ADAPTIVE X-TRACKING FOR LINEAR SYSTEMS WITH HIGHER RELATIVE DEGREE -THE CONTINUOUS ADAPTATION CASE

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Abstract

This paper presents a simple adaptive controller which universally achieves so-called X-tracking for linear systems where only little structural information about the system to be controlled is needed. The paper extends previous results to the case of systems with higher relative degree. Stability and convergence of the adaptation is proven for tracking arbitrary but sufficiently smooth reference trajectories. The design of the controller is very simple and intuitive and only few parameters have to be tuned. The robustness is increased by the introduction of a dead-zone in the adaptation, whose width λ can be chosen by the user.

In this paper a continuous adaptation law is used as opposed to the discrete law suggested in earlier papers. There are several advantages in using a continuous adaptation: Besides displaying a simpler structure the necessary gain to achieve the control goal will also be significantly lower in general. To demonstrate the performance the controller is applied to the model of a ball and plate experiment.

1 Introduction

A popular method for the robust stabilization of control systems is adaptive control. On the one hand, online identification techniques are used for tuning the controller (see (Åst95; Nar91) for a survey). In *non-identifier-based adaptive control,* **on** the other hand, the controller is directly tuned without estimating the parameters of the plant, usually by increasing it as long as the control objective has not yet been achieved (see (Ilc91) for a survey of earlier works). The first, rather complicated non-identifier-based controller was proposed in 1978 by Feuer and Morse (Feu78). In the mid-80's several authors improved and simplified the adaptation and the controller structure, see (Ilc91) for a list of references. These controllers achieve stabilization via adapting a gain continuously or in a step-wise manner. The latter has the drawback there is that the height of the steps has to grows exponentially. Usually, the gain adaptation reduces to increasing the gain as long as the control objectives, for example stabilization, have not been achieved.

To increase the robustness especially when output noise is present, a dead-zone (of width λ) in the gain adaptation is introduced in (Mil91; Ilc94). This is usually called X-stabilization or X-tracking as the objective is to control the output or the tracking error no longer to zero but to a X-neighborhood of zero. Thus, an output error of amplitude smaller than the width of the dead-zone does not increase the adaptation parameter. While in (Ilc94) a continuous adaptation is used for systems of relative degree one, (Mil91) and later (Bul99b) who are dealing with the higher relative degree case need an adaptation parameter that is increasing in a step-wise manner.

Our contribution is to propose a simple adaptation scheme allowing X-tracking with a continuously increasing adaptation parameter for any minimum-phase linear system with known relative degree while keeping the simplicity of the controller. This leads to a simple, robust controller that does not need the discontinuities in the adaptation parameter and the tracking error is guaranteed to converge to the interval $[0, \lambda]$. Another approach for treating the higher relative degree case with a continuous adaptation has recently been proposed in (Ye99), but due to the fact that there the X-tracking controller is derived via backstepping, the controller design and the resulting controller are more complicated.

To demonstrate the applicability of the X-tracker the controller is applied to the model of a ball and plate laboratory setup.

The paper is organized as follows. After stating the system class and explaining the structure of the controller in Section 2, the theory and an outline of the proof are presented in Section 3. Section 4 presents the example.

2 Preliminaries

System class

We consider linear systems with known relative degree r > 1, having one input and one output and stable zero-dynamics, therefore being stabilizable and detectable. They can be described by the differential equation

$$\dot{x}(t) = Ax(t) + bu(t), \quad A \in \mathbb{R}^{n\times n}, \tag{1a}$$

$$y(t) = cx(t), \qquad z(t),\ b, c^T \in \mathbb{R}^n \tag{1b}$$

with

$$cA^i b = 0 \qquad \text{for } i = 0, \ldots, r-2 \tag{2a}$$

$$CA^{r-1} b = g \geq \bar{g} > 0, \tag{2b}$$

$$\det \begin{bmatrix} sI_n - A & b \\ c & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}}_+ \tag{2c}$$

where (2a), (2b) are the relative degree conditions and (2c) guarantees minimum phaseness.

Objective

The control objective is to track a reference signal $y_{ref}(\cdot)$ asymptotically while tolerating a tracking error smaller than a user-defined λ. All states should remain bounded, i.e. $x \in L$,. $y_{ref}(\cdot)$ is in $W^{r,\infty}$, the set of all bounded functions that are absolutely continuous on compact subintervals and whose r first derivatives are essentially bounded. This set includes almost all practically relevant signals.

For this an adaptive output-feedback controller is designed in the state-space. It consists of an adaptive high-gain observer and an adaptive high-gain controller, both described in the following.

Observer

The observer is an adaptive version of the high-gain observer introduced by Nicosia and Tornambè (Nic89) as in (Bu197). A state-space representation is

$$\dot{\hat{x}}(t) = \hat{A}_\kappa \hat{x}(t) + \&e(t) \tag{3a}$$

$$e(t) = y(t) - y_{ref}(t) \tag{3b}$$

with $\hat{x} \in \mathbb{R}^r$ and

$$\hat{A}_\kappa = \begin{bmatrix} -p_{r-1} & \kappa & 1 & 0 & & \\ -p_{r-2}\cdot\kappa^2 & 0 & 1 & & & \\ : & & & & \ddots & \\ -p_1\cdot\kappa^{r-1} & 0 & 0 & & & 1 \\ -p_0\cdot\kappa^r & 0 & 0 & & & 0 \end{bmatrix}, \quad \hat{b}_\kappa = \begin{bmatrix} p_{r-1}\cdot\kappa \\ p_{r-2}\cdot\kappa^2 \\ \\ p_1\cdot\kappa^{r-1} \\ p_0\cdot\kappa^r \end{bmatrix}$$

Note that if the parameters p_i are chosen such that $s^r + \sum_{i=0}^{r-1} p_i s^i$ is a Hurwitz polynomial, then for any positive value of the observer gain κ, the spectrum of $\hat{A}$, lies in the open left half plane, $\sigma(\hat{A}_\kappa)$ c $\mathbb{C}_-$ and the observer dynamics are stable. No further knowledge of the model besides that of the relative degree is needed for the observer design. The observer gain κ is adapted according to the adaptation law described below.

Controller

The controller is an observer-state feedback

$$u = -q_k \hat{x}, \tag{4}$$

where

$$q_k = \left[q_0 \cdot k^r, \ \cdots \ , \ q_{r-1}\cdot k\right]$$

The parameters q_i are chosen such that $s^r + g\sum_{i=0}^{r-1} q_i s^i$ is a strongly Hurwitz polynomial. Then for any positive value of the controller gain k, the spectrum of $A - bq$, lies in the open left half plane. Of the model, only the relative degree and a lower bound of the high-frequency gain are needed for the controller design. The adaptation law for the controller gain k is described below.

Gain Adaptation

The adaptation for the observer gain κ and the controller gain k is chosen in such a way that the gains are increased as long as the amplitude of the tracking error e is larger than the user-defined bound λ from the control objectives.

Let, for $\lambda > 0$, $\gamma > 0$, $k(0) = k_0 > 0$,

$$\dot{k}(t) = d_\lambda(e(t), k(t))^2; d_\lambda(e,k) = \frac{\gamma}{k^r}\begin{cases} |e| - \lambda & \text{for } |e| \geq \lambda, \\ 0 & \text{for } |e| \leq \lambda. \end{cases} \tag{5}$$

In order to guarantee the achievement of the objectives the observer gain κ has to grow sufficiently faster than the controller gain k for large k's. For simplicity, we choose as a simple case $\kappa = k^2$.

This adaptation law ensures a monotonical increases of the observer and controller gains.

In Section 3 we need the following definition.

Definition 1 *A polynomial* $p(s) = s^r + \sum_{i=0}^{r-1} p_i s^i$ *is called* strongly Hurwitz if *p(s) is a Hurwitz polynomial and there exists* a *symmetric, positive definite* matrix *P such that the companion* **matrix**

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -p_0 & \cdots & & -p_{r-1} \end{bmatrix}$$

satisfies for $\Psi_r = \text{diag}\{1, 2, \ldots, r\}$ *the inequalities*

$$A^T \cdot P + P \cdot A < 0 \tag{6a}$$

$$\Psi_r \cdot P + P \cdot \Psi_r > o. \tag{6b}$$

A matrix with a strongly Hurwitz characteristic polynomial will be called strongly Hurwitz.

Remark 1 *As shown in (Bul99c), the strongly Hurwitz condition is not a restrictive assumption.*

Remark 2 Systems *with* $cA^{r-1}b = g \leq \bar{g} < 0$ *instead of* (2b) can easy *be treated by changing the sign of the controller.*

3 Results

As stated in the following theorem we can prove that combining the adaptive observer (3) with the adaptive controller (4) and using the adaptation law (5) with $\kappa = k^2$ to close the loop of an arbitrary system of class (1), (2) yields that the tracking error asymptotically converges to the λ-strip, that the adaption converges, that all states remain bounded and that no finite escape time can occur.

Theorem 3 ***If for all*** $\gamma \geq \bar{g}$, $s^r + \sum_{i=0}^{r-1} p_i s^i$ ***and*** $s^r + \gamma \sum_{i=0}^{r-1} q_i s^i$ ***are strongly Hurwitz polynomials then the application of the X-tracker*** (3), (4), ***(5) with*** $\kappa = k^2$ ***to any stabilizable system of the class*** (1), ***(2) and to any reference signal*** $y_{ref}(\cdot) \in W^{r,\infty}$ ***results in a closed-loop system which, independently of the initial values*** $x(0) \in \mathbb{R}^n$, ***i(0)*** $\in \mathbb{R}^r$, ***k(0) > 0 has a*** unique ***solution which exists on the whole half axis t*** $\in [0, \infty)$ ***and,*** moreover,

a) $(x(\cdot), \hat{x}(\cdot), k(\cdot)) \in L_\infty(0, \infty)$,

b) $\lim_{t\to\infty} \mathrm{dist}(|e(t)|, [0, \lambda]) = 0$.

Remark 4 ***Theorem 3 states that for*** any linear system ***with known relative*** degree ***and lower bound of the high-frequency gain, a X-tracking controller*** can ***be designed with the guarantee that all states and adaptation*** parameters ***remain bounded and that the tracking error y*** $- y_{ref}$ ***asymptotically converges to the X-strip. The width of this strip is a*** parameter ***which can be chosen by the user and will usually depend on the specifications, on model uncertainties and on the quality of the*** measurement.

Remark 5 ***The motivation for such an observer-based controller comes from the fact that for fixed adaptation parameters k and*** κ ***the*** transfer ***function from y to*** $\hat{x}_i$, ***the i-th state of the observer,***

$$\hat{x}_i(s) = s^{i-1}\left(1 - \frac{\left(\frac{s}{\kappa}\right)^n + \ldots + \left(\frac{s}{\kappa}\right)^{n-i+1} p_{i-1}}{\left(\frac{s}{\kappa}\right)^n + \left(\frac{s}{\kappa}\right)^{n-1} p_1 + \ldots + p_n}\right) y(s),$$

is at "low" frequencies, i.e. for small $\left(\frac{s}{\kappa}\right)$, ***approximately*** a series ***of differentiators,***

$$\hat{x}_i(s) \approx s^{i-1} y(s)$$

with a band width proportional to the observer gain. Thus, the observer ***states approximate the derivatives of the output y Therefore, the controller is approximately a*** $PD\ldots D^{r-1}$ ***controller:***

$$u(s) \approx -k \sum_{i=0}^{r-1} q_i s^i k^{r-1-i} y(s) = -k^r \sum_{i=0}^{r-1} q_i \left(\frac{s}{k}\right)^i y(s).$$

Proof

The following lemma, proven in (Bul99b) will be used in the proof of Theorem. 3.

Lemma 6 (Normal form) ***Every stabilizable*** linear system ***(A, b, c) of*** order ***n and relative degree r as given in (1) and (2)*** is ***similar to the following state space representation*** $(\bar{A}, \bar{b}, \bar{c})$:

$$\left[\begin{array}{c|c} \bar{A} & \bar{b} \\ \hline \bar{c} & 0 \end{array}\right] = \left[\begin{array}{cccccccc|c} 0 & 1 & & & & & & & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & & & 1 & & & & & 0 \\ -\alpha_0 & \cdots & & -\alpha_{r-1} & \cdots & & & -\alpha_{n-1} & g \\ & & & & \tilde{A}_{22} & & & & 0 \\ & & & * & \cdots & * & & & \vdots \\ & & & \vdots & \ddots & \vdots & \tilde{A}_{33} & & \vdots \\ 1 & & & * & \cdots & * & & & 0 \\ \hline 1 & 0 & \cdots & & & & & 0 & 0 \end{array}\right] \tag{7}$$

where $\sigma(\tilde{A}_{22}) \subset \mathbb{C}_-$, $\sigma(\tilde{A}_{33}) \subset \mathbb{C}_-$ ***and the*** stars ***indicate for real entries. All other entries are zero.***

Proof (of Theorem 3)

Outline of the Proof. We will prove Theorem 3 in four steps. First, it is shown that ***k*** cannot go to infinity on the maximal time interval where all states remain bounded. Then, it is proven that the solution of the differential equations exists for all times and thus that the maximal time interval of existence of the solution is infinite and that there are no peaking effects. A consequence of the first two steps is that the controller gain ***k*** converges. In the third part boundedness of the observer states $\hat{x}$ and thus of the plant input u as well as boundedness of the plant states x is shown. The proof concludes by showing that the tracking error converges to the X-strip.

1) Boundedness of the adaption parameters. By Lemma 6, we may assume that the system is given in the normal form (7). The nonlinear closed-loop system (7), (3), (4) and (5) is of the form

$$\dot{x} = Ax - bq_k\hat{x}, \qquad x(0) = x_0 \in \mathbb{R}^n \tag{8a}$$

$$\dot{\hat{x}} = \hat{A}_\kappa \hat{x} + \hat{b}_\kappa e, \qquad \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^r \tag{8b}$$

$$\dot{k} = d_\lambda(e, k)^2, \qquad k(0) = k_0 > 0, \tag{8c}$$

$$e = cx - y_{ref}. \tag{8d}$$

Using the coordinates $\bar{x}^T = [\xi^T, z^T, \hat{e}^T]$ where $\xi^T = [x_1 - y_{ref},\ x_2 - \dot{y}_{ref},\ \ldots,\ x_r - y_{ref}^{(r-1)}]$ denotes the tracking error and its derivatives, $z = [x_{r+1}, \ \ldots, \ x_n]$ are the uncontrollable, but stable states and those of the zero-dynamics, $[\tilde{x}_{r+1}, \ldots, \tilde{x}_n]$ in (7), and e $= [x_1, \ldots, x_r]^T - \hat{x} \in \mathbb{R}^r$ is the observer error, the closed loop (8) can be written as

$$\dot{\bar{x}} = \bar{A}\bar{x} - \bar{B}y_{ref} =: f(\bar{x}, t). \tag{9}$$

With

$$\bar{\bar{x}} = K^{-1}\left[\xi^T, \quad z^T, \quad \hat{e}^T\right]^T \tag{10}$$

where $K = \mathrm{diag}\{K_r, I_m, K_r^2\}$, $K_r = \mathrm{diag}\{k, k^2, , k^r\}$ and setting $\kappa = k^2$, the closed-loop differential equation is

$$\dot{\bar{\bar{x}}} = D_k \bar{\bar{A}}\bar{\bar{x}} - \bar{\bar{B}}y_{ref} - \frac{\dot{k}}{k}\Psi x^2$$

with

$$\Psi = \operatorname{diag}[\Psi_r, 0_m, \Psi_r], \quad \Psi_r = \operatorname{diag}[1,\ 2, \ldots, r],$$
$$\Delta = \operatorname{diag}\{\Delta_1, \Delta_2, \Delta_3\} = \operatorname{diag}\{kI_r, I_m, k^2 I_r\},\ \bar{\bar{A}} =$$

$$\left[\begin{array}{cccc|ccc|cccc}
0 & 1 & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & \\
0 & & & 1 & & & & & & & \\
-\tilde{q}_0 - \frac{\alpha_0}{k^r} & \cdots\cdots & & -\tilde{q}_{r-1} - \frac{\alpha_{r-1}}{k} & -\frac{\alpha_r}{k^r} & \cdot & -\frac{\alpha_{n-1}}{k^r} & \tilde{q}_0 & \tilde{q}_1 k & \cdots & \tilde{q}_{r-1}k^{r-1} \\
\hline
 & & & & & \breve{A} & & & & & \\
1 & & & & & & & & & & \\
\hline
 & & & & & & & -p_{r-1} & 1 & & \\
 & & & & & & & \vdots & & \ddots & \\
 & & & & & & & -p_1 & & & 1 \\
-\frac{\tilde{q}_0}{k^r} - \frac{\alpha_0}{k^{2r}} & & \cdots & -\frac{\tilde{q}_{r-1}}{k^r} - \frac{\alpha_{r-1}}{k^{r+1}} & -\frac{\alpha_r}{k^{2r}} & \cdot & -\frac{\alpha_{n-1}}{k^{2r}} & -p_0 + \frac{\tilde{q}_0}{k^r} & \frac{\tilde{q}_1}{k^{r-1}} & \cdots & \frac{\tilde{q}_{r-1}}{k}
\end{array}\right]$$

and $\bar{\bar{B}} = k^{-(r-1)} \boldsymbol{e}_r[\alpha_0, \ldots, \alpha_{r-1}, 1]$,

$$\boldsymbol{y}_{ref}^T = [y_{ref}, \dot{y}_{ref}, \ldots, y_{ref}^{(r)}], \quad \tilde{q}_i = q_i \,.\, g.$$

$\boldsymbol{e}_r$ is the r-th unit vector and $\breve{A} \in \mathbb{R}^{(n-r)\times(n-r)}$ is Hurwitz as $\breve{A}$ is block-triangular with the stable matrices $\tilde{A}_{22}, \tilde{A}_{33}$ on the diagonal.

By assumption, the matrices $\bar{\bar{A}}_{22} = \breve{A}$,

$$\bar{\bar{A}}_{11} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\tilde{q}_0 & \cdots\cdots & & -\tilde{q}_{r-1} \end{bmatrix}, \quad \bar{\bar{A}}_{33} = \begin{bmatrix} -p_{r-1} & 1 & & \\ \vdots & & \ddots & \\ -p_1 & & & 1 \\ -p_0 & 0 & \ldots & 0 \end{bmatrix}.$$

are strongly Hurwitz. Therefore, there exist unique symmetric, positive definite solutions P_1, P_2, P_3 of the Lyapunov equations

$$\bar{\bar{A}}_{ii}^T P_i + P_i \bar{\bar{A}}_{ii} = -Q_i \qquad i = 1,2,3, \tag{11a}$$
$$P_i \Psi_r + \Psi_r P_i > \boldsymbol{0} \qquad i = 1,3. \tag{11b}$$

for some matrices $Q_i \geq$ I, $i = 1,2,3$. The state space partitioning of $\bar{\bar{\boldsymbol{x}}}$ and $\bar{\bar{B}}$ into $\bar{\bar{\boldsymbol{x}}}_i$ and $\bar{\bar{B}}_i$ respectively with $i = 1,2,3$ corresponds to the one for $\bar{\bar{A}}$.

Boundedness of the adaptation parameter is done via contradiction. Due to lack of space, this part of the proof is not shown here but can be found in (Bul99a).

2) Global existence of a unique solution. Applying the boundedness of $\boldsymbol{k}$ to (9), it can be seen that there exist constants c, $\boldsymbol{d}$, s.t.

$$|\boldsymbol{f}(\bar{\boldsymbol{x}}, t)| \leq c\|\bar{\boldsymbol{x}}\| + \boldsymbol{d} \text{ and } \boldsymbol{f}(\cdot) \in C^1.$$

Thus, $\bar{\boldsymbol{x}}(t)$ exists for all $t \in \mathbb{R}$ (Ha180).

3) Boundedness of the observer states. As $k(\cdot)$ is bounded, $d_\lambda(\cdot) \in L_2(0,$ co). From that, (5) and the Hölder inequality follows that

$$\gamma^{-1} k^r(\cdot) d_\lambda(\cdot) \in L_2(0, \infty).$$

Combining this with

$$|e(\cdot)| - \gamma^{-1} k^r(\cdot) d_\lambda(e(t), k(t)) \in L_\infty(0, \infty),$$

yields that

$$|e(\cdot)| = \underbrace{|e(\cdot)| - \gamma^{-1} k^r(\cdot) d_\lambda(\cdot)}_{L_\infty(0,\infty)} + \underbrace{\gamma^{-1} k^r(\cdot) d_\lambda(\cdot)}_{\in L_2(0,\infty)} \tag{12}$$

Defining $\boldsymbol{\xi}(t) = K_r^{-2}(t)\boldsymbol{x}(t), \hat{A} = \hat{A}_{\kappa=1}, \hat{\boldsymbol{b}} = \hat{\boldsymbol{b}}_{\kappa=1}$, (8b) is transformed to

$$\dot{\boldsymbol{\xi}} = k^2 \hat{A} \boldsymbol{\xi} + k^2 \hat{\boldsymbol{b}} e - \frac{\dot{k}}{k} 2 \Psi_r \boldsymbol{\xi}$$

define $\hat{A}_1 = k_\infty^2 \hat{A}, \hat{A}_2 = \hat{A} - \hat{A}_1, \hat{A}_3 = -2\frac{\dot{k}}{k}\Psi_r$

$$\dot{\boldsymbol{\xi}} = \hat{A}_1 \boldsymbol{\xi} + \hat{A}_2 \boldsymbol{\xi} + \hat{A}_3 \boldsymbol{\xi} + k^2 \hat{\boldsymbol{b}} e. \tag{13}$$

A_1 is a constant Hurwitz matrix and there exist $t_0 >$ 0, $M_1 > 0$, $M_2 > 0$, $M_3 > 0$ such that

$$\|A_2(t)\| \leq M_1 \text{ for all } t \geq t_0,$$
$$\int_{t_0}^{\infty} \|\hat{A}_3(t)\| dt = \|\Psi\| \log(k)|_{t_0}^{\infty} =: M_2,$$
$$M_3 = \|\hat{\boldsymbol{b}}\|, \text{ and}$$
$$M_4, \mu > 0 : \|e^{\hat{A}_1(t-t_0)}\| \leq M_4 e^{-my(t-t_0)} \forall t \geq t_0.$$

Variation of Constants, see e.g. (Bel53), to (13) yields

$$\begin{aligned}
\|\boldsymbol{\xi}(t)\| &\leq M_4 e^{-\mu(t-t_0)} \|\boldsymbol{\xi}(t_0)\| + \int_{t_0}^{t} M_4 e^{-\mu(\tau - t_0)} \\
&\quad \left(\|A_2(\tau)\| + \|A_3(\tau)\| + k^2(\tau) M_3 |e(\tau)|\right) d\tau \\
&\leq M_4 \|\boldsymbol{\xi}(t_0)\| + \frac{M_4}{\mu}(M_1 + M_2) \\
&\quad + M_4 M_3 k_\infty^2 \int_{t_0}^{t} |e(\tau)| e^{-\mu(\tau - t_0)} d\tau.
\end{aligned}$$

Combined with (12), this yields that $\boldsymbol{\xi}(\cdot)$ is bounded (Des75), and by the boundedness of $\boldsymbol{k}$ it follows that

$$\hat{\boldsymbol{x}}(\cdot) \in L_\infty(0, \infty) \quad \text{and} \quad u = -\tilde{\boldsymbol{q}}_k \hat{\boldsymbol{x}}(\cdot) \in L_\infty(0, \infty). \tag{14}$$

4) Boundedness of the states of the plant. Since (c, $\boldsymbol{A}$) is detectable, there exists some $k \in \mathbb{C}^{n\times 1}$ such that $\sigma(A - \boldsymbol{kc})$ c $\mathbb{C}^-$. Now consider the observer

$$\dot{\tilde{\boldsymbol{x}}} = A\tilde{\boldsymbol{x}} + bu + \boldsymbol{k}c(\boldsymbol{x} - \tilde{\boldsymbol{x}}) = (\boldsymbol{A}\text{-}\boldsymbol{kc})\% + bu + \boldsymbol{k}(e - y_{ref}).$$

By (12), (14), and the exponential stability of $(\boldsymbol{A} - \boldsymbol{kc})$ it, holds that $\tilde{\boldsymbol{x}} \in \boldsymbol{L}$,. Exploiting exponential stability of

$$\frac{d}{dt}(\boldsymbol{x} - 2) = (\boldsymbol{A} - \boldsymbol{kc})(\boldsymbol{x} - \tilde{\boldsymbol{x}})$$

again and boundedness of $\tilde{\boldsymbol{x}}$ yields $\boldsymbol{x}(\cdot) \in L_\infty(0, co)$ and also $e(\cdot) = c\boldsymbol{x}(\cdot) - y_{ref}(\cdot) \in L_\infty(0, \infty)$.

5) Convergence of the tracking error. It remains to show b). For this we prove that $\lim_{t\to\infty} d_\lambda(e(t), k(t)) = 0$. Since $e(\cdot)$ and $k(\cdot)$ are bounded it follows that $\dot{k}(\cdot) = d_\lambda(\cdot)^2 \in L_\infty(0,\infty)$. From $\dot{e} = \boldsymbol{c}\,[A\boldsymbol{x} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}}] - \dot{y}_{ref}$ and previous results we conclude $\dot{e}(\cdot) \in L_\infty(0,\infty)$. Now

$$\frac{d}{dt} d_\lambda(t)^2 = 2 d_\lambda(t) \left(\frac{e(t)\dot{e}(\cdot)}{|e(t)|} - r\frac{\dot{k}}{k} d_\lambda(t) \right) \in L_\infty(0,\infty)$$

and hence $d_\lambda(\cdot)^2$ is uniformly continuous. This, together with $d_\lambda(\cdot)^2 \in L_1(0,\infty)$ yields, by Barbălat's Lemma (Bar59) that $\lim_{t\to\infty} d_\lambda(t)^2 = 0$.

This completes the proof. ■

4 Example

To demonstrate the applicability of the proposed λ-tracker the controller is applied in simulation to the identified model of a ball and plate laboratory setup (Her97). The system is depicted in Figure 1. The objective is to make a steel ball follow a user-defined trajectory by applying voltages to the motors controlling the angles of a plate that can be rotated about two axis (α and β). The dimensions are approximately 80 cm x 80 cm for the plate and the radius of the ball is 5cm The maximal angles of the plate are about 6” for each of the axes α, β.

Following (Her97) the ball and plate system can well be approximated by two linear, decoupled systems of the form

$$x_i(s) = \frac{a_i}{s^2(s^2 + b_i s + c_i)} u_i(s), \qquad \begin{array}{l} a_i > 0, b_i > 0, c_i > 0 \\ i = \alpha, \beta \end{array}$$

due to the fact that the maximum angles are small. Therefore, this system with two inputs and outputs can be treated as two decoupled single input single outputs systems, for which controllers can be designed independently. x_α and x_β are the ball positions relative to the plate in direction of the α and β axis, respectively (see also Figure 1). Thus, the dimension of each system and the relative degree are both 4, implying that the system has trivial zero dynamics and is thus minimum phase. We assume that the high frequency gain, here equal to a_i, is larger than 20. For the controller design we make no further assumptions about the model. In particular, we assume that the values a_i, b_i, c_i are not known. The parameters of both controllers have been chosen as the following strongly Hurwitz polynomials: parameters $\boldsymbol{p}$ and $\boldsymbol{q}$ are such that the poles of the respective polynomials lie at 1,2,3,4 for the observer and for the controller, both for a controller gain of 1. For both axes, the controller parameters are chosen to be $k_0 = 1$, $\gamma = 1000$, $\lambda = 1$ cm. The reference trajectory consists of a circle with radius of 20 cm for the first minute, 10 cm for the second, and 15cm for the last. The observer and the model are both initialized at the center of the plate, i.e. at zero.

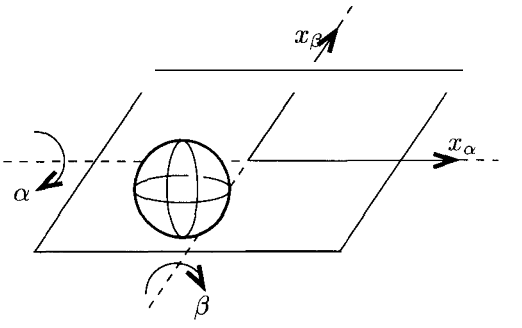

Figure 1: Sketch of the Ball and Plate system.

For the simulations, model (4) with the numerical values identified in (Her97) as $a_\alpha = 34.2$, $b_\alpha = 16s^{-1}, c_\alpha = 400s^{-2}$ respectively $a_\beta = 37.9$, $b_\beta = 19s^{-1}, c_\beta = 640s^{-2}$ has been used to represent the plant. Again, these parameters were not used for the controller design.

The result can be seen in Figure 2. In Figure 2.a an xy-plot of the ball's trajectory over three minutes starting at the origin is shown. After the transient the ball follows quite well the reference trajectory, depicted by plus signs. In Figure 2.b the tracking error is depicted. It can be seen that it is slowly decreasing towards the X-strip. There, it keeps oscillating. There are fast increases of the controller gain during the first seconds and then smaller ones each time the corresponding tracking error leaves the X-strip, as shown in Figure 2.c.

5 Conclusions

[Section] In this paper a high-gain adaptive controller for linear, minimum-phase systems of arbitrary, but known relative degree is presented. This is the first step in the direction of X-tracking of high relative degree nonlinear systems. The assumption of knowledge of a bound on the high frequency gain should be removed in the future. Future research will focus on nonlinear systems.

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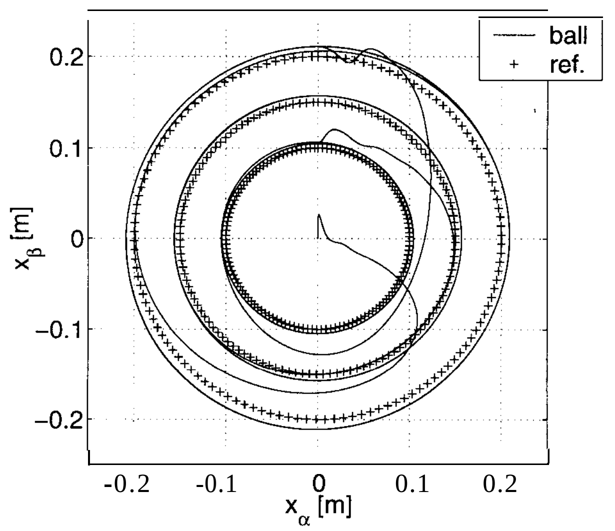
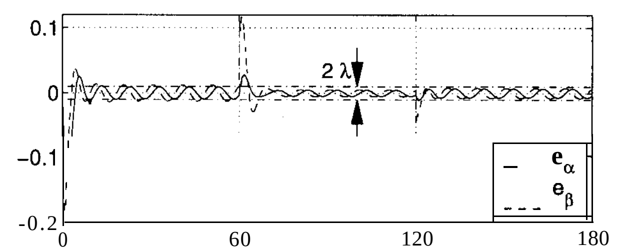
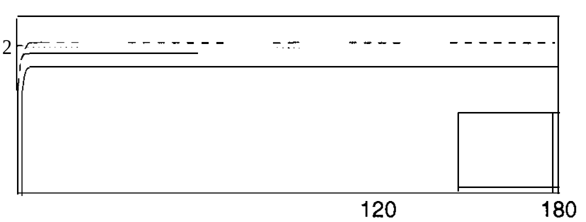


(a) Ball position and reference.



(b) Tracking errors.



(c) Controller gains.

Figure 2: Ball and Plate with X-tracking control.

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