Generation of spatially correlated wind histories

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Abstract

Turbulent wind is a natural and therefore random phenomenon. The analysis of flexible structures subjected to turbulent wind requires, in a finite element approach, the resolution of a system of stochastic differential equations. It is supposed that the characteristics of the structure are perfectly known; thus the stochastic aspect of the problem comes from the random loading only. As the turbulent wind is most commonly characterized by its frequency content, the resolution of this equation is generally performed in the frequency domain. All governing equations must therefore be linear. If this can be reasonably supposed for the structure, this hypothesis is not mathematically justified for the wind loading.

When the non linear behaviour of any part of the structure has to be accounted for, the resolution in the frequency domain is not valid anymore. It is then necessary to compute, with a step-by-step dynamic analysis, the response of the structure to a series of generated wind histories. In a finite element approach, this is for example achieved thanks to Newmark's or Wilson's methods. Averaging across the ensemble of the computed time-varying responses allows then to recover statistical results that a stochastic analysis would provide. This analysis method is known as a stochastic Monte Carlo simulation.

After having emphasised the need to represent correctly the coherence between wind pressures at different points of the structure, this paper will present a method for generating a set of correlated wind histories that must be used in a finite element analysis. Finally an example of application of the presented method will prove its efficiency.

Generation, coherent, correlated, multidimensional, target power spectral density

Introduction

Structures studied in the civil engineering field are subjected to natural phenomena as wind pressures, ground accelerations, thermal effects or wave pressures. The main characteristic of these loadings is their randomness. Depending on the kind of random properties which are taken into account

(variables or functions), the imperfectly defined character of the loading is treated by means of random variables, e.g. wave height, or random processes, e.g. ground acceleration. In this paper, we will focus only on this latter kind of randomness.
The main scope of application of these not perfectly defined functions is the dynamic analysis of light soft structures. Indeed, for such analyses, the loading (earthquake, wind pressures) is time dependent and has a fixed frequency content. The loading as well as the displacements or internal forces of the structure are random processes and are then characterized by power spectral densities (*psd*). The purpose of the analysis is to determine, from the *psd* of the applied force, the *psd's* of the displacements and internal forces.
This kind of dynamic analysis can be realized trough two very different ways. The first one is the stochastic approach studying the structure in the frequency domain. Frequency after frequency the multiplication of the transfer function by the *psd* of the applied force gives the *psd* of the displacements. This method allows to appreciate very fast the random characteristics of the response but has the great disadvantage to be limited to the analysis of linear structures.
When non linearity's have to be accounted for, the analysis in the frequency domain is not valid anymore and it must be replaced by a step-by-step analysis in the time domain. Samples of the random process are generated in order to correspond the best to the (target) *psd* of the applied force. For each of these samples, a dynamic analysis is performed. The statistical structure of the response is finally obtained by averaging the *psd's* computed from each response.
This paper presents a method to generate a set of samples of correlated random processes. The examples are presented in the context of the generation of correlated wind histories but the method could also be applied to the generation of different (but correlated) ground accelerations at distant piles of a long span bridge. Other applications are the generation of the surface roughness along a road or the evolution of the water level.
The finite element method is not directly present but the sample generator developed will point out all its efficiency when coupled with a finite element program. Indeed a Monte Carlo simulation needs to loop outside the finite element program, going from the generator (to create the applied forces) to the finite element analysis.

Generation of 1-dimensional processes

Before trying to generate a set of samples, we will briefly see how to generate one sample only. Two methods are generally presented : the Fourier decomposition and the autoregressive (*AR*) - and sometimes with a moving average (*ARMA*) - filtering of a white noise. These methods are nowadays commonly used and then presented more into details in many reference books of fundamental dynamics ([1], [2]).

Fourier decomposition

The power spectral density of a sample can be expressed in terms of its Fourier transform :

$$S_i^x = \frac{2}{N\Delta n}\left|X_i\right|^2 \tag{1}$$

where N, Δn and X_i represent respectively the number of points in the sample, the frequency step and the Fourier transform of the signal at frequency $n_i = (i-1)\Delta n$. This relation shows that the *psd* of a sample is an image of its frequency content. It seems then obvious to do the generation in the frequency domain.

Let us note $S^{(t\arg)}$ the target *psd* of the random process and let us represent it for N discretized frequencies. For example, if the generation concerns the wind velocity, this discretized target *psd* can be expressed by :

$$S_i^{(t\arg)} = \frac{4L_u\sigma_u^2}{\left[1+70.8\left(\frac{n_i L_u}{\overline{U}}\right)^2\right]^{5/6}} \quad ; \quad n_i = (i-1)\Delta n \tag{2}$$

where L_u, σ_u and $\overline{U}$ represent respectively the length scale, the standard deviation of the wind velocity and the mean wind speed.

The sample can then be generated by choosing :

$$X_i = \sqrt{\frac{N\Delta n}{2}}\sqrt{S_i^{(t\arg)}}\,e^{j\varphi_i} \tag{3}$$

where φ_i is a random delay uniformly distributed between 0 and 2π. The *psd* of the so-generated signal matches exactly the target *psd* at the discretized frequencies.

Generally, the sample must be expressed in the time domain, which can be done by Fourier-inverting the signal generated in the frequency domain :

$$x_i = ifft(X_i) \tag{4}$$

Since there is a good matching between target and generated *psd's*, this method is very accurate but it could be criticized to be quite slow.

ARMA-filtering

As the sample to generate must usually be expressed in the time domain, generation methods have also been imagined in this domain. The most general one is the *ARMA*-filtering of an existing signal. It is nevertheless often presented with two simplifications : an *AR* filter is used instead of an *ARMA* filter and the existing signal is a white noise, which is of course very easy to generate. The signal to generate y_i is expressed in terms of this white noise w_i :

$$y_i = b_0 w_i - \sum_{k=1}^{p} a_k y_{i-k} \tag{5}$$

where b_0 and $a_1,...a_p$ are the autoregression parameters. This way of expressing the new sample allows to connect the *psd's* of the white noise S_i^w and of the process to generate S_i^y :

$$S_i^y = |H_i|^2 S_i^w = |H_i|^2 \tag{6}$$

where

$$H_i = \frac{b_0}{1 + a_1 e^{-2j\pi\frac{i}{N}} + ... + a_1 e^{-2pj\pi\frac{i}{N}}} \tag{7}$$

is the transfer function between the white noise and the target process.
The key of the generation consists then in finding the autoregression parameters such that the squared norm of the transfer function $|H_i|^2$ corresponds the best to the target *psd* $S^{(targ)}$. Once these parameters are determined, a white noise of a sufficient length has to be generated and then filtered. The most time-consuming part of this method consists in the optimisation. This method is thus useful when several samples have to be generated for the same process.

Generation of multi-dimensional processes

This paragraph presents a method to generate correlated samples caracterized by their *psd's* (which could be different from one to another sample) and coherence functions. The method presented here is based on a matrix eigenvalue decomposition. It is the numerical equivalent (and therefore adopted to finite element applications) of a continuous analytical approach presented by Carassale and Solari ([3], [4]) who develop, for simple analytical target fields, closer forms of the eigenvalue decomposition.

For simplicity in the notations, this presentation will treat two samples only ; it could however be applied for the generation of a larger amount of correlated samples.

Basic concepts

The *cross-psd* between two samples x and y can be expressed in terms of their Fourier transforms :

$$S_i^{xy} = \frac{2}{N\Delta n} X_i \overline{Y}_i \tag{8}$$

where $\overline{Y}_i$ represents the conjugate of the Fourier transform of signal y. This complex function of the frequency represents the coherence between signals x and y. Because it has more physical interest, the coherence function is generally preferred to this one :

$$\gamma_i^{xy} = \frac{S_i^{xy}}{\sqrt{S_i^x S_i^y}} \quad ; \quad -1 \le \gamma_i^{xy} \le 1 \tag{9}$$

For brevity in the notations, the *psd's* and the *cross-psd* associated to samples x and y are generally collected in the power spectral density matrix (*psdm*) :

$$\left[S^{xy}\right] = \begin{bmatrix} S^x & S^{xy} \\ S^{xy} & S^x \end{bmatrix} \tag{10}$$

Let us consider the eigen value decomposition of this target *psdm* :

$$\left[S^{xy}\right] = [\Phi][\Omega][\Phi]^T \tag{11}$$

where $[\Phi]$ and $[\Omega]$ are respectively the matrix of the eigen modes and the diagonal matrix of the eigen values. As the *psdm* is a function of the frequency, both of these matrices are also frequency dependent. This modal decomposition is thus time consuming since it has to be repeated for a large number of frequency steps (*N* times).

As the matrix $[\Omega]$ is diagonal, its diagonal functions are power spectral densities of two uncoupled processes; they will be called the modal processes. Both methods presented here over for the generation of 1-dimensional processes can be used to generate separately two samples whose target *psd's* are these modal ones :

$$\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \rightarrow \begin{cases} N_1(n) \leftrightarrow \nu_1(t) \\ N_2(n) \leftrightarrow \nu_2(t) \end{cases} \tag{12}$$

Depending on the generation method used, the two modal samples are first expressed in the time ($\nu_1(t)$, $\nu_2(t)$) or in the frequency domain ($N_1(n)$, $N_2(n)$). These two expressions are anyway equivalent.

Let us built now new samples by introducing nodal processes obtained as linear combinations of the modal processes :

$$\begin{Bmatrix} X(n) \\ Y(n) \end{Bmatrix} = [\Phi(n)] \begin{Bmatrix} N_1(n) \\ N_2(n) \end{Bmatrix} \tag{13}$$

It can be shown that the *psd's* of these two couples of samples are connected by the relation :

$$[S^{xy}] = [\Phi][S^{N_1N_2}][\Phi]^T = [\Phi][\Omega][\Phi]^T \tag{14}$$

The comparison of Equations 11 and 14 shows thus than this way of forming samples x and y enables to give them the expected *psd's* and the required coherence.

Summary of the method

1. Discretize the *psdm* of the target processes with *N* frequencies equally spaced (necessary for *FFT*-transforms);
2. For each of these frequencies, make an eigenvalue decomposition of the *psdm*;
3. Utilize the eigen values to generate modal (uncoupled) samples;

4. In the frequency domain, combine these samples with the coefficients of the eigen modes to obtained the expected correlated samples;
5. If needed, come back to the time domain by *FFT*-inverting each sample generated.

Example 1

This first example consists in the generation of correlated samples whose *psd's* (Figure 1) are given by :

$$S_1(n) = ne^{-2n}$$
$$S_2(n) = ne^{-3n} \quad (15)$$

The target correlation between the processes is given by this coherence function :

$$\gamma(n) = e^{-n/2} \quad (16)$$

It is desired to generate a 600-second sample discretized with N=2^{13}=8192 points. The frequency step and the cut-off frequency are then respectively equal to $\Delta n = 1.67.10^{-3} Hz$ and $n_c = 13.65.Hz$.

Figures 2 and 3 respectively show the eigen values, i.e. the target modal *psd's*, and the eigen modes of the target *psdm*.

The shape of the target coherence function is such that signals are perfectly coherent for low frequencies ($\gamma = 1$) and perfectly independents for high frequencies ($\gamma = 0$). This confirms the physical meaning of the coherence; it is then not amazing that an exponential shape is usually chosen for the coherence. Under these two conditions for the limit values of the coherence, it is possible to show that :

- for low frequencies, one eigen value only is non zero; all the other (if more than two) are equal to zero. In other words, the rank of the target *psdm* is equal to 1;
- for high frequencies, each eigen value corresponds to a target *psd* of a node. If all the target *psd's* have the same asymptotic behaviour for high frequencies, the eigen values will thus all be the same.

Figure 4 shows a Cartesian reproduction of the target modal *psd's* and the *psd's* of the modal samples generated. The 1-dimensional generation has been realized

with the Fourier decomposition method (presented here over) which gives accurate results.
In the upper-right corner, the coherence function between these two signals is represented. As the samples have been generated separately, their coherence function is almost equal to zero.
Equation 13 allows then to form the “nodal” samples which have now the expected coherence. Figure 5 is the nodal equivalent of Figure 3; it shows that both *psd’s* and the cross-*psd* of the generated samples correspond to the target ones.

Improvement of the basic algorithm (1)

The main objection which could be formulated to this method is that the target *psdm* has to be decomposed in its eigen values for a large amount of frequencies (8192 in the previous example !). With today’s computers, it is not really a problem for the generation of a small number of samples. It would however be desirable for this generation procedure to be also applicable to the generation of a larger amount of signals. This is for example important in the study of long-span bridges subjected to wind loading. A set of wind histories can for example be composed of about one thousand coherent signals !
For these kinds of generation, it is obvious that one can’t afford to use the basic method presented before.
The first idea to reduce the computation efforts is to work with a limited number of modes. Indeed, the first example presented has shown that, at least for lower frequencies, the first target modal *psd* was much larger than the second one. This second mode seems then to bring a small contribution to the formation of the nodal samples.

Example 2

The data’s relative to this second example are exactly the same as for the first example, excepted that, in this case, one mode only will be computed. Figure 6 presents directly the nodal results of the generation. As expected, it shows that using a reduced number of modes doesn’t affect significantly the accuracy of the nodal *psd’s*. However, as the generated signals are now proportional to a same

third, they are perfectly coherent. Using a limited number of modes doesn't allow to give the expected coherence to the target signals.
The problem may look a bit simplistic here because one mode only (half of the number of modes) is kept for the generation. Generation of larger amounts of wind histories have shown that a reduction of the number of modes was conceivable but, if a good accuracy on the generated coherence is important, 90% of the modes at least must be computed and used.
The above remarks on the shape of the coherence function can give a good interpretation of the modes. For low frequencies, one mode only is enough to represent the nodal sample but for higher frequencies, as the eigen values have a similar asymptotic bevahiour, several (and at the limit, all) modes a necessary to represent correctly the nodal sample and the coherence existing between them.
We can then conclude that the first mode (or the first modes for the study of very large structures) gives their forms to the target nodal *psd's* and that higher modes are only there to give small variations around the tendency imposed by the first one(s). These variations allows then to give the desired coherence between the generated processes.

Improvement of the basic algorithm (2)

Another improvement of the method can be imagined when looking at the shape of the eigen values and eigen modes of the *psdm*. For the previous two examples, the eigenvalue decomposition was realized for an important number of frequencies (8192 !). This number seems to be very large to represent correctly so slow-varying functions.
A first idea is to systematically make the eigenvalue decomposition once every ten steps and then to interpolate the eigen values and eigen modes between the computed values. This allows to limit the number of decomposition (820).
Another method has been imagined for which the user doesn't have to choose the number of steps to skip between each decomposition ([5]). This method is based on the curvature of the target modal *psd's* (eigen values). Indeed, as the intermediate values will be computed by interpolation, the optimum algorithm shouldn't realize any decompositions in the linear zones (the beginning and the end are enough) and should realize more decompositions in the zones of large curvature.

Figure 7 illustrates the philosophy of this new procedure. f_1, f_2 and f_3 are three frequencies for which the decomposition has been realized. The corresponding values V_1, V_2 and V_3 represent one of the eigen values, computed for these three frequencies. The decomposition for frequency f_3 would have been useless if the points (f_1, V_1), (f_2, V_2) and (f_3, V_3) were on the same line. An estimator of the interest of point 3 can be expressed by the ratio between the value for frequency f_3 obtained by extrapolation of the two previous ones (A) and the value really computed (B). For each mode, we define an error estimator :

$$\varepsilon_m = \frac{\overline{AB}}{\overline{BC}} = \frac{V_1 + \frac{V_2 - V_1}{f_2 - f_1}(f_3 - f_1) - V_3}{V_3} \tag{17}$$

Then, for each decomposition frequency, we can also define a general error estimator by :

$$\varepsilon = \max\left(\left|\varepsilon_m\right|\right) \tag{18}$$

If this estimator is small, then the eigen values are linear in the vicinity of frequency f_3 and the frequency step (f_4-f_3) can be increased. On the contrary, if this estimator is large, at least one of the target modal *psd's* has a larger curvature in the vicinity of f_3 and the frequency step should be decreased. Regarding these physical desires, a frequency step amplification factor (Figure 8) has been imagined :

$$\frac{f_4 - f_3}{f_3 - f_2} = 1.5e^{-10\varepsilon} \tag{20}$$

The choice of this function is justified by considering that :

- it is important that the size of the step decreases quickly with increasing ε;
- the value at the origin must be limited in such a way that the step doesn't increase indefinitely when all eigen value are linear;
- it is also important that the step increases in linear zones only; the relation has thus been designed so that the step is increased for values of ε smaller than 4%.

So far, all the generations realized using this step amplification factor have proved good results. These generations concerned mainly wind histories which are

characterized by important curvatures. This relation could eventually be adapted for smoother target *psd's*.

Example 3

The target characteristics of the generation are the same as for examples 1 (both eigen modes are computed and used). Figure 9 shows the eigenvalue decomposition : the decomposition frequencies are represented with crosses. This figure shows that the presented method enables to represent correctly the target modal *psd's*. The optimisation is worthwhile because 110 frequencies only are used for representation of the functions.
Figure 10 represents the size of the frequency step as a function of the frequency. It shows that the step grows from the original step (very small), tends to be reduced in the vicinity of the maximum in the first mode, increases then in the linear decreasing part and is finally reduced when the curvature increases again.

Conclusions

After having briefly summarized the 1-dimensional generation procedures commonly used, this paper has presented a numerical method for the generation of correlated samples of random fields. Unlike analytical developments, this method enables to treat any difficult shape of random processes. The numerical approach makes also this method well adapted to finite element analyses.
When all is said and done, the basic procedure is quite laborious since it is based on a large number of eigenvalue decompositions. Two improvements have been proposed. An illustrated example have shown that the first one, concerning a reduction of the modal basis, should be done very carefully. The second improvement, based on a representation of the eigen values with a non constant frequency step, is much more efficient and allows to reduce drastically the number of modal decompositions. Thanks to this non uniform representation, this generation procedure can be applied to the generation of a very large number of correlated processes.

References

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Figures Captions

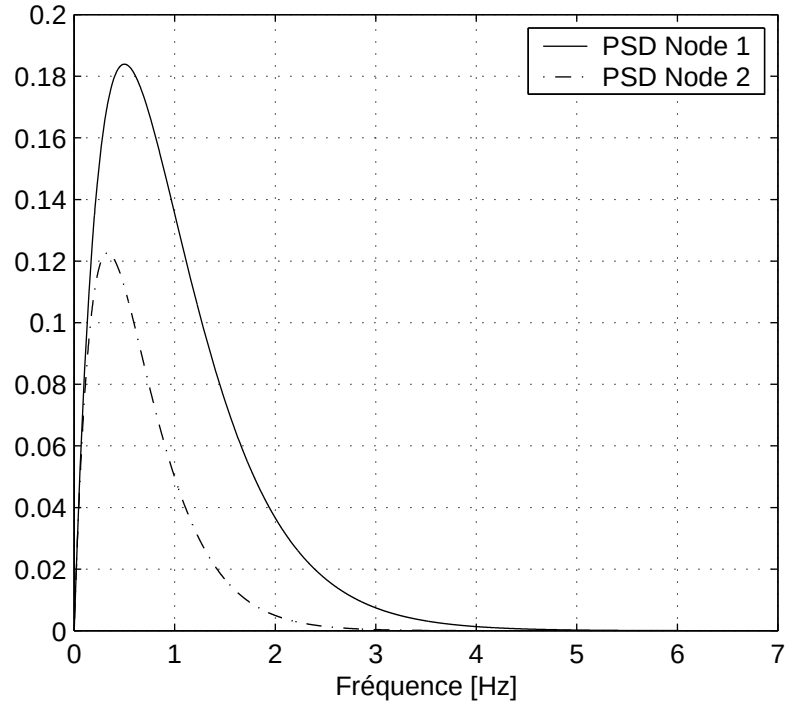

Figure 1 : Example 1 : Target power spectral densities

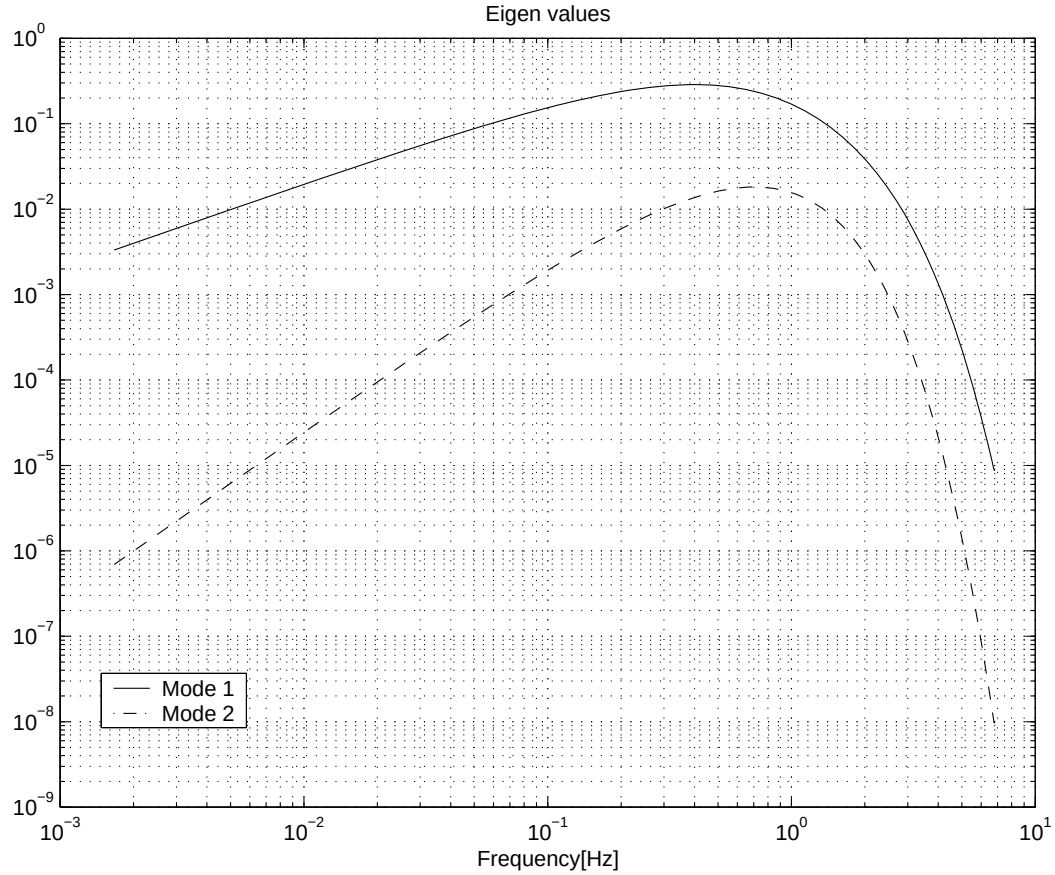

Figure 2 : Example 1 : Target modal *psd's*

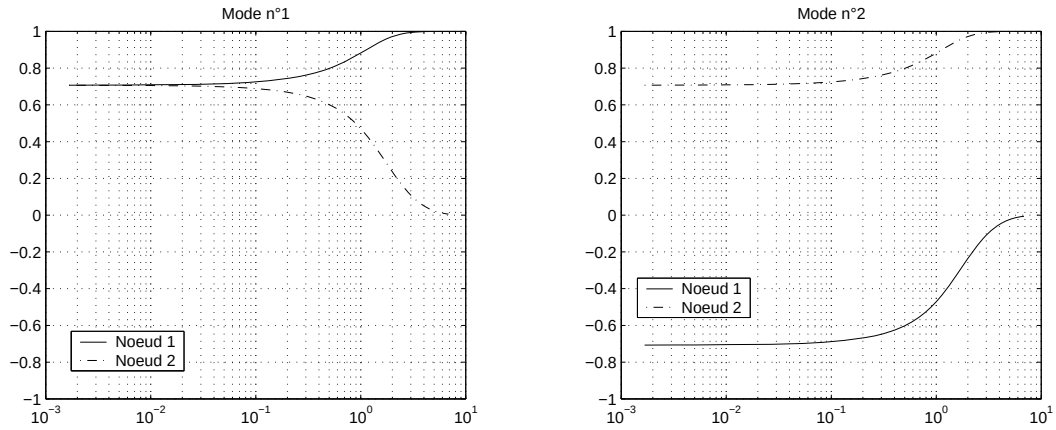

Figure 3 : Example 1 : Eigen modes of the target *psdm*

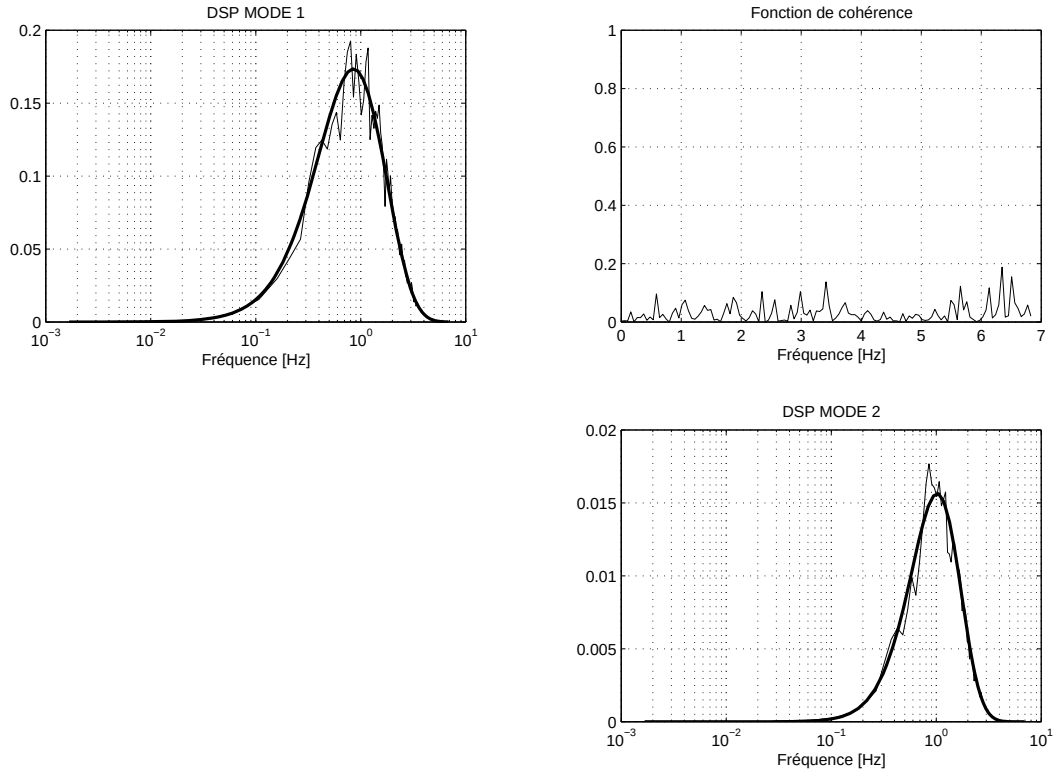

Figure 4 : Example 1 : Frequency characteristics of the generated uncoupled samples

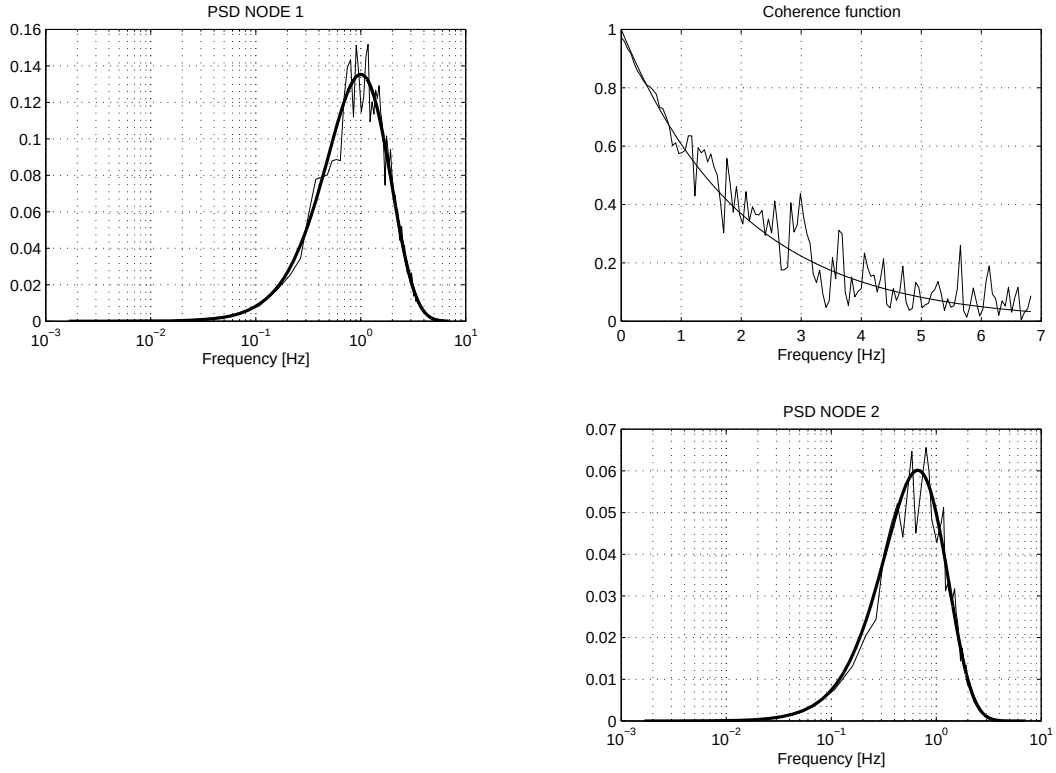

Figure 5 : Example 1 : Frequency characteristics of the generated coupled samples

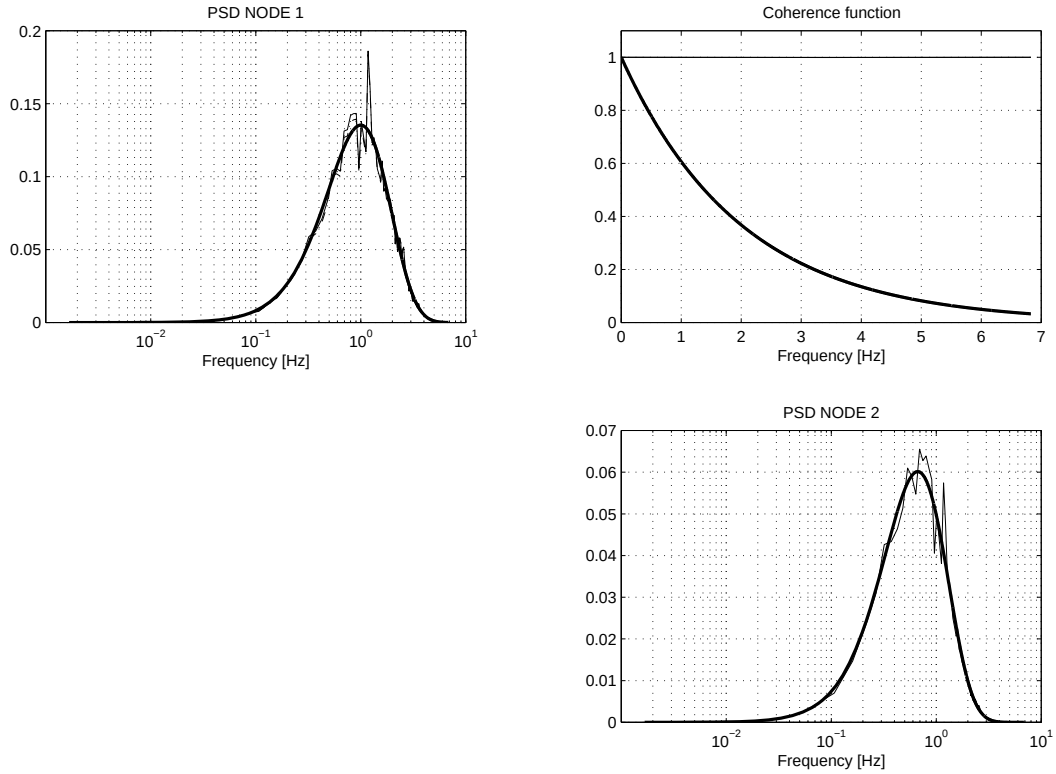

Figure 6 : Example 2 : Illustration of the reduction of the size of the modal basis

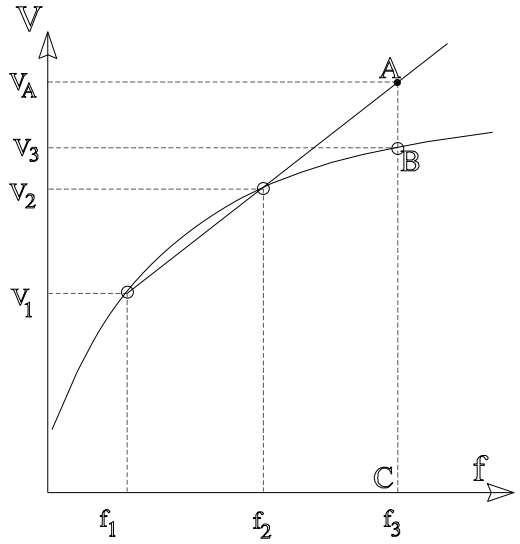

Figure 7 : Illustration of the error estimator

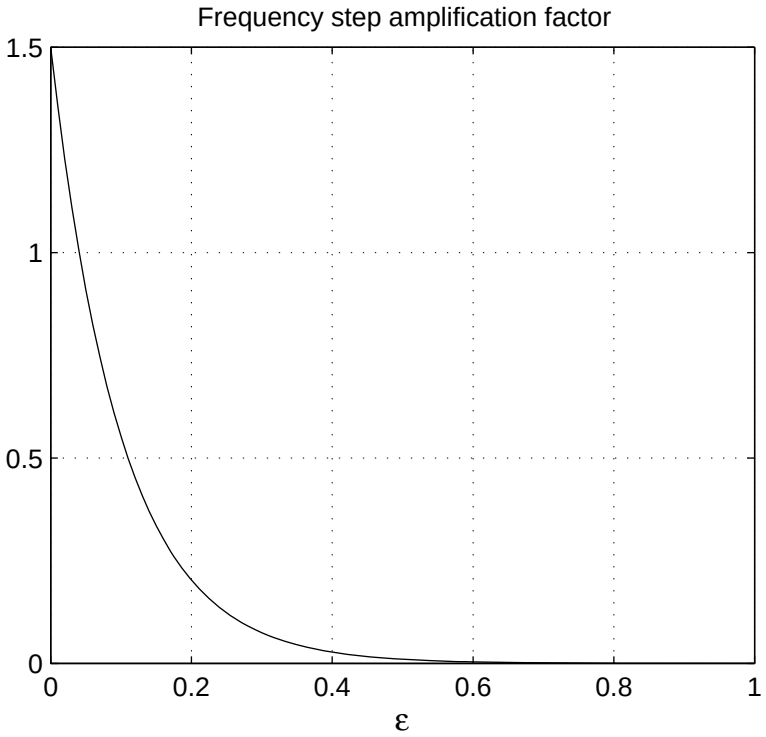

Figure 8 : Frequency step amplification factor

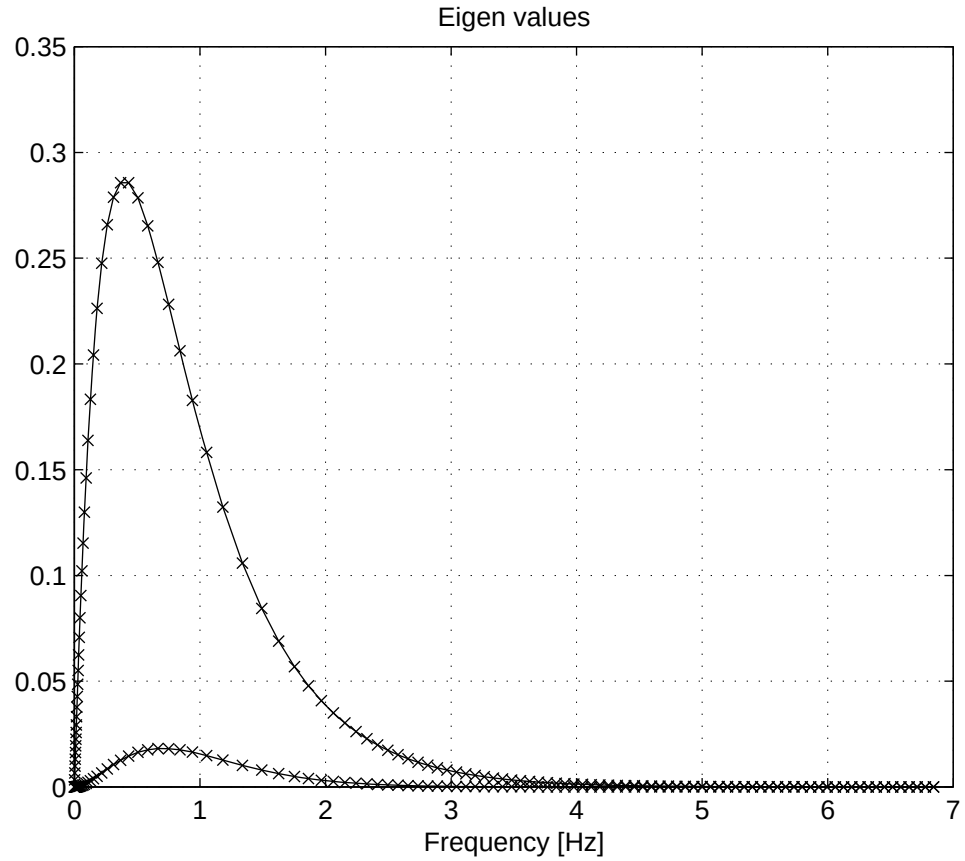

Figure 9 : Example 3 : Representation of the decomposition frequencies

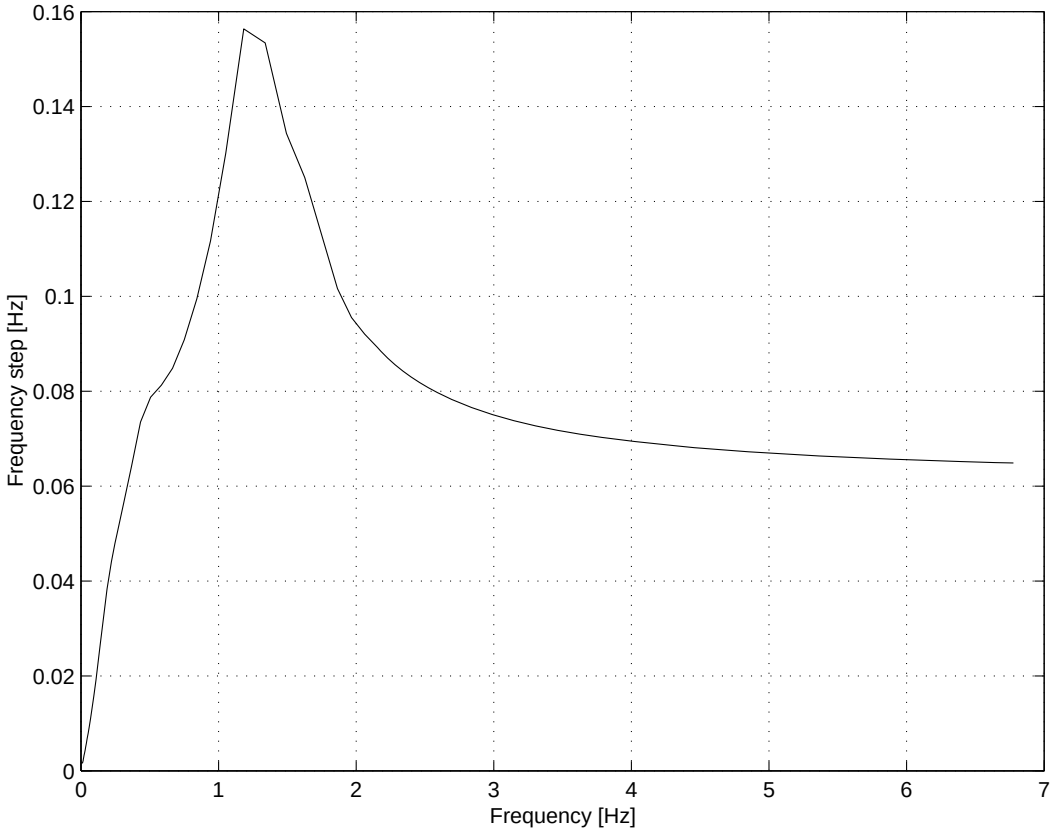

Figure 10 : Example 3 : Evolution of the frequency step for the representation of the eigenvalues