holds, which directly implies (13).

Let us now prove the second part of Proposition 8. Let $J \in \mathbf{N}$ and $n_0 \in \mathbf{N}$ such that $j_{n_0} \leq J \leq j_{n_0+1}-1$. Let us first show that property (14) is satisfied. By definition, we have

$$\begin{aligned}\sum_{j=j_1}^{J} 2^{Mj}\theta(2^{-j}) &= \sum_{n=0}^{n_0-1}\sum_{j=j_n}^{j_{n+1}-1} 2^{Mj}\inf(2^{-j_n\alpha}, 2^{j_{n+1}(M-\alpha)}2^{-jM}) \\ &\quad + \sum_{j=j_{n_0}}^{J-1} 2^{Mj}\inf(2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(M-\alpha)}2^{-jM}).\end{aligned}$$

Therefore,

$$\sum_{j=j_1}^{J} 2^{Mj}\theta(2^{-j}) \leq \sum_{n=0}^{n_0-1} j_{n+1}2^{j_{n+1}(M-\alpha)} + J\inf(2^{MJ}2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(M-\alpha)}),$$

that is

$$\begin{aligned}\sum_{j=j_1}^{J} 2^{Mj}\theta(2^{-j}) &\leq j_{n_0}2^{j_{n_0}(M-\alpha)} + J\inf(2^{MJ}2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(M-\alpha)}) \\ &\leq 2J\inf(2^{MJ}2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(M-\alpha)}),\end{aligned}$$

which shows that property (14) holds.

We now check inequality (15). Since

$$\theta(2^{-j}) \leq 2^{-j_n\alpha}$$

for any $n \geq n_0$ and any $j \in \{j_n, \dots, j_{n+1}-1\}$, we have

$$\begin{aligned}\sum_{j=J}^{\infty} \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} &\leq \sum_{j=J}^{j_{n_0+1}-1} \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} \\ &\quad + \sum_{n=n_0+1}^{\infty}\sum_{j=j_n}^{j_{n+1}-1} \frac{2^{-j_n\alpha}|\log(2^{-j_n\alpha})|^\beta}{j^\beta}\end{aligned}$$

$$= \sum_{j=J}^{j_{n_0+1}-1} \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} + C\sum_{n=n_0+1}^{\infty} 2^{-j_n\alpha} j_n^\beta \sum_{j=j_n}^{j_{n+1}-1} \frac{1}{j^\beta}.$$

Using equality (12), we get

$$\sum_{j=J}^{\infty} \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} \le C \sum_{j=J}^{j_{n_0+1}-1} \frac{\inf(j_{n_0}^\beta 2^{-j_{n_0}\alpha}, j^\beta 2^{j_{n_0+1}(M-\alpha)} 2^{-jM})}{j^\beta} + C\sum_{n=n_0+1}^{\infty} j_n^\beta 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} \frac{1}{j^\beta}. \tag{18}$$

Moreover, since

$$\sum_{n=n_0+1}^{\infty} j_n^\beta 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} \frac{1}{j^\beta} \le \sum_{n=n_0+1}^{\infty} j_n 2^{-j_n\alpha} \le j_{n_0+1} 2^{-j_{n_0+1}\alpha},$$

inequality (18) yields

$$\begin{aligned}\sum_{j=J}^{\infty} \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta} &\le C \sum_{j=J}^{j_{n_0+1}-1} \frac{\inf(j_{n_0}^\beta 2^{-j_{n_0}\alpha}, j^\beta 2^{j_{n_0+1}(M-\alpha)} 2^{-jM})}{j^\beta} \\ &\quad + C j_{n_0+1} 2^{-j_{n_0+1}\alpha} \\ &\le C'(\inf(j_{n_0} 2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(M-\alpha)} 2^{-JM}) \\ &\quad + j_{n_0+1} 2^{-j_{n_0+1}\alpha}) \\ &\le C' J^\beta \theta(2^{-J}).\end{aligned}$$

Since $M > \alpha$, relation (16) is straightforward. □

*Remark 4* The concept of modulus of continuity has been used in [21] to deal with a more general notion of uniform Hölderian regularity than the usual one, induced by the Hölder spaces. For a given $M$ and a given modulus of continuity $\theta$, a wavelet characterization of the property

$$\omega_f^M(r) \le C\theta(r) \tag{19}$$

for any $r \ge 0$ is provided under the two following assumptions on $\theta$: for any $J \ge 0$,

$$\sum_{j=0}^{J} 2^{jM}\theta(2^{-j}) \le C' 2^{JM}\theta(2^{-J}) \tag{20}$$

and

$$\sum_{j=J}^{\infty} 2^{j(M-1)}\theta(2^{-j}) \le C' 2^{J(M-1)}\theta(2^{-J}). \tag{21}$$

Properties (20) and (21) are much stronger than properties (14), (15) and (16), which concern the weak uniform regularity of a function $f$.

### 4.2 Proof of Proposition 5

We shall split the proof into two parts.

**Proposition 9** *Let* $\alpha > 0$; *if* $f \in C^\alpha_w(\mathbf{R}^d)$ *then, for any* $C > 0$, *there exists a strictly increasing sequence of integers* $(j_n)_{n\in\mathbf{N}}$ *such that for any* $n \geq 0$ *and any* $j \in \{j_n, \dots, j_{n+1} - 1\}$,

$$\sup_{|\lambda|=2^{-j}} |c_\lambda| \leq C' C\theta(2^{-j}),$$

*for some* $C' > 0$ *depending only on the chosen wavelet basis, where* $\theta$ *is the function defined by equality* (12).

*Proof* Assume that $f$ belongs to $C^\alpha_w(\mathbf{R}^d)$ and let $C > 0$. By Proposition 7, we have for any $r$ sufficiently small,

$$\omega^M_f(r) \leq C\theta(r). \tag{22}$$

If $d = 1$, let us recall (see [20]) that if the wavelet basis belongs to $C^M(\mathbf{R}^d)$ then there exists a function $\Psi_M$ with fast decay and such that $\psi = \Delta^M_{\frac{1}{2}}\Psi_M$. In dimension $d > 1$, we use the tensor product wavelet basis:

$$\psi^{(i)}(x) = \Psi^{(1)}(x_1)\cdots\Psi^{(d)}(x_d),$$

where for all $i$, $\Psi^{(i)}$ are either $\psi$ or $\phi$ but at least one of them must equal $\psi$. For example, assume that $\Psi^{(1)} = \psi$. Then, for any $i \in \{1, \dots, 2^d - 1\}$, any $j \geq 0$ and any $k \in \mathbf{Z}^d$,

$$c^{(i)}_{j,k} = 2^{jd} \int_{\mathbf{R}^d} f(x)\Psi^{(1)}(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)\,dx.$$

We thus have

$$\begin{aligned} c^{(i)}_{j,k} &= 2^{jd} \int_{\mathbf{R}^d} f(x)\Delta^M_{1/2}\Psi_M(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)\,dx \\ &= 2^{jd} \int_{\mathbf{R}^d} \Delta^M_{1/2^{j+1}e_1} f(x)\Psi_M(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)\,dx, \end{aligned}$$

with $e_1 = (1, 0, \dots, 0)$ and therefore

$$|c^{(i)}_{j,k}| \leq 2^{jd} \int_{\mathbf{R}^d} |\Delta^M_{1/2^{j+1}e_1} f(x)||\Psi_M(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)|\,dx.$$

We thus get, using inequality (22),

$$|c^{(i)}_{j,k}| \leq C2^{jd}\theta(2^{-(j+1)}) \int_{\mathbf{R}^d} |\Psi_M(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)|\,dx.$$

Setting $y = 2^j x - k$ in the last integral, we obtain

$$2^{jd} \int_{\mathbf{R}^d} |\Psi_M(2^j x_1 - k_1)\cdots\Psi^{(d)}(2^j x_d - k_d)|\,dx = \|\Psi_M \otimes \cdots \otimes \Psi^{(d)}\|_{L^1(\mathbf{R}^d)}.$$

Since $\theta$ is a non-decreasing function, we can write

$$|c_{j,k}^{(i)}| \leq C\theta(2^{-j})\|\Psi_M\|_{L^1(\mathbf{R}^d)},$$

which ends the proof. □

From now on in this section, we suppose that $f$ is uniformly Hölderian and that property (8) is satisfied. For the second part of the proof, we need to introduce the following notations:

$$f_{-1}(x) = \sum_{k\in\mathbf{Z}^d} C_k\varphi(x-k), \qquad f_j(x) = \sum_{i=1}^{2^d-1}\sum_{k\in\mathbf{Z}^d} c_{j,k}^{(i)}\psi(2^j x - k), \tag{23}$$

with $j \geq 0$. Since $f$ is uniformly Hölderian, $f_j$, as defined by equality (23), converges uniformly on any compact to a limit which has the same regularity as the wavelets. Furthermore $\sum_{j\geq -1} f_j(x)$ converges uniformly on any compact. The proof is based on the following lemma which provides an upper bound for $\|\partial^\gamma f_j(x)\|_{L^\infty(\mathbf{R}^d)}$, for any $|\gamma| \leq M$.

**Lemma 10** *Let $m \in \{0,\dots,M\}$; there exists some $C' > 0$ depending only on $m$ and on the chosen wavelet basis such that for any $\gamma \in \mathbf{N}^d$ satisfying $|\gamma| = m$ and for $j$ sufficiently large,*

$$\|\partial^\gamma f_j(x)\|_{L^\infty(\mathbf{R}^d)} \leq C'C2^{jm}\inf\left(\theta(2^{-j}), \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta}\right),$$

*where $\theta$ is the function defined by equality (12).*

*Proof* Since $f$ satisfies Property (8), one has

$$|c_{j,k}^{(i)}| \leq C\theta(2^{-j}), \tag{24}$$

for $j$ sufficiently large. Furthermore, since $f$ is uniformly Hölderian,

$$\left|\log|c_{j,k}^{(i)}|\right| \geq C'j, \tag{25}$$

for some $C' > 0$ and $j$ sufficiently large. Now, using the trivial relation

$$|c_{j,k}^{(i)}| = \inf\left(|c_{j,k}^{(i)}|, \frac{|c_{j,k}^{(i)}||\log|c_{j,k}^{(i)}||^\beta}{|\log|c_{j,k}^{(i)}||^\beta}\right),$$

inequalities (24) and (25) leads to

$$|c_{j,k}^{(i)}| \leq \inf\left(\theta(2^{-j}), \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta}\right).$$

Therefore, for any integer $p > d$,

$$\begin{aligned}|\partial^\alpha f_j(x)| &= \left|\sum_{i=1}^{2^d-1}\sum_{k\in\mathbf{Z}^d} c_{j,k}^{(i)} 2^{jm}\partial^\alpha\psi^{(i)}(2^j x - k)\right| \\ &\leq C'C2^{jm}\sum_{i=1}^{2^d-1}\sum_{k\in\mathbf{Z}^d}\frac{\inf(\theta(2^{-j}), \frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta})}{(1+|2^j x - k|)^p},\end{aligned}$$

using the fast decay of the wavelets. The use of the classical bound

$$\sup_{x\in\mathbf{R}^d}\sum_{k\in\mathbf{Z}^d}\frac{1}{(1+|2^jx-k|)^p}<\infty$$

ends the proof of this lemma. □

**Proposition 11** *Let $\alpha>0$; if $f$ is uniformly Hölderian and if for any $C>0$, there exists a strictly increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that (8) holds, let $h\in\mathbf{R}^d$ and define $J=\sup\{j_n:|h|<2^{-j_n}\}$. We have, for h sufficiently small,*

$$|\Delta_h^M f(x)|\leq C'J^\beta\theta(2^{-J}), \tag{26}$$

*where $\theta$ is the function defined by equality (12).*

*Proof* Let us set

$$g_1=\sum_{j=-1}^{j_1-1}f_j(x),\qquad g_2=\sum_{j=j_1}^{J-1}f_j,\quad\text{and}\quad g_3=\sum_{j=J}^{\infty}\Delta_h^M f_j(x).$$

For any $j\geq -1$, $f_j$ has the same regularity as the wavelets and so does $g_1$. Therefore, we can suppose that $g_1$ belongs to $C^\eta(\mathbf{R}^d)$ with $M<\eta\notin\mathbf{N}$ and for any $r>0$,

$$\omega_{g_1}^M(r)\leq C'r^M,$$

(see e.g. [25]). Using relation (16), we get that inequality (26) holds for $f=g_1$.

Let us now consider the case $f=g_2$. Lemma 10 with $m=M$ leads to the inequality

$$|\partial^\gamma f_j(x)|\leq C'C2^{jM}\theta(2^{-j})$$

for any $\gamma$ such that $|\gamma|=M$ and for any $j_1\leq j\leq J-1$. Furthermore, for any $j$, $f_j\in C^\eta(\mathbf{R}^d)$ which can be considered as a subset of the homogeneous Hölder space $\dot{C}^\eta(\mathbf{R}^d)$ (see e.g. [27]). Therefore,

$$|\Delta_h^M f_j(x)|\leq |h|^M\sum_{|\gamma|=M}\|\partial^\gamma f_j\|_{L^\infty(\mathbf{R}^d)},$$

for any $j\geq j_1$. We thus have

$$\left|\sum_{j=j_0}^{J-1}\Delta_h^M f_j(x)\right|\leq C'C|h|^M\sum_{j=j_0}^{J-1}2^{jM}\theta(2^{-j}).$$

Using relation (14), we get

$$\left|\sum_{j=j_0}^{J-1}\Delta_h^M f_j(x)\right|\leq C'C|h|^MJ2^{JM}\theta(2^{-J})\leq C'CJ\theta(2^{-J}).$$

We have thus proved that the function $g_2$ satisfies inequality (26).

For $g_3$, let us apply lemma 10 with $m=0$ to obtain

$$\left|\sum_{j=J}^{\infty}\Delta_h^M f_j(x)\right|\leq C'C\sum_{j=J}^{\infty}\frac{\theta(2^{-j})|\log\theta(2^{-j})|^\beta}{j^\beta}.$$

By inequality (15), we have

$$\left|\sum_{j=J}^{\infty} \Delta_h^M f_j(x)\right| \leq C'CJ^\beta\theta(2^{-J}).$$

The results concerning $g_1$, $g_2$ and $g_3$ put together show that the function $f$ satisfies inequality (26), which ends the proof. □

### 4.3 Proof of Lemma 6

Let us show that property (8) is equivalent to the negation of property (9). Indeed by definition, the wavelet coefficients of $f$ satisfy property (8) if and only if for any $C > 0$, there exists an increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that

$$\sup_{i,k} |c_{j,k}^{(i)}| \leq C \inf(2^{-j_n\alpha}, 2^{j_{n+1}(M-\alpha)}2^{-jM}),$$

for any $n \in \mathbf{N}$ and any $j \in \{j_n, \dots, j_{n+1} - 1\}$. This statement can be reformulated as follows: for any $C > 0$, there exists an increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that for any $n \in \mathbf{N}$,

$$\sup_{\ell\geq j_n} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{-j_n\alpha}$$

and

$$\sup_{j_0\leq\ell\leq j_{n+1}} 2^{\ell M} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{j_{n+1}(M-\alpha)}.$$

Let us set

$$n_0 = \inf\left\{n \in \mathbf{N} : \sup_{0\leq\ell\leq j_0} \left(2^{\ell M} \sup_{i,k} |c_{\ell,k}^{(i)}|\right) \leq C2^{j_{n+1}(M-\alpha)}\right\}.$$

Replacing the sequence $j_n$ by $\ell_n = j_{n+n_0+1}$, property (8) is equivalent to the existence, for any $C > 0$, of a strictly increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that for any $n \in \mathbf{N}$,

$$\sup_{\ell\geq j_n} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{-j_n\alpha},$$

and

$$\sup_{\ell\leq j_n} 2^{\ell M} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{j_n(M-\alpha)}.$$

To conclude, observe that the last property is equivalent to the existence, for any $C > 0$ and any $j_0 \in \mathbf{N}$, of some $j_1 > j_0$ such that

$$\sup_{\ell\geq j_1} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{-j_1\alpha}$$

and

$$\sup_{\ell\leq j_1} 2^{\ell M} \sup_{i,k} |c_{\ell,k}^{(i)}| \leq C2^{j_1(M-\alpha)}.$$

Since this is the negation of relation (9), the lemma is proved.

## Appendix: Optimality of the Assumptions of Theorem 3

We prove here the optimality of the assumptions of Proposition 5 and thus of Theorem 3. To this end we use two counter-examples already introduced in [19].

### A.1 A Uniform Irregular Function Satisfying Property (8)

Let $\alpha \in (0,1)$, $\ell_0 \in \mathbf{N}$ and define the two following sequences of integers $(j_n)_{n\in\mathbf{N}}$ and $(j_{n,\alpha})_{n\in\mathbf{N}}$ as

$$\begin{cases} j_1 = \ell_0, & \\ j_{n+1} = [\frac{1}{1-\alpha} 2^{j_n\alpha} - j_n\alpha], & \forall n \geq 1, \\ j_{n,\alpha} = [2^{j_n\alpha}], & \forall n \geq 1. \end{cases}$$

We aim at proving the following result.

**Proposition 12** *Let us assume that the multiresolution analysis is compactly supported. Let $\varepsilon \in (0,1)$ and $\ell_0$ be such that* $\operatorname{supp}(\psi) \subset [-2^{\ell_0}, 2^{\ell_0}]$. *Furthermore, let us assume that $\psi(0) \neq 0$. The function $f$ defined as*

$$\begin{aligned} f(x) = {} & \sum_{n=0}^{\infty} 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n,\alpha}} \sum_{\ell=j+2}^{j_{n,\alpha}} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big) \\ & + \sum_{n=0}^{\infty} 2^{j_{n+1}(1-\alpha)} \sum_{j=j_{n,\alpha}+1}^{j_{n+1}-1} \sum_{\ell=j+2}^{j_{n+1}} 2^{-\ell} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big) \\ & + \sum_{n=0}^{\infty} 2^{-j_{n+1}\alpha} \sum_{j=j_{n,\alpha}+1}^{j_{n+1}-1} \sum_{\ell=j_{n+1}}^{j_{n+1,\alpha}} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big) \end{aligned}$$

*satisfies the following properties*:

1. *$f$ is not a uniformly Hölderian function*,
2. *the wavelet coefficients of $f$ satisfy property* (8),
3. *$f$ is uniformly irregular with exponent $\beta$, where*

$$\beta = \max\left(\alpha\varepsilon, \frac{\alpha\varepsilon}{(1-\alpha)+\alpha\varepsilon}\right) < \alpha. \tag{27}$$

*Proof* The two first properties being straightforward, we just have to prove that $f$ is uniformly irregular with exponent $\beta$. Let $n \in \mathbf{N}$ and define

$$f_j(x) = \sum_{\ell=j+2}^{j_{n,\alpha}} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big),$$

for $j \in \{j_n, \dots, j_{n,\alpha}\}$ and

$$f_j(x) = \sum_{\ell=j+2}^{j_{n+1}} 2^{-\ell} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big) + \sum_{\ell=j_{n+1}}^{j_{n+1,\alpha}} \ell^{-\varepsilon} \psi\big(2^{\ell}(x - 2^{-(j-\ell_0)})\big),$$

for $j \in \{j_{n,\alpha}, \dots, j_{n+1}-1\}$. We need to estimate

$$f(2^{-(j-\ell_0)}) - f(0) = f(2^{-(j-\ell_0)})$$

for any $j \in \mathbf{N}$. First, observe that for $j \neq j'$, $\operatorname{supp}(f_j) \cap \operatorname{supp}(f_{j'}) = \emptyset$. Indeed for any $j$, we have

$$\operatorname{supp}(f_j) \subset [3.2^{-(j+2-\ell_0)}, 5.2^{-(j+2-\ell_0)}]$$

and hence $f(2^{-(j-\ell_0)}) - f(0) = f_j(2^{-(j-\ell_0)})$ for any $j \in \mathbf{N}$.

We now distinguish two cases. Let us first assume that $j \in \{j_n, \dots, j_{n,\alpha}\}$; we have

$$f(2^{-(j-\ell_0)}) = 2^{-j_n\alpha} \sum_{\ell=j+2}^{j_{n,\alpha}} \ell^{-\varepsilon}\psi(0) \geq 2^{-j_n\alpha}((j_{n,\alpha}+1)^{1-\varepsilon} - (j+2)^{1-\varepsilon})$$

Therefore, if $j_n \leq j \leq j_{n,\alpha}/2$,

$$f(2^{-(j-\ell_0)}) \geq 2^{-j_n\alpha}(j_{n,\alpha}+1)^{1-\varepsilon}(1-2^{-(1-\varepsilon)}) \geq C' 2^{-j\alpha\varepsilon},$$

whereas if $j_{n,\alpha}/2 \leq j \leq j_{n,\alpha}$,

$$f(2^{-(j-\ell_0)}) \geq 2^{-j_n\alpha} j_{n,\alpha}^{-\varepsilon} \geq j^{-1-\varepsilon}.$$

Gathering these inequalities, we have, for any $j \in \{j_n, \dots, j_{n,\alpha}\}$,

$$f(2^{-(j-\ell_0)}) \geq C' 2^{-j\alpha\varepsilon}. \tag{28}$$

Let us now consider the second case, where $j \in \{j_{n,\alpha}+1, \dots, j_{n+1}-1\}$ for some $n \in \mathbf{N}$. We have

$$f(2^{-(j-\ell_0)}) = \left(2^{j_{n+1}(1-\alpha)} \sum_{\ell=j+2}^{j_{n+1}} 2^{-\ell}\ell^{-\varepsilon} + 2^{-j_{n+1}\alpha} \sum_{\ell=j_{n+1}}^{j_{n+1,\alpha}} \ell^{-\varepsilon}\right)\psi(0).$$

If one remarks that

$$\begin{aligned} f(2^{-(j-\ell_0)}) &\geq C'(2^{j_{n+1}(1-\alpha)}2^{-j}j^{-\varepsilon} + 2^{-j_{n+1}\alpha} j_{n+1,\alpha}^{1-\varepsilon}) \\ &= C'(2^{j_{n+1}(1-\alpha)}2^{-j}j^{-\varepsilon} + 2^{-j_{n+1}\alpha\varepsilon}), \end{aligned}$$

then for any $j_{n,\alpha}+1 \leq j \leq ((1-\alpha)+\alpha\varepsilon)j_{n+1}$, we get

$$f(2^{-(j-\ell_0)}) \geq C' 2^{j\frac{1-\alpha}{(1-\alpha)+\alpha\varepsilon}} 2^{-j} j^{-\varepsilon} = C' 2^{-j\frac{\alpha\varepsilon}{(1-\alpha)+\alpha\varepsilon}} j^{-\varepsilon}, \tag{29}$$

whereas if $((1-\alpha)+\alpha\varepsilon)j_{n+1} \leq j \leq j_{n+1}-1$,

$$f(2^{-(j-\ell_0)}) \geq C' 2^{-j_{n+1}\alpha\varepsilon} \geq C' 2^{-j\frac{\alpha\varepsilon}{(1-\alpha)+\alpha\varepsilon}}. \tag{30}$$

Inequalities (28), (29) and (30) together imply $f \in UI^\beta(\mathbf{R}^d)$. □

## A.2 Necessity of the Logarithmic Correction in the Wavelet Criteria

Let $\varepsilon, \alpha \in (0,1)$, $\beta > 1$ and define $(j_n)_{n\in\mathbf{N}}$ as

$$j_n = [\beta^n],$$

for any $n \in \mathbf{N}$. Let us also define the function $f_{\alpha,\beta,\varepsilon}$ on $\mathbf{R}$ as follows,

$$f_{\alpha,\beta,\varepsilon}(x) = \sum_{n=0}^{\infty} \sum_{j=j_n+1}^{j_{n+1}} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \sin(2^j \pi x). \tag{31}$$

We first give an estimation of the wavelet coefficients $(c_{j,k})$ of $f_{\alpha,\beta,\varepsilon}$.

**Proposition 13** *Assume that the multiresolution analysis is the Meyer multiresolution analysis. Then for $n \geq 1$, any $j \in \{j_n, \dots, j_{n+1}-1\}$ and any $C > 0$,*

$$\sup_{k\in\mathbf{Z}} |c_{j,k}| \leq C \inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j}), \tag{32}$$

*for $n$ sufficiently large.*

*Proof* Let $n \in \mathbf{N}$ and $\ell \in \{j_n, \dots, j_{n+1}-1\}$. By definition of the wavelet coefficients of a bounded function, we have

$$c_{\ell,k} = 2^{\ell} \int_{\mathbf{R}^d} f_{\alpha,\beta,\varepsilon}(x)\psi(2^{\ell}x - k)\,dx.$$

Since the trigonometric series $f_{\alpha,\beta,\varepsilon}$ is uniformly converging on any compact,

$$c_{\ell,k} = 2^{\ell} \sum_{n=0}^{\infty} \sum_{j=j_n+1}^{j_{n+1}} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \int_{\mathbf{R}^d} \sin(2^j\pi x)\psi(2^{\ell}x - k)\,dx,$$

or

$$c_{\ell,k} = \frac{2^{\ell}}{2i} \sum_{n=0}^{\infty} \sum_{j=j_n+1}^{j_{n+1}} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \times \int_{\mathbf{R}^d} (e^{i2^j\pi x} - e^{-i2^j\pi x})\psi(2^{\ell}x - k)\,dx,$$

that is,

$$c_{\ell,k} = \sum_{n=0}^{\infty} \sum_{j=j_n+1}^{j_{n+1}} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \times \frac{e^{i2^{j-\ell}k\pi}\hat{\psi}(2^{j-\ell}k) - e^{-i2^{j-\ell}k\pi}\hat{\psi}(-2^{j-\ell}k)}{2i}. \tag{33}$$

Since the Meyer wavelet belongs to the Schwartz class, its Fourier transform is symmetric and compactly supported with

$$\operatorname{supp}(\hat{\psi}) \subset \left[-\frac{8\pi}{3}, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right],$$

the sum in equality (33) contains at most five terms corresponding to

$$k \in \{\ell - \log_2(k), \ell - \log_2(k) + 1, \ell - \log_2(k) + 2, \ell - \log_2(k) + 3, \ell - \log_2(k) + 4\}.$$

One directly checks that for any $n \in \mathbf{N}$, $j \in \{j_n, \dots, j_{n+1} - 1\}$, this implies inequality (32). □

Let us now prove the uniform irregularity properties of the functions $f_{\alpha,\beta,\varepsilon}$.

**Proposition 14** *For any $\beta > 1$ and any $(\alpha, \varepsilon) \in (0, 1)^2$, $f_{\alpha,\beta,\varepsilon} \in UI^{\alpha}_{1-\varepsilon}(\mathbf{R})$.*

*Proof* Let us remark that it is sufficient to prove that for any $\ell \in \mathbf{N}$,

$$f_{\alpha,\beta,\varepsilon}(2^{-\ell}) \geq 2^{-\alpha\ell}\ell^{1-\varepsilon}. \tag{34}$$

Let $n_0 \in \mathbf{N}$ and $\ell \in \{j_{n_0+1}, \dots, j_{n_0+1}\}$. By definition, we have

$$\begin{aligned} f_{\alpha,\beta,\varepsilon}(2^{-\ell}) &= \sum_{n=0}^{n_0-1} \sum_{j=j_n+1}^{j_{n+1}} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \sin(2^j 2^{-\ell}\pi) \\ &\quad + \sum_{j=j_{n_0}+1}^{\ell-1} \frac{\inf(2^{-j_{n_0}\alpha}, 2^{j_{n_0+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} \sin(2^j\pi 2^{-\ell}). \end{aligned}$$

The classical inequality $\sin(x) \geq (2/\pi)x$ valid for any $x \in [0, \pi/2]$ leads to the following inequality if $j_{n_0} + 1 \leq \ell \leq j_{n_0+1}$,

$$\begin{aligned} f_{\alpha,\beta,\varepsilon}(2^{-\ell}) &\geq \sum_{j=j_{n_0}+1}^{\ell-1} \frac{\inf(2^{-j_n\alpha}, 2^{j_{n+1}(1-\alpha)}2^{-j})}{j^{\varepsilon}} 2^{j-\ell} \\ &\geq 2.2^{-\ell} \inf(2^{-j_{n_0}\alpha}2^{\ell}\ell^{1-\varepsilon}, \ell^{1-\varepsilon}2^{j_{n_0+1}(1-\alpha)}) \\ &\geq 2\inf(\ell^{1-\varepsilon}2^{-j_{n_0}\alpha}, \ell^{1-\varepsilon}2^{-\ell}2^{j_{n_0+1}(1-\alpha)}). \end{aligned}$$

Let $t \in (1, \beta)$ such that $\ell = tj_{n_0}$, that is $j_{n_0} = \ell/t$. We get

$$f_{\alpha,\beta,\varepsilon}(2^{-\ell}) \geq 2\inf(\ell^{1-\varepsilon}2^{-\ell\frac{\alpha}{t}}, \ell^{1-\varepsilon}2^{-\ell(1-\frac{\beta\ell}{t}+\frac{\alpha\beta\ell}{t})}).$$

Since

$$\sup_{t\in[1,\beta]} \max(\alpha/t, 1 - \beta\ell/t + \alpha\beta\ell/t) \leq \alpha,$$

inequality (34) is satisfied for any $\ell \in \mathbf{N}$. □

Propositions 13 and 14 together imply the following proposition.

**Proposition 15** *For any $(\alpha, \varepsilon, \beta) \in (0, 1)^2 \times (1, +\infty)$, the functions $f_{\alpha,\beta,\varepsilon}$ defined by the relation (31) are uniformly Hölderian, satisfy (8) and belong to $UI^{\alpha}_{1-\varepsilon}(\mathbf{R})$.*

## References


1. Adler, R.J.: The Geometry of Random Field. Wiley, New York (1981)
2. Berman, S.M.: Gaussian sample functions: uniform dimension and Hölder conditions nowhere. Nagoya Math. J. **46**, 63–86 (1972)

3. Berman, S.M.: Local nondeterminism and local times of Gaussian processes. Indiana Univ. Math. J. **23**(1), 69–86 (1973)
4. Bousch, T., Heurteaux, Y.: On oscillations of Weierstrass-type functions. Manuscript (1999)
5. Bousch, T., Heurteaux, Y.: Caloric measure on domains bounded by Weierstrass type graphs. Ann. Acad. Sci. Fenn. Math. **25**(2), 501–522 (2000)
6. Bardet, J.M., Bertrand, P.: Definition, properties and wavelet analysis of multiscale fractional Brownian motion. Fractals **15**(1), 73–87 (2007)
7. Benassi, A., Bertrand, P., Cohen, S., Istas, J.: Identification of the Hurst index of a step fractional Brownian motion. Stat. Inference Stoch. Process. **3**(1), 101–111 (2000)
8. Clausel, M.: Lacunary fractional Brownian motion. To appear in ESAIMPS (2010)
9. Clausel, M., Nicolay, S.: Wavelet techniques for pointwise anti-Hölderian irregularity. Constr. Approx. **33**(1), 41–75 (2011)
10. Clausel, M., Nicolay, S.: Some prevalent results about strongly monoHölder functions. Nonlinearity **23**(9), 2101–2116 (2010)
11. Coeurjolly, J.F.: Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. Stat. Inference Stoch. Process. **4**, 199–227 (2001)
12. Coeurjolly, J.F.: Hurst exponent estimation of locally self-similar Gaussian processes using sample quantiles. Ann. Stat. **36**, 1404–1434 (2008)
13. Daubechies, I.: Orthonormal bases of compactly supported wavelets. Commun. Pure Appl. Math. **41**, 909–996 (1988)
14. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
15. Flandrin, P.: Wavelet analysis of fractional Brownian motion. IEEE Trans. Inf. Theory **38**(2), 910–917 (1992)
16. Heurteaux, Y.: Weierstrass function with random phases. Trans. Am. Math. Soc. **355**(8), 3065–3077 (2003)
17. Heurteaux, Y.: Weierstrass function in Zygmund's class. Proc. Am. Math. Soc. **133**(9), 2711–2720 (2005)
18. Istas, J., Lang, G.: Quadratic variations and estimation of the Hölder index of a Gaussian process. Ann. Inst. Poincaré Probab. Stat. **33**, 407–436 (1997)
19. Jaffard, S.: Construction et propriétés des bases d'ondelettes. Remarques sur la controlabilité exacte. Ph.D. thesis (1989)
20. Jaffard, S.: Wavelet techniques in multifractal analysis, fractal geometry and applications. In: Proc. Symp. Pure Math. AMS, Providence (2004)
21. Jaffard, S., Meyer, Y.: Wavelets methods for pointwise regularity and local oscillations of functions. Mem. Am. Math. Soc. **123**, 587 (1996)
22. Kahane, J.P.: Geza Freud and lacunary Fourier series. J. Approx. Theory **46**(1), 51–57 (1986)
23. Ken, J.T., Wood, A.T.A.: Estimating the fractal dimension of a locally self-similar Gaussian process using increments. J. R. Stat. Soc. Ser. B **59**, 679–700 (1997)
24. Khintchine, A.: Ein Satz der Wahrscheinlichkeitsrechnung. Fundam. Math. **6**, 9–20 (1924)
25. Krantz, S.G.: Lipschitz spaces, smoothness of functions, and approximation theory. Expo. Math. **3**, 193–260 (1983)
26. Mallat, S.: A Wavelet Tour of Signal Processing. Academic Press, San Diego (1998)
27. Meyer, Y.: Ondelettes et Opérateurs. Hermann, Paris (1990)
28. Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Chapman & Hall, London (1994)
29. Stoev, S., Taqqu, M., Park, C., Michailidis, G., Marron, J.S.: LASS: a tool for the local analysis of self-similarity. Comput. Stat. Data Anal. **50**, 2447–2471 (2006)
30. Weiss, M.: On the law of iterated logarithm for lacunary trigonometric series. Trans. Am. Math. Soc. **91**, 444–469 (1959)
31. Xiao, Y.: Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. Probab. Theory Relat. Fields **109**, 129–157 (1997)
32. Xiao, Y.: Properties of local nondeterminism of Gaussian and stable random fields and their applications. Ann. Fac. Sci. Toulouse Math. **XV**, 157–193 (2005)