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"ON THE APPROXIMATION OF INCOMPRESSIBLE SOLIDS IN THE DISPLACEMENT METHOD"

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ON THE APPROXIMATION OF INCOMPRESSIBLE MATERIALS IN THE DISPLACEMENT METHOD

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ABSTRACT

A mathematical analysis of the numerical approximation to incompressibility with nearly incompressible displacement finite elements is presented. It explains why, as observed by many authors, convergence to the incompressible solution is not necessarily obtained when Poisson's ratio is increased up to 0.5. It also allows predicting under which conditions convergence of the nearly incompressible approach is guaranteed.

1. Introduction

It is obvious that when incompressibility is assumed, the stiffness matrix of a displacement finite element model is no longer definite. Thus, the analysis of incompressible structures cannot be performed exactly using the displacement method.

From an engineer point of view it seems reasonable to expect that to a continuous change in material behaviour is associated a continuous change in the solution. In our case, the nearly incompressible solution can thus be expected to be not very different from the exact incompressible one.

Some numerical experiments with the nearly incompressible approach, however, have lead to somewhat discouraging results, specially in plane strain problems. The failures of the method have generally been attributed to the fact that the stiffness matrix become singular as ν tends to 0.5.

A short presentation of the displacement method applied to nearly incompressible materials is given first, and a detailed analysis of the inaccuracy so introduced follows which shows that ill-conditioning of the stiffness matrix cannot be responsible for the discrepancies observed on the results. The proof is next given that the lack of convergence is essentially due to an improper discretization of the displacement field. This conclusion is enhanced by a properly chosen example.

2. Presentation of the method

The strain energy can be written in the general form

$$W(\varepsilon) = G \{ \varepsilon_{ij} \varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{\ell\ell} \varepsilon_{ii} \} , \qquad (1)$$

where $\varepsilon_{ij} = \frac{1}{2} (D_i u_j + D_j u_i)$, ν is Poisson's ratio and G Coulomb's modulus Of course, it is impossible to set ν equal to 0.5, since the stiffness matrix would not be definite. But it seems logical to approximate the problem by setting $\nu = 0.5 - \varepsilon$, ε being an arbitrary small position constant. The problem can then be solved numerically, and one may hope that the results will give a good approximation to the incompressible solution.

The exact variational problem for an incompressible structure is

$$\int_V G\, \varepsilon_{ij}\, \varepsilon_{ij}\, dV - \int_{S_2} \overline{t}_i\, u_i\, dS \qquad \min_{u \in I} \quad , \tag{2}$$

where I is the subspace of all admissible displacements verifying the incompressibility equation $\mathrm{div}\, \vec{u} = 0$.

The approximate problem that we will effectively solve is

$$\int_V G\, \varepsilon_{ij}\, \varepsilon_{ij}\, dV + \frac{\nu}{1-2\nu} \int_V G\, (\mathrm{div}\, \vec{u}\,)^2\, dV - \int_{S_2} \overline{t}_i\, u_i\, dS \quad \min_{u \in H} \tag{3}$$

where H denotes the whole space of admissible displacements.
Comparing the two functionals shows that they only differ through the additional term

$$\frac{\nu}{1-2\nu} \int_V G\, (\mathrm{div}\, \vec{u}\,)^2\, dV \ . \tag{4}$$

(4) is precisely the norm of the function that must be zero multiplied by the factor $\frac{\nu}{1-2\nu}$. As the latter becomes infinite for $\nu = 0.5$, the additional term (4) can thus be regarded as a <u>penalty functional</u> [3] , [4].

However, some authors have used this method and obtained unexpectedly poor results. These inaccuracies have generally been attributed to the fact that the stiffness matrix becomes nearly singular. In the following example, we shall demonstrate that the problem is essentially of an other nature.

3. Analysis of an example

Let us consider the plane strain problem of a pressurized thick-walled cylinder made of an incompressible material contained in a thick elastic case (fig. 1). This problem is idealized by triangular torus elements, as represented on fig. 2. The elastic characteristics of the cylinder and its container are

$$E = 23.0769 \text{ kg/mm}^2 , \quad E_c = 2.1 \times 10^4 \text{ kg/mm}^2 , \quad \nu_c = 0.3.$$

The same problem has been analyzed with two types of elements. The first type is a classical displacement element where the displacements are discretized as polynomials of the coordinates [7] . The second type is a mixed element following Herrmann's variational principle [6] . In Herrmann's formulation the variable are thus the displacements and the mean pressure.It allows for an exact treatment of incompressible structures and will thus be useful for a comparison.

The problem has been analyzed successively with the following values of Poisson's ratio: ν = 0.49, 0.499, 0.4999, 0.49999, 0.499999, and using linear, quadratic and cubic elements. For each analysis we consider the following quantities : the total energy, the circumferential force in the shell, and the ratio between the largest error on the reactions and the mean load. This ratio is a good measure of numerical conditioning of the system. When lower than 10^{-8} it is assumed equal to 0. The results are compared to those of an analysis made with Herrmann's incompressible elements of third degree.

Table 1. Variation of potential energy (X 10^{-5})

degree	element	ν= 0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999	ν=0.5
1	H	3.358223	2.560309	2.460611	2.450394	2.449370	–
	C	2.377885	0.647630	0.085141	0.008806	0.000884	–
2	H	3.433747	2.601914	2.498709	2.488142	2.487082	–
	C	3.352144	2.330433	1.714860	0.678295	0.098106	–
3	H	3.438266	2.604381	2.500965	2.490377	2.489316	2.489193
	C	3.432883	2.582317	2.430949	2.286522	1.534954	–

Table 2. Variation of circumferencial force $N_{\theta\theta}$

degree	element	ν=0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999	ν=0.5
1	H	231.6	275.1	280.5	281.1	281.1	–
	C	167.1	70.90	9.885	1.029	0.1033	–
2	H	236.5	279.6	284.8	285.4	285.4	–
	C	231.2	250.6	195.5	77.81	11.26	–
3	H	236.8	279.3	285.1	285.6	285.7	285.7
	C	236.5	277.5	277.1	262.3	176.2	–

Table 3. Relative error in the reactions (Element C)

degree	ν=0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999
1	0. (*)	0.	0.	0.	0.
2	0.	0.	0.	0.	0.
3	0.	0.	0.	2.10^{-8}	1.10^{-6}

(*) 0. means $<10^{-8}$

The results obtained from both approaches are collected in tables 1, 2 and 3. The evolution of the potential energy is represented on figure 3, and the variation of the circumferential force on figure 4. The displacement and Herrmann's formulations are noted "C" and "H", respectively.

It can be seen that Herrmann's solution behaves continuously as $\nu \rightarrow 0.5$, and that even the lower degree approximation give fairly accurate results. It singularly contrasts with the behaviour of the various displacement approaches. The latter provide with accurate results when Poisson's ratio remains sufficiently small. When the structure tends to the incompressible one, the three solutions seem to converge to zero.

It is important to point out this evolution is such that the solution is better for higher degrees. But this contradicts the idea that the discrepancies are related to the ill-conditioning of the stiffness matrix. Indeed in this case, the lower the system is, i.e. for the first degree, the better the numerical results would be. In addition, it can be seen that the relative errors on the reactions are always very little, indicating thus that numerical inversion is not responsible for the discrepancy observed.

Thus it is obvious that the problem is not essentially a numerical problem and that another phenomenon must exist which explains this tremendous divergence. In the next section the polynomial discretization of the displacement field will be identified as the main source of discrepancy.

4. Analysis of the problem

When there are no kinematical modes, the admissible displacements form a Hilbert space H on which the following scalar product can be defined :

$$(\vec{u},\vec{v}) = \int_V 2G\, \varepsilon_{ij}(\vec{u})\, \varepsilon_{ij}(\vec{v})\, dV \qquad (5)$$

The displacement modes which verify the incompressibility equation

$$\operatorname{div} \vec{u} = 0 \tag{6}$$

form a linear subspace I. Of course, if $\vec{u} \in I$, the quadratic functional

$$b(\vec{u},\vec{u}) = \int_V 2G(\operatorname{div} \vec{u})^2 \, dV \tag{7}$$

vanishes.

Let us next consider a linear subspace $S \subset H$. The quantity

$$e(S) = \min_{u \in S} \frac{\int_V 2G(\operatorname{div} \vec{u})^2 dV}{||\bar{u}||^2} = \min_{\substack{u \in S \\ ||\bar{u}|| = 1}} b(\vec{u},\vec{u}) \tag{8}$$

has the following property

$$\left\{ \begin{array}{l} e(S) > 0 \quad \text{if and only if} \quad S \cap I = \{0\} \\ e(S) = 0 \quad \text{if } S \cap I \neq \{0\} \end{array} \right\} \tag{9}$$

The set of linear fucntionals $f(\vec{u})$ which are continuous and verify the inequality

$$||f||_{S^*} = \max_{\vec{u} \in S} \frac{|f(\vec{u})|}{||\bar{u}||} < \infty \tag{10}$$

is called dual of S [8] . We shall denote it S^* . The quantity $||f||_{S^*}$ is the norm of f in S^*. Of course, if S and T are two linear spaces and if $S \subset T$, the norms $||f||_{S^*}$ and $||f||_{T^*}$ verify the inequality

$$||f||_{S^*} \leq ||f||_{T^*} \tag{11}$$

For simplicity, we suppose in the following that the kinematical conditions are homogeneous. The extension to other boundary conditions would be straightforward.

The exact incompressible problem (2) is of the form

$$\frac{1}{2}\,||\vec{u}||^2 - f(\vec{u}) \qquad \min_{\vec{u}\in I}$$

where $\frac{1}{2}\,||\vec{u}||^2$ is the strain energy, and the linear functional $f(\vec{u})$, the potential energy of external loads.

If we decompose the displacement vector in the following form

$$\vec{u} = \lambda\,\vec{u}_1 \quad , \tag{12}$$

where

$$\lambda = ||\vec{u}||$$

$$\vec{u}_1 = \frac{\vec{u}}{||\vec{u}||} \quad ,$$

the variational problems (2) may be rewritten in the form

$$\left\{ \left| \frac{1}{2}\,\lambda^2 - \lambda f(\vec{u}_1) \right| \quad \min_{\lambda} \right\} \qquad \min_{\vec{u}_1} \tag{13}$$

Let us first minimise on $\vec{u}_1$:

$$f(\vec{u}_1) \quad \max \quad \begin{cases} \vec{u}_1 \in I \\ ||\vec{u}_1|| = 1 \end{cases} \tag{14}$$

By virtue of (10), the solution $\vec{u}_1$ is such that

$$f(\vec{u}_1) = ||f||_{I^*} \tag{15}$$

Then, the minimisation on λ shows that the solution $\vec{u}$ verify the following relations

$$\left\{ \begin{array}{l} ||\vec{u}|| = ||f||_{I^*} \\ ||\vec{u}||^2 = f(u) = ||f||^2_{I^*} \end{array} \right\} \tag{16}$$

The finite element method consists to select a finite dimensional subspace $S \subset H$ and to minimize the functional in this subspace. Any displacement vector lying in S may be decomposed in the finite element basis

$$\vec{u} = \sum_{i=1}^{N} q_i \vec{u}_i \quad , \tag{17}$$

where N is the dimension of S. It is easy to extract the linear combinations of $\vec{u}_i$ which lie in I : they are the solution of the system

$$\sum_{i=1}^{N} A_{ij} q_j = 0, \tag{18}$$

where

$$A_{ij} = \int_V \mathrm{div}(\vec{u}_i) \, \mathrm{div}(\vec{u}_j) \quad dV \tag{19}$$

The linear closure S_I of these combinations is naturally a linear subspace of S. If we orthogonalize the basis $\{\vec{u}_1, \ldots, \vec{u}_N\}$ to the basis of S_I we obtain the basis of an other subspace S_c orthogonal to S_I.

Any displacement of S can thus be decomposed in the form

$$\vec{u} = \vec{u}_I + \vec{u}_c \quad , \tag{20}$$

where $\vec{u}_I \in S_I$ and $\vec{u}_c \in S_c$. |10|

The approximate problem (3) is

$$\frac{1}{2}||\vec{u}||^2 + \frac{\frac{\nu}{2}}{1-2\nu} b(\vec{u},\vec{u}) - f(\vec{u}) \quad \min_{\vec{u} \in S} \tag{22}$$

The decomposition (20) gives

$$\frac{1}{2}||\vec{u}_I||^2 + \frac{1}{2}||\vec{u}_c||^2 + \frac{\frac{\nu}{2}}{1-2\nu} b(\vec{u}_c,\vec{u}_c) - f(\vec{u}_I) - f(\vec{u}_c) \quad \min_{\substack{\vec{u}_I \in S_I \\ \vec{u}_c \in S_c}} \tag{23}$$

This variational problem can thus be split in the following ones

$$\frac{1}{2}||\vec{u}_I||^2 - f(\vec{u}_I) \quad \min_{u_I \in S_I} \tag{24}$$

and

$$\frac{1}{2}||\vec{u}_c||^2 + \frac{\frac{\nu}{2}}{1-2\nu} b(\vec{u}_c,\vec{u}_c) - f(\vec{u}_c) \quad \min_{u_c \in S_c} \tag{25}$$

The first problem is the discretization of the exact incompressible problem (2). Following (16), at the solution, we have

$$|| \vec{u}_I|| = ||f||_{S_I^*} \tag{26}$$

According to Clapeyron's theorem , the solution of the second problem is

$$||\vec{u}_c||^2 + \frac{\nu}{1-2\nu} \; b\,(\vec{u}_c,\vec{u}_c) = f\,(\vec{u}_c)$$

But

$$|f(\vec{u}_c)| \leq ||f||_{S_c^*} \quad ||\vec{u}||$$

$$b(\vec{u}_c,\vec{u}_c) \geq ||\vec{u}_c||^2 \;\; e(S_c)$$

Thus, the following inequality holds for the compressible part u_c of the displacement u_c

$$||\vec{u}_c||^2 \;\; \left|1 + \frac{\nu}{1-2\nu} \;\; e(S_c)\right| \;\; \leq \;\; ||f||_{S_c^*} \quad ||\vec{u}_c||$$

and

$$||\vec{u}_c|| \leq \frac{||f||_{S_e^*}}{\left|1+ \frac{\nu}{1-2\nu}\, e(S_c)\right|} \tag{27}$$

Since $\frac{\nu}{1-2\nu}$ when $\nu \to 0.5$,
the compressible term u_c converges to zero when $\nu \to 0.5$. The factor $e(S_c)$ shows that this convergence is particularly fast when the discretization is poor.

This is precisely where the difficulty lies. If the subspace S is such that $S_I = 0$, the finite element solution is contained in S_c. <u>Then, according to (27) it must converge to zero when $\nu \to 0.5$</u>
It is easy to check that it was the case in the preceding example. In an axisymmetric plane strain problem, the incompressibility condition takes the form

$$\frac{1}{r} \; \frac{d}{dr} \; (ru) = 0 \tag{28}$$

The solutions of which obviously form the linear one-dimensional subspace.

$$I = \{u \mid u = \frac{\alpha}{r} , \alpha = \text{constant} .\}$$

As the finite element subspaces that we have considered do not contain the functions $\frac{\alpha}{r}$, the solution will then converge to zero.
The lower the degree is, the faster the convergence to zero will be, since in this case $e(S_c)$ is the larger. It is exactly the phenomenon that we have observed in the numerical application above.

Let us still make the following remark about the convergence of the potential energy. Since

$$||\vec{u}_I|| = ||f||_{S_I^*}$$

and

$$||\vec{u}_c|| \leq \frac{||f||_{S_c^*}}{\left|1 + \frac{\nu}{1-2\nu} e(S_c)\right|}$$

$$f(\overline{u}) = f(\overline{u}_I) + f(\overline{u}_c) \leq ||f||^2_{S_I^*} + ||f||_{S_c^*} \; ||\overline{u}_c|| \qquad (30)$$

$$\leq ||f||_{S_I^*} + \frac{||f||^2_{S_c^*}}{\left|1+ \frac{\nu}{1-2\nu} e(S_c)\right|}$$

It appears that $f(\overline{u}) \to ||f||_{S_I^*}$ when $\nu \to 0.5$.

For $S_I \subset I$,

$$||f||_{S_I^*} \leq ||f||_{I^*} \qquad (31)$$

i.e. the potential energy converges to a lower bound to the exact value. But the solution of the approximated problem does not necessarily give a lower bound of the energy since the second term of (30) can happen to be larger than the difference $||f||_{I^*} - ||f||_{S_I^*}$.

To conclude this section, the approximated functional (2) may be used, but the convergence of the results depend on the representation of the incompressible subspace by its finite element discretization. If no incompressible function lies in the finite element model, the solution will converge to zero, but if the incompressible subspace is sufficiently represented, a satisfactory displacement finite element solution can then be expected.

5. An example of good convergence

As an illustration of the theory, we consider next the case of a cylinder whose lateral surface is clamped. At one end, the "inlet end" , a pressure of 100 kg/cm^2 is applied. The "outlet end" is free. All radial displacements are fixed. It can easily be shown that this problem is identical to Stokes flow in an infinite pipe [5] . Its exact analytical solution gives

$$u_{max} = \frac{\Delta p . D^2}{16\ Gl} ,$$

where u_{max} is the displacement on the axis, $\Delta p/l$ the pressure gradient, D the diameter, and G Coulomb's modulus.

Similarly we obtain for the potential energy

$$P = p . \overline{u} . \pi . r^2$$

where $\bar{u} = u_{max} / 2$.

With the following numerical values of material constant, geometrical characteristics and applied load

$\ell = 40.$, $D=20.$, $E=10.$; $p = 100.$,

the solution is thus

$u_{max} = 18.75$

$P = 2.945243 \times 10^5$.

This structure has been analysed for the following values of Poisson's ratio.
$= 0.49$, 0.499, 0.4999, 0.49999, 0.499999
with second degree elements. The results are collected in the following table.

Table 4

	ν=0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999
u_{max} (inlet)	26.19736	19.52696	18.82769	18.75777	18.75078
u_{max} (outlet)	14.76267	18.32613	18.70767	18.74577	18.74958
Potential energy ($\times 10^{-5}$)	4.553300	3.128204	2.963807	2.947102	2.945429

These results are plotted on figures 6 and 7, which emphasize the very good convergence to the incompressible solution.
For compressible structures, u_{max} (outlet) $\neq$ u_{max} (inlet) but for incompressible structures their value must be the same.
This result is restored with a good precision for ν= 0.4999

Conclusion

The approximation of incompressible structures by displacement nearly incompressible elements may be regarded as a penalty method. The results depend essentially on the representation of the incompressible subspace by the finite element model. If there are no incompressible modes in the idealisation the solution converges to zero as Poisson's ratio tends to 0.5. This phenomenon is the main cause of the difficulties encountered in many experiments. If the incompressible subspace is correctly represented, however, it is possible to obtain fairly accurate results. This theoretical conclusion is of great practical interest because it justifies the use of displacement models in many incompressible structures.

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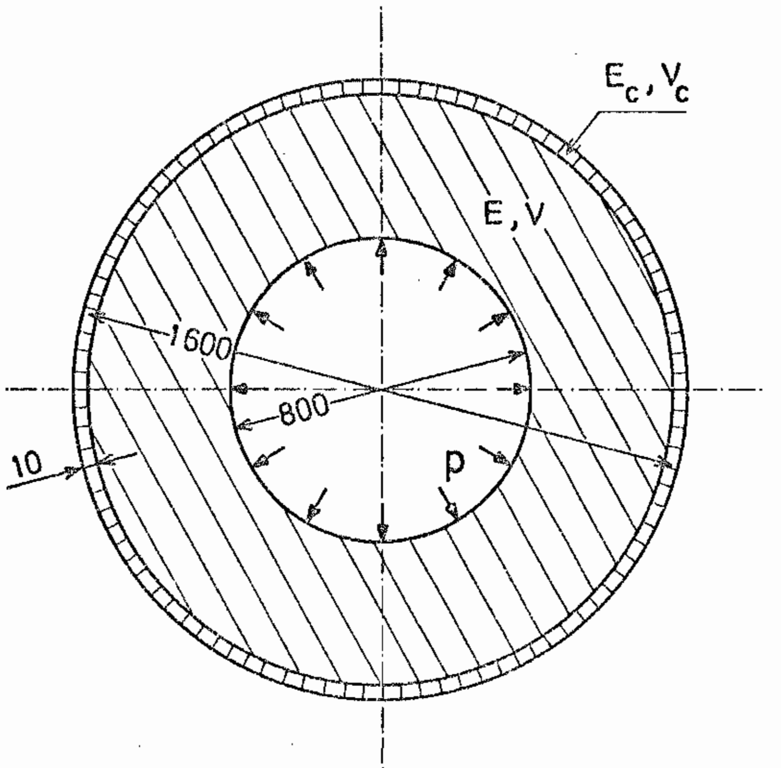


FIG. 1 THICK-WALLED CYLINDER IN A THIN CASE

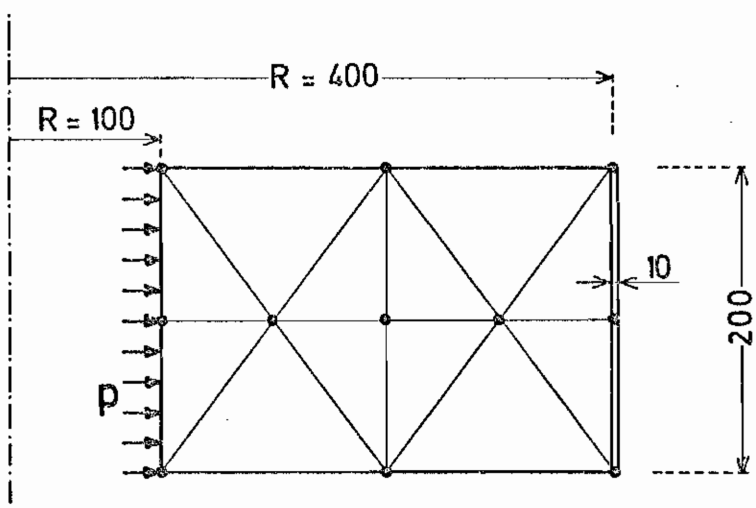


FIG. 2 IDEALIZATION BY TRIANGULAR TORUS ELEMENTS

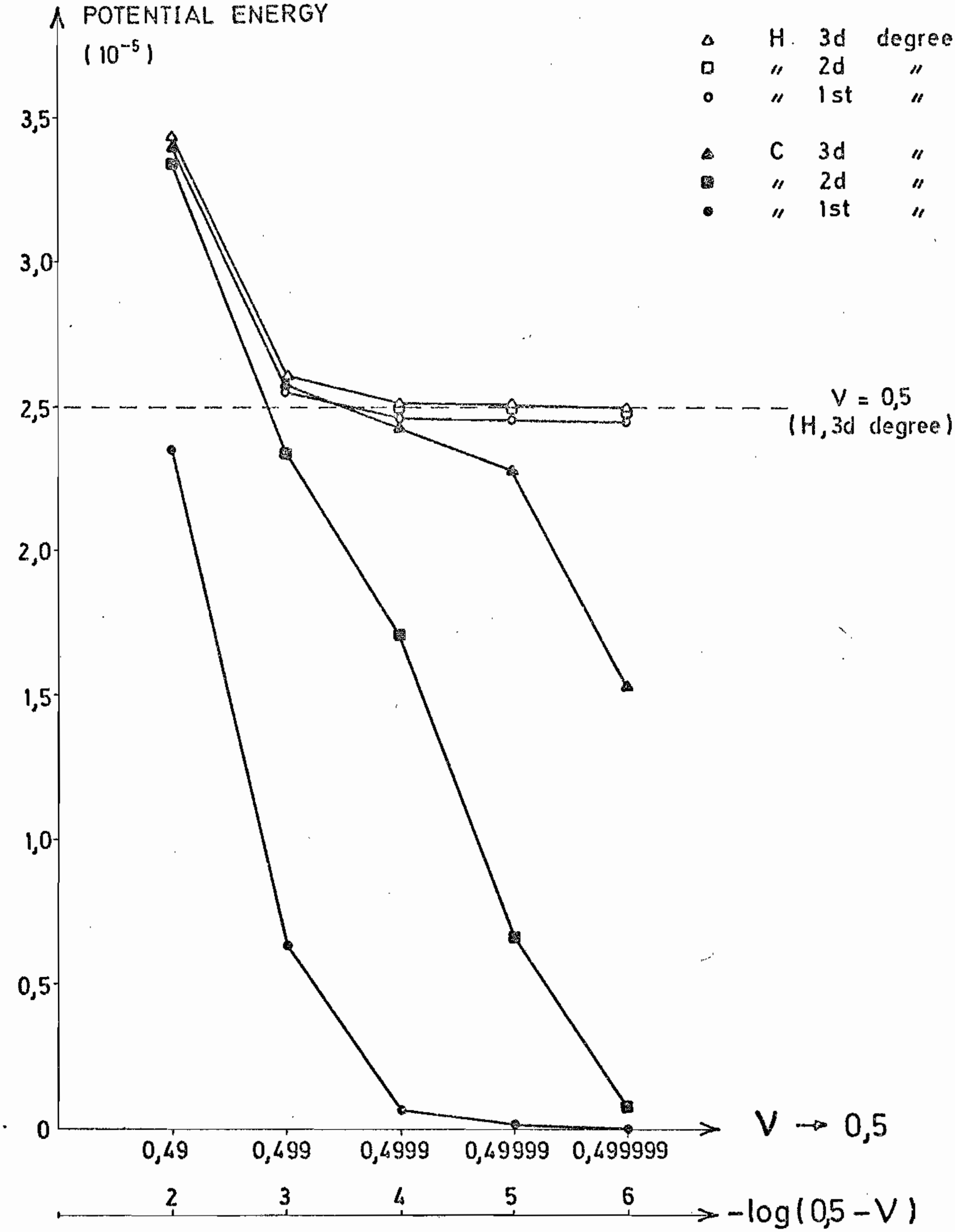


FIG. 3 POTENTIAL ENERGY

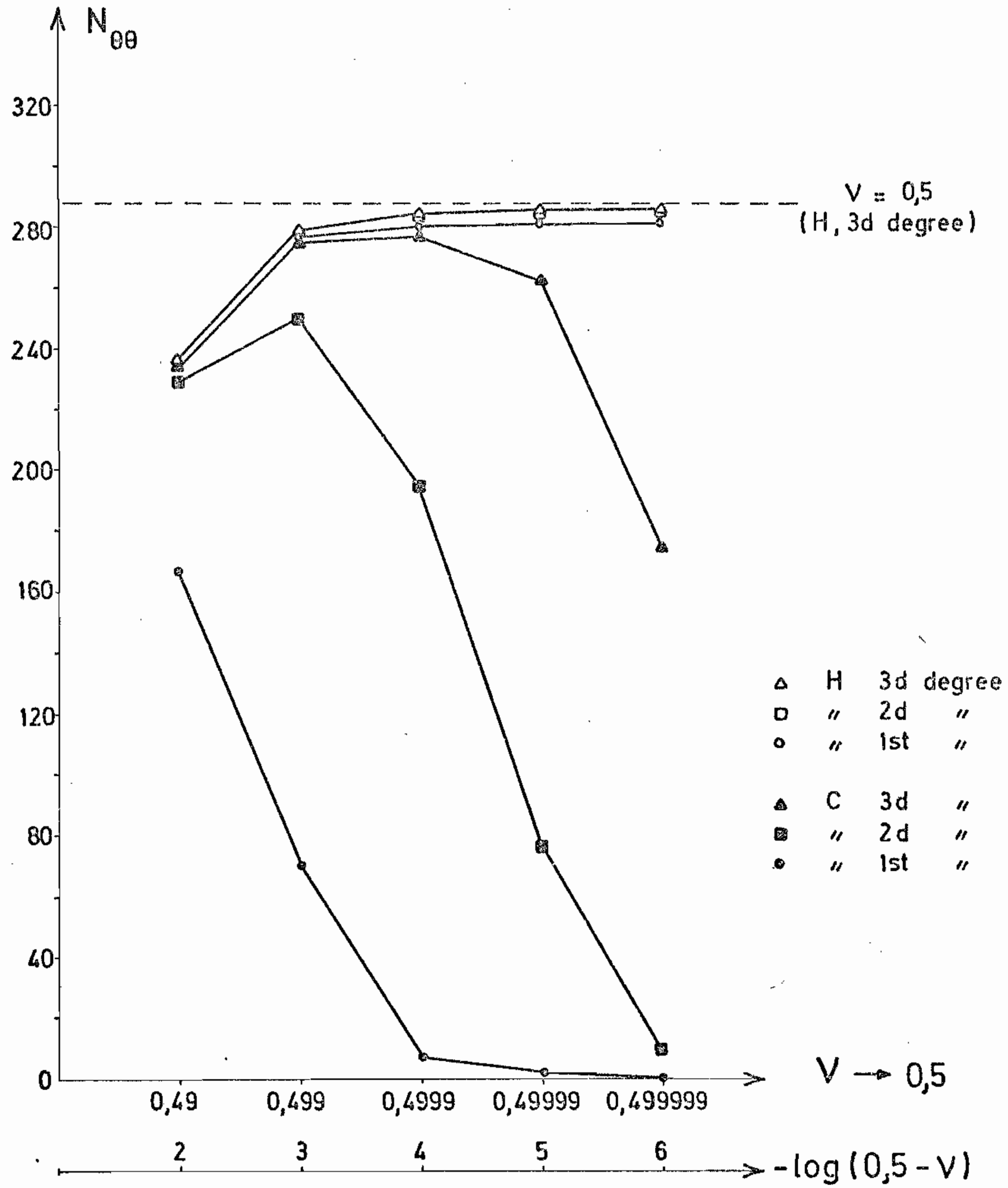


FIG. 4 CIRCUMFERENCIAL FORCE IN THE SHELL

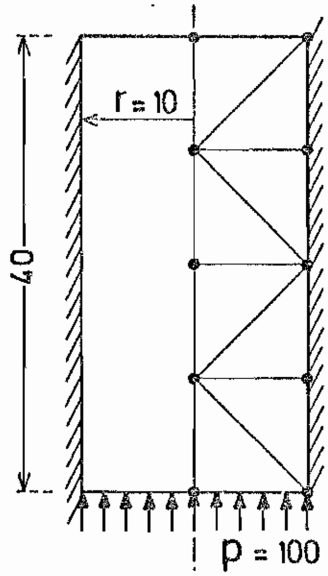
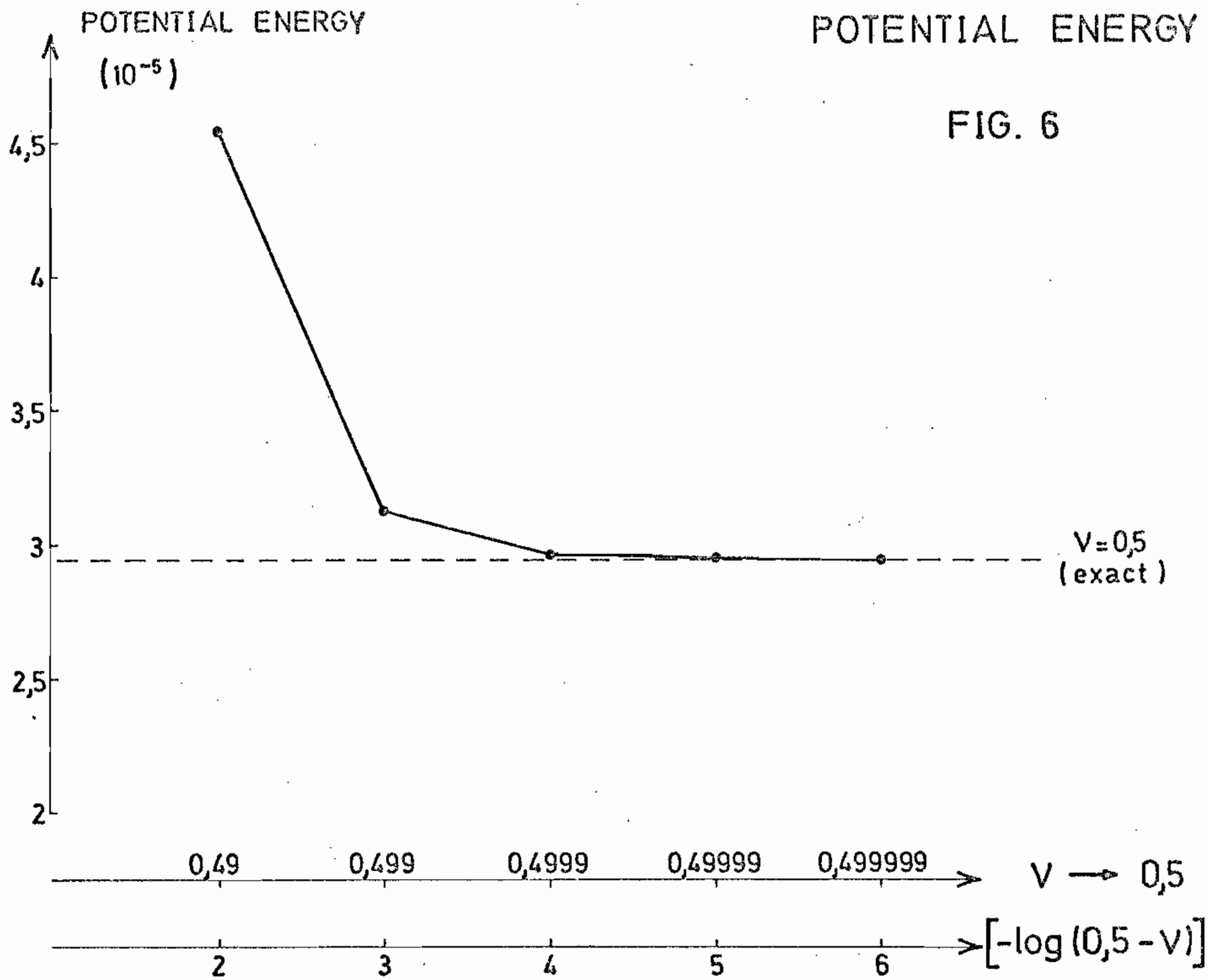


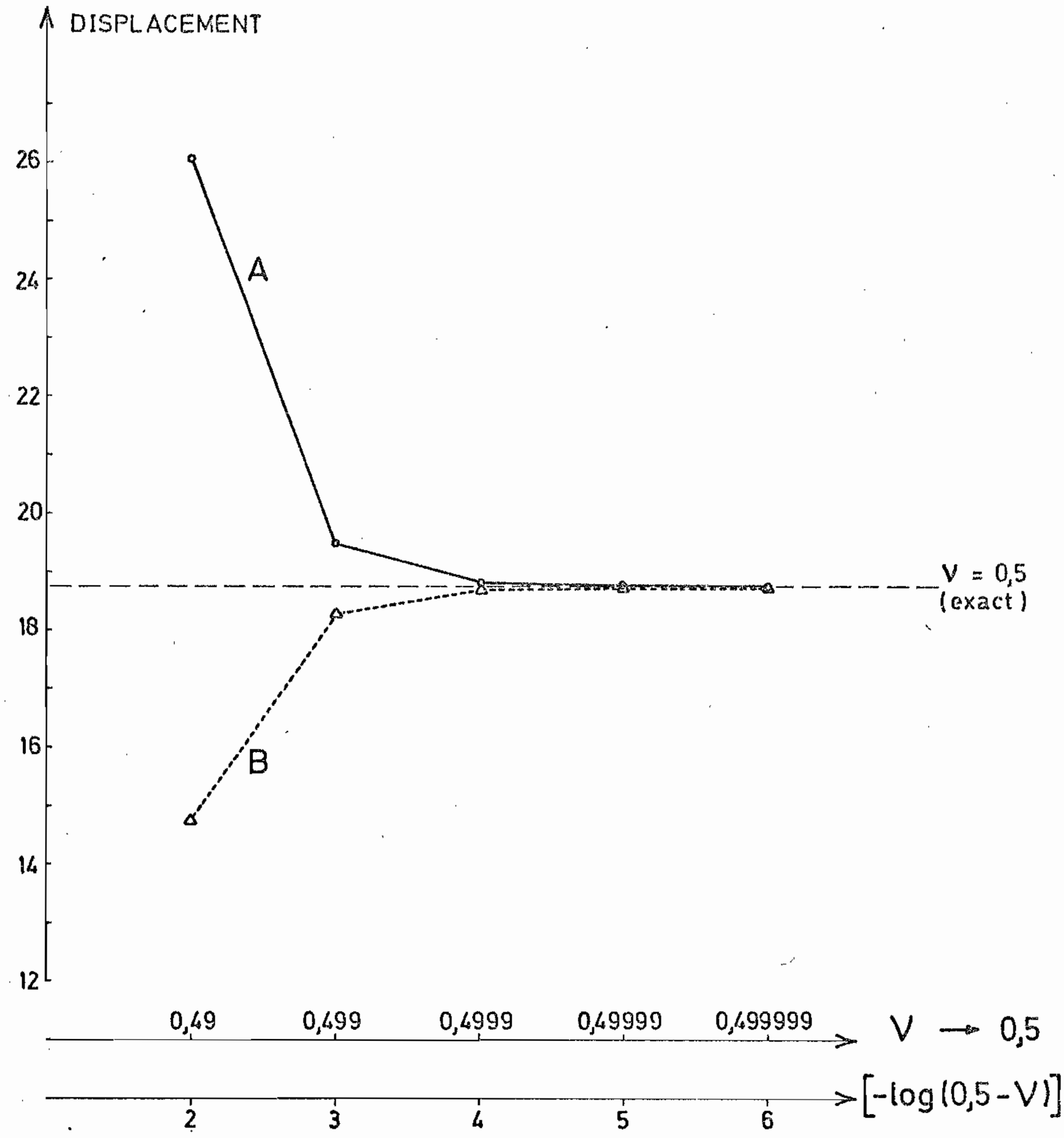
FIG. 5

AXISYMMETRICAL CYLINDER



POTENTIAL ENERGY

FIG. 6



DISPLACEMENT at r = 0 SECTION A and B

FIG. 7