Antiwindup regulation of saturated linear systems

F. Forni, L. Zaccarian, R. Sepulchre

***Abstract*— We consider the output regulation problem for linear systems subject to actuators saturation. A state-feedback unconstrained regulator is transformed into an error-feedback bounded regulator by introducing an observer which is then decomposed into the sum of *unconstrained* and *antiwindup* dynamics. The unconstrained dynamics are regulated towards a predefined reference trajectory, by a control signal u which may violate the saturation bounds during transients. The antiwindup dynamics transiently store the mismatch between unconstrained and constrained dynamics. The antiwindup design also applies to a predefined error-feedback dynamic regulator for the unconstrained system, as in standard antiwindup setting. As a particular case of distinct interest, the design provides a new global asymptotic stabilizer for saturated null-controllable systems.**

I. Introduction

The performance of any control methodology is limited by the finite range of the actuators. This issue is particularly relevant for regulation problems in the presence of magnitude saturated inputs. The dependence of the feedforward control action on exogenous reference or disturbance signals, combined with input saturations, may reduce or invalidate the regulation action of the controller.

Several solutions to the regulation problem for saturated linear systems can be found in the literature. Semiglobal and global solutions to the regulation problem for saturated null controllable linear systems are provided in [10], [6], [19], [3]. They rely on a low-gain approach [10], [6], [19], or on a predefined global bounded stabilizer [3]. A slow convergence rate to the desired output is the main drawback of these constructions. Approaches that combine low-gain and high-gain controllers [13], [14], [2], [24] take into account saturation bounds while improving the convergence rate.

A different perspective is provided by antiwindup methods (see for example [9], [20], [23]), which address constrained stabilization through augmentation of the closed-loop state, by way of the so-called "antiwindup dynamics". Antiwindup methods secure a controller that performs well in the absence of saturation (the so-called "unconstrained controller") against performance degradation in the presence of input saturation. They do so by "storing" in the antiwindup state the transient mismatch between the constrained and unconstrained trajectories.

The present paper proposes an antiwindup solution to the constrained linear regulation problem (i.e. the regulation problem for systems with input saturation $\sigma_M(\cdot)$ of magnitude $M \in \mathbb{R}_{>0}$). The approach is based on the introduction of a dynamic controller ξ, a typical step in the general solution of the regulator problem via error feedback [7, Chapter 8]. Mimicking the solution proposed in [15], the controller dynamics are decomposed as the sum of the *unconstrained* dynamics and the *antiwindup* dynamics, a typical step in antiwindup approaches. This decomposition is tailored to regulation problems: the unconstrained dynamics are regulated towards the desired trajectory $\pi(\omega)$ (ω is the state of the exogenous dynamics), by a signal u which is allowed to violate the saturation constraints during transients, while the antiwindup dynamics are stabilized to zero, to ensure asymptotic regulation of the overall closed-loop system.

By initializing the antiwindup dynamics at zero, the proposed design recovers the classical antiwindup feature of preserving locally, that is, as long as the control does not saturate, the unconstrained response, while it guarantees global regulation through the stabilization of the antiwindup dynamics.

State feedback stabilization is a particular case of special interest of bounded regulation theory. Our general construction provides a new dynamical state-feedback global stabilizer for saturated null-controllable linear systems, which differs from earlier contributions in the literature ([21], [18], [22], [5], [11], [25]). The proposed stabilizer has the advantageous feature of allowing for an arbitrary state feedback in the nonsaturated region of the state-space, augmented with antiwindup dynamics that depend on a single tuning parameter.

The paper is organized as follows. In Section II, we transform a state-feedback unconstrained regulator into an error-feedback bounded regulator by introducing and decomposing an observer/dynamic controller. An antiwindup schemes is provided in Section III, for null-controllable saturated linear systems. The approach directly applies to observer based stabilizing controllers. In Section IV we extend the antiwindup design to closed loops for which a predefined error-feedback regulator is provided. The design is illustrated on a PI controller. In Section V we show how to transform a state-feedback unconstrained stabilizing controller into a saturated (global) stabilizing controller. The design is illustrated on a triple integrator. Conclusions follow.

Notation: For a given vector s, s_i denotes the ith element of s, $|s| = \sqrt{s^T s}$, and $|s|_\infty = \max_i(|s_i|)$. For any given bound

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$M \in \mathbb{R}_{\geq 0}$, the *saturation* function $\sigma_M : \mathbb{R}^n \to \mathbb{R}^b$ satisfies $\sigma_M(s) = \left[\min(M,|s_1|)\frac{s_1}{|s_1|} \; \dots \; \min(M,|s_n|)\frac{s_n}{|s_n|} \right]^T$, for each $s \in \mathbb{R}^n$. $\sigma(u)$ denotes the saturation function with $M = 1$. $\sigma_\infty(u) = u$ for each $u \in \mathbb{R}^n$. dz_M denotes the function $\mathrm{dz}_M(u) = u - \sigma_M(u)$. Matrices are denoted by capital letters. I_n denotes the identity matrix of dimension n. For simplicity of the exposition, the main quantities adopted in the paper are summarized in the next table.

Symbol	Meaning
x	controlled plant state
ω	exogenous system state
$\xi = (x, \omega)$	aggregate state
$\hat{\xi} = (\hat{x}, \hat{\omega})$	observed states
$\xi_{nc} = (x_{nc}, \omega_{nc})$	unconstrained states
$\xi_{aw} = (x_{aw}, \omega_{aw})$	antiwindup states
y, r, d	output, reference, and disturbances
e	mismatch between output and reference
u	control input
$v = (v^{(x)}, v^{(\omega)}), v_{nc}, v_{aw}$	output injection
A, S, B, C, W, Q	plant and exogenous system matrices
$\overline{A}, \overline{B}, \overline{C}$	extended system matrices
Π, Γ	solution of the regulator equation
$K, \overline{L}$	control and observer gains

II. A GENERAL ANTIWINDUP REGULATOR

We introduce the notation used in the paper by recalling the formulation of an observer-based regulator for the unconstrained regulation problem [7] ($M = \infty$). Consider the saturated linear system

$$\begin{aligned} \dot{x} &= Ax + B\sigma_M(u) + W\omega \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and $\omega \in \mathbb{R}^m$ is an exogenous signal. Following [7, Chapter 8] and [3], we assume that ω is generated by a neutrally stable exogenous system

$$\begin{aligned} \dot{\omega} &= S\omega \\ r &= Q\omega \end{aligned} \tag{2}$$

with $S = -S^T$ (skew symmetric). We consider the goal of asymptotically stabilize the regulation error

$$e = Cx - Q\omega. \tag{3}$$

The unconstrained regulator is obtained when a solution exists to the regulator equations

$$\Pi S = A\Pi + B\Gamma + W \tag{4a}$$

$$0 = C\Pi - Q. \tag{4b}$$

which, following [7, Chapter 8], guarantee respectively that the set $\mathcal{I} = \{(x, \omega) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, x = \Pi\omega\}$ is invariant for (1),(2), and that $e = 0$ for each $(x, \omega) \in \mathcal{I}$. The controller is formulated by introducing the observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\sigma_M(u) + W\hat{\omega} + v^{(x)} \\ \dot{\hat{\omega}} &= S\hat{\omega} + v^{(\omega)}, \end{aligned} \tag{5}$$

that is compactly rewritten as

$$\begin{aligned} \dot{\hat{\xi}} &= \overline{A}\hat{\xi} + \overline{B}\sigma_M(u) + v \\ \hat{e} &= \overline{C}\hat{\xi} \\ \overline{A} = \left[\begin{smallmatrix} A & W \\ 0 & S \end{smallmatrix}\right]&, \; \overline{B} = \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right], \; \overline{C} = \left[\begin{smallmatrix} C & -Q \end{smallmatrix}\right], \; v = \begin{bmatrix} v^{(x)} \\ v^{(\omega)} \end{bmatrix}. \end{aligned} \tag{6}$$

Then, assuming detectability of the pair $(\overline{C}, \overline{A})$, global exponential stability of the mismatch $\hat{\xi} - \xi$ (for any input u) is enforced by the output injection

$$v = \overline{L}(\overline{C}\hat{\xi} - e) \tag{7}$$

with $\overline{A} + \overline{LC}$ Hurwitz. Finally, for $M = \infty$ regulation is achieved by selecting the control input u as

$$\alpha(\hat{\xi}) = \Gamma\hat{\omega} + K(\hat{x} - \Pi\hat{\omega}) \tag{8}$$

where K guarantees that $(A + BK)$ is Hurwitz. Explicitly, using $\delta x = \hat{x} - \Pi\hat{\omega}$ and $\sigma_\infty(u) = u$, we have

$$\begin{aligned} \dot{\delta x} &= A\hat{x} + W\hat{\omega} + B\alpha(\hat{x}, \hat{\omega}) - \Pi S\hat{\omega} + v^{(x)} + \Pi v^{(\omega)} \\ &= (A + BK)\delta x + v^{(x)} + W v^{(\omega)} + \underbrace{[A\Pi\hat{\omega} + P\hat{\omega} + B\Gamma\hat{\omega} - \Pi S\hat{\omega}]}_{=0}. \end{aligned} \tag{9}$$

Regulation of x to $\Pi\omega$ is ensured by the convergence of $\hat{\xi}$ to ξ. In fact, (9) is input to state stable (ISS) and its state δx converges to zero as the input v converges to zero.

In order to account for the saturation constraint ($M < \infty$), we decompose the observer dynamics into $\hat{\xi} = \xi_{nc} - \xi_{aw}$ where $\xi_{nc} = \left[x_{nc}^T \; \omega_{nc}^T \right]^T$ and $\xi_{aw} = \left[x_{aw}^T \; \omega_{aw}^T \right]^T$ are, respectively, the *unconstrained* and the *antiwindup* components. Their dynamics are given by

$$\begin{aligned} \dot{\xi}_{nc} &= \overline{A}\xi_{nc} + \overline{B}u + v_{nc} \\ \dot{\xi}_{aw} &= \overline{A}\xi_{aw} + \overline{B}[u - \sigma_M(u)] + v_{aw} \\ u &= \alpha(\xi_{nc}) \end{aligned} \tag{10}$$

where v_{nc} and v_{aw} are output injections that must satisfy

$$v = v_{nc} - v_{aw} \tag{11}$$

to preserve the convergence between $\hat{\xi} = \xi_{nc} - \xi_{aw}$ and ξ. In what follows we characterize explicitly v_{aw}, while v_{nc} is implicitly defined through (11) and (7).

The rationale for the design of the antiwindup is as follows. For $v_{aw} = 0$, $u = \alpha(\xi_{nc})$ regulates the unconstrained dynamics towards the desired output while the mismatch between ξ_{nc} and $\hat{\xi}$, caused only for large enough transients by the presence of saturation, is stored within the state ξ_{aw} of the antiwindup dynamics. Therefore, regulation of the saturated closed loop is achieved by stabilizing ξ_{aw} to zero. Designing the stabilizer v_{aw} to guarantee global regulation can then be stated in terms of the overall dynamics (1),(2),(10),(7), whose state is $(\xi, \xi_{nc}, \xi_{aw})$, with $\xi = (x, \omega)$, as the goal of stabilizing asymptotically the set

$$\mathcal{A} = \{(\xi, \xi_{nc}, \xi_{aw}) = (\left[\begin{smallmatrix} \Pi\omega \\ \omega \end{smallmatrix}\right], \left[\begin{smallmatrix} \Pi\omega \\ \omega \end{smallmatrix}\right], 0)\}. \tag{12}$$

A block diagram of the closed loop is shown in Figure 1.

It is evident that when A is a Hurwitz matrix, $v_{aw} = 0$ guarantees regulation: convergence between ξ and $\hat{\xi}$ is ensured by v, from any initial condition and control signal u; assuming $|\Gamma\omega| < M$ (feasible feedforward term) and initializing $\omega_{aw} = 0$, we have that $\omega_{aw}(t) \equiv 0$ for $t \geq 0$, (x_{nc}, ω_{nc}) converges to $(\Pi\omega, \omega)$, and the control value $u = \alpha(\xi_{nc})$ eventually enters the saturation bounds; then, the mismatch dynamics of $x_{nc} - \hat{x} = x_{aw}$ become $\dot{x}_{aw} = Ax_{aw}$, which converges exponentially to zero.

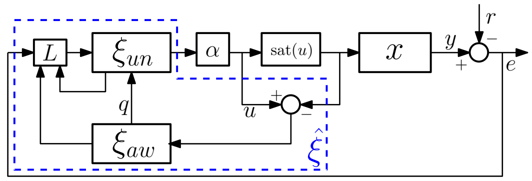

Fig. 1. Antiwindup regulation for saturated null-controllable linear systems: *Unconstrained* and *antiwindup* components ($b = |x_{nc} - \Pi\omega_{nc}|$).

When A is not Hurwitz (or for initial values of $\omega_{aw} \neq 0$), v_{aw} is used to enforce asymptotic stability of the antiwindup dynamics. One may further wish to inject a *correction* signal c with the goal of modifying the unconstrained reference dynamics to improve the transient behavior of the closed loop in the presence of saturation. A simple design for v_{aw} is based on the next lemma, which will be used in next section to achieve global regulation for null-controllable systems.

Lemma 1: Suppose that A has no eigenvalue on the open right half plane. Consider the transformation T such that $J = T^{-1}AT$ is a matrix in real Jordan form. Then, the system $\dot{x} = Ax + q$ is globally asymptotically stabilized at zero by $q = -\rho T\sigma(T^{-1}x)$, for any given $\rho > 0$. ⌟

Proof of Lemma 1: Consider a coordinate transformation $x = Tz$ such that $J = T^{-1}AT$ is a matrix in real Jordan form, that is, whose Jordan blocks $J^{(i)}$ have either the form
$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \text{ or } \begin{bmatrix} D & I_2 & & \\ & \ddots & \ddots & \\ & & D & I_2 \\ & & & D \end{bmatrix}, D = \begin{bmatrix} \lambda & \theta \\ -\theta & \lambda \end{bmatrix}.$$
Using the coordinate transformation, rewrite the system $\dot{x} = Ax + q$ as $\dot{z} = Jz - \rho\sigma(z)$. The stability of z is analyzed by looking separately at each subsystem $\dot{z}^{(i)} = J^{(i)}z^{(i)} - \rho\sigma(z^{(i)})$ associated to the ith Jordan block.

If $J^{(i)}$ has eigenvalues with negative real part, the input $-\rho\sigma(z^{(i)})$ improves the speed of convergence to zero, preserving the stability for any given ρ. Suppose that $J^{(i)}$ has real eigenvalues at zero. Using $z^{(i)} = [z_1, \ldots, z_\nu]^T$, we have
$$\begin{array}{rcl} \dot{z}_1 & = & -\rho\sigma(z_1) + z_2 \\ \dot{z}_2 & = & -\rho\sigma(z_2) + z_3 \\ & \cdots & \\ \dot{z}_\nu & = & -\rho\sigma(z_n) \end{array} \tag{13}$$
where we neglected the index i for simplicity of the exposition, and ν is the dimension of the Jordan block $J^{(i)}$.

Asymptotic stability of the system above is established by induction. • As a base case consider the subsystem Σ_ν given by the z_ν dynamics. Take $V_\nu = \frac{1}{2}z_\nu^2$ then $\dot{V}_\nu = -\rho\sigma(z_\nu)z_\nu$ which establishes GAS (and LES) of $z_\nu = 0$. • Suppose now that the subsystem $\Sigma_{j+1,\nu}$ whose states are $(z_{j+1}, \ldots, z_\nu)$ is GAS at 0. Since $\Sigma_{j,\nu} : (z_{j+1}, \ldots, z_\nu)$ is the cascade of $\Sigma_{j+1,\nu}$ and $\Sigma_j : \dot{z}_j = -\rho\sigma(z_j) + z_{j+1}$, we prove GAS of $\Sigma_{j,\nu}$ by showing that Σ_j is GAS and its solutions are forward complete [17]. To this aim, consider the function $V_j = \frac{1}{2}z_j^2$ which satisfies $\dot{V}_j = -\rho\sigma(z_j)z_j + z_j z_{j+1}$. Then, for $z_{j+1} = 0$ Σ_j is GAS at zero, while $\dot{V}_j \leq z_j^2 + z_{j+1}^2 \leq 2V_j + z_{j+1}^2$, that is, $V_j(t)$ remains bounded for any bounded signal $z_{j+1}(t)$ [8, Lemma A.1], which establish forward completeness of the solutions to Σ_j for any bounded $z_{j+1}(t)$.

A similar inductive argument proves the case of real Jordan blocks with imaginary eigenvalues. We show only the base case. Consider the subsystem $\dot{z}_i = \left[\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right] z_i - \rho\sigma(z_i)$ where now $z_i = \left[\begin{smallmatrix} z_{i,a} & z_{i,b} \end{smallmatrix}\right]^T \in \mathbb{R}^2$. Take the function $V_i = \frac{1}{2}z_i^T z_i$. Then, $\dot{V}_i = -\rho\sigma(z_{i,a})z_{i,a} - \rho\sigma(z_{i,b})z_{i,b}$ which establish GAS of the zero state. ■

III. Main Result

Output feedback global regulation of saturated null-controllable linear systems is achieved by the following design.

Theorem 1: Consider a minimal realization (A, B, C) of (1) and suppose that all the eigenvalues of A in (1) lie in the closed left-half plane. For a given $0 \leq \rho < 1$, consider an invariant set Ω such that for each $\omega(0) \in \Omega$ the solution to (2) satisfies $|\Gamma\omega(t)| \leq \rho M$ for all $t \geq 0$ (feasibility of the feedforward term). Consider gains K such that $A + BK$ is an Hurwitz matrix. Finally, consider a transformation T such that $J_A = T^{-1}AT$ is a matrix in real Jordan form. Then, given any $0 < M < \infty$, there exists $k_1 > 0$ sufficiently small such that, for any $k_2 > 0$ and $\gamma > 0$, the selection
$$\begin{array}{rcl} v_{aw} & = & \begin{bmatrix} -k_1 T\sigma(T^{-1}x_{aw}) \\ -k_2\sigma(\omega_{aw}) \end{bmatrix} \\ u & = & \alpha(\xi_{nc}) \;=\; \Gamma\omega_{nc} + K(x_{nc} - \Pi\omega_{nc}) \end{array} \tag{14}$$
ensures global asymptotic stability of $\mathcal{A}$ in (12) restricted to $\omega \in \Omega$ for the dynamics for (1),(2),(10),(7),(11),(14). ⌟

The design of Theorem 1 is an antiwindup design in the following sense: for feasible feedforward terms $|\Gamma\omega(t)| \leq \rho M$ the regulator (7), (10), (11), (14) provides a solution to the bounded regulation problem ($0 < M < \infty$). Initializing the antiwindup dynamics at zero, the regulator preserves locally, that is, as long as $\sigma_M(u) = u$, the responses induced on the plant by the unconstrained controller (7), (5), (8), while it guarantees global regulation through the stabilization of the antiwindup dynamics.

Remark 1: The terms $-k_1 T\sigma(T^{-1}x_{aw})$ and $-k_2\sigma(\omega_{aw})$ in (14) can be replaced by $-k_1 T\sigma(\gamma T^{-1}x_{aw})$ and $-k_2\sigma(\gamma\omega_{aw})$ with $\gamma > 1$ to improve the convergence to zero near the origin. ⌟

Remark 2: Theorem 1 applies to (possible unstable) plants of the form
$$\dot{x} = Ax + B\sigma_N(\nu) + B\psi(y) + W\omega, \quad N \in \mathbb{R}_{>0}, \tag{15}$$
provided that $|\psi(y)|_\infty < \delta N$, for some $0 < \delta < 1$, and that $\dot{x} = Ax + B\sigma_N(\nu)$ is an null-controllable system. In fact, the precompensation $\nu = u - \psi(y)$ transforms (15) into (1) with $M = (1-\delta)N$. ⌟

Proof of Theorem 1 (sketch): We use the subvectors $v^{(x)}$, $v_{aw}^{(x)}$ of dimension n, and $v^{(\omega)}$, $v_{aw}^{(\omega)}$ of dimension m, to denote $v = \left[\begin{smallmatrix} v^{(x)} \\ v^{(\omega)} \end{smallmatrix}\right]$ and $v_{aw} = \left[\begin{smallmatrix} v_{aw}^{(x)} \\ v_{aw}^{(\omega)} \end{smallmatrix}\right]$.

(i) The dynamics $x_e = x_{nc} - \Pi\omega_{nc}$ converges to a ball of radius $\overline{\lambda}|v_{aw}^{(x)}|$ for $\overline{\lambda} = \frac{\lambda_{\max}(P)}{1-\delta}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$. In fact, v

converges to zero (by construction), and there exists $V = \frac{1}{2}x_e^T P x_e$, $P = P^T > 0$, such that for any $0 < \delta < 1$,

$$\dot{V} \le -\delta |x_e|^2 \quad \text{if } |x_e| \ge \tfrac{\lambda_{\max}(P)}{1-\delta}(|v_{aw}^{(x)}|+|v^{(x)}|+|\Pi v^{(\omega)}|). \tag{16}$$

(ii) ω_{nc} asymptotically converges to ω. Consider $\delta\omega = \omega_{nc} - \omega$. From $\omega_{aw} = \omega_{nc} - \hat{\omega} = \delta\omega + (\omega - \hat{\omega})$, we get

$$\begin{aligned} \dot{\delta\omega} &= S\delta\omega - k_2\sigma(\omega_{aw}) + v^{(\omega)} \\ &= S\delta\omega - k_2\sigma(\delta\omega + (\omega - \hat{\omega})) + v^{(\omega)}, \end{aligned} \tag{17}$$

that is, an exponentially stable linear dynamics forced by $\omega - \hat{\omega}$ and $v^{(\omega)}$, both converging asymptotically to zero (by construction).

(iii) From (i) and (ii), for any initial condition $(\xi(0), \xi_{nc}(0), \xi_{aw}(0))$, there exist $\bar{t} > 0$ and (a sufficiently small) $k_1 > 0$, such that $u(t) - \sigma_M(u(t)) = 0$ for all $t \ge \bar{t}$. In fact, from (16), (14), and from the fact that v converges to 0, taking $k_1 < \frac{M(1-\rho)}{|K|\overline{\lambda}}$ where ρM is the bound on $|\Gamma\omega(t)|$, we get $\dot{V} \le -\delta|x_e|^2$ for $|x_e| \ge \frac{\lambda_{\max}(P)}{1-\delta}|k_1|$. Thus, from (i), x_e converges asymptotically to a ball radious $\overline{\lambda}k_1$. Since $\overline{\lambda}k_1 < \frac{M(1-\rho)}{|K|}$, there exists a time $\bar{t} > 0$ such that

$$\begin{aligned} |u(t)| &\le |\Gamma\omega(t)| + |K(x_{nc}(t) - \Pi\omega_{nc}(t))| \\ &\le \rho M + |K|\overline{\lambda}k_2 \ \le\ M \quad \text{for each } t \ge \bar{t}. \end{aligned} \tag{18}$$

(iv) For $u - \sigma_M(u) = 0$, ξ_{aw} converges asymptotically to zero. In fact, $\frac{d}{dt}|\xi_{aw}|^2 \le \gamma_1|\xi_{aw}|^2 + \gamma_2|\mathrm{dz}_M(u)|^2 + \gamma_3|k_2|^2$, for some $\gamma_1, \gamma_2, \gamma_3 > 0$, which establish forward completeness by [8, Lemma A.1]. By forward completeness, the fact that v converges to 0, and eventually $u - \sigma_M(u) = 0$, ξ_{aw} is GAS. In fact, it is the cascade of two asymptotically stable systems: ω_{aw} satisfies $\frac{d}{dt}\omega_{aw}^T\omega_{aw} = -2k_2\omega_{aw}^T\sigma(\omega_{aw})$, and x_{aw} asymptotically converges to zero by Lemma 1 (for $\omega_{aw} = 0$).

For $\omega \in \Omega$, (i)-(iv) guarantee global asymptotic convergence to the set $\mathcal{A}$. In fact, if ξ_{aw} converges to zero then, from (16), also x_e converges to zero. Global convergence and local exponential stability (near the origin (1),(2),(10),(7),(11),(14) is a linear closed loop) guarantee that $\mathcal{A}$ restricted to $\omega \in \Omega$ is globally asymptotically stable. ■

Remark 3: Replacing the vector $v_{aw}^{(x)} = -k_1 T\sigma(T^{-1}x_{aw})$ in (14) by $v_{aw}^{(x)} = -\max(k_3|x_{nc} - \Pi\omega_{nc}|, k_1)T\sigma(T^{-1}x_{aw})$ still guarantees regulation (for k_1, k_3 sufficiently small) while extending the control authority of $v_{aw}^{(x)}$ for large feedback errors $|x_{nc} - \Pi\omega_{nc}|$. This improves the stabilization to zero of the antiwindup dynamics. ⌟

Remark 4: The vector $v_{aw}^{(\omega)} = -k_2\sigma(\omega_{aw})$ of v_{aw} in Theorem 1 asymptotically stabilizes the antiwindup subsystem ω_{aw} when $\omega_{aw}(0) \neq 0$. In fact, for $\xi_{aw}(0) = 0$, selecting $v_{aw}^{(\omega)} = 0$ still guarantees regulation, since $u - \sigma_M(u)$ does not enter directly the ω_{aw} dynamics, which remains at zero for all $t \ge 0$.

The stabilizing role of $v_{aw}^{(\omega)}$ can be generalized by adding a *correction* term to it in order to modify the trajectories of ξ_{nc} (stored in ξ_{aw}, which is temporarily driven away from zero), to improve the transient performance of the overall closed loop. For example, $v_{aw}^{(\omega)} = -k_2\sigma(\omega_{aw}) + c$, $c = -k_3|\mathrm{dz}_{\rho M}(\Gamma\omega_{nc})|\omega_{nc}$, $k_3 > 0$, enforces a dynamical shift of the exosystem solutions whenever the feedforward constraint ρM is violated. In fact, from the assumption on the feasibility of the feedforward term $\Gamma\omega(t)$ in Theorem 1, if $|\Gamma\omega_{nc}| > \rho M$ then ω_{nc} is certainly an inaccurate estimate of ω. A fast correction of this estimate is one of the goals of the antiwindup scheme. For example, the estimation of an unknown constant disturbance in PI regulation may exceed the control authority, in which case the closed-loop dynamics would benefit from an immediate reduction of the estimated disturbance. ⌟

IV. ANTIWINDUP ON PREDEFINED REGULATORS

The error feedback antiwindup regulator of Section II and III is based on the decomposition of an observer into unconstrained and antiwindup dynamics. In this section we apply the decomposition directly to an error-feedback unconstrained dynamic regulator, thus showing the potential of the antiwindup approach on predefined unconstrained controllers.

For the plant and the exogenous system (1), (2), (3), suppose that the dynamic controller

$$\begin{aligned} \dot{\eta} &= F\eta + Ge \\ u &= H\eta, \end{aligned} \tag{19}$$

where F has no eigenvalues with positive real parts, is an error-feedback regulator for $M = \infty$. The antiwindup regulator is then constructed by introducing an antiwindup augmentation for (1), (19) and by modifying the interconnection between plant and controller, as follows:

$$\begin{aligned} \dot{x}_{aw} &= Ax_{aw} + B(u - \sigma_M(u)) + v_{aw}^{(x)} \\ \dot{\eta}_{aw} &= F\eta_{aw} + GCx_{aw} + v_{aw}^{(\eta)} \\ \dot{\eta} &= F\eta + G(e + Cx_{aw}) \\ u &= H(\eta + \eta_{aw}) \end{aligned} \tag{20}$$

In fact, considering the coordinate transformation $(x_{nc}, x_{aw}, \eta_{nc}, \eta_{aw}) := (x + x_{aw}, x_{aw}, \eta_{nc}, \eta_{aw})$, and using $e_{nc} = Cx_{nc} - Q\omega = e + Cx_{aw}$, we get

$$\begin{aligned} \dot{x}_{nc} &= Ax_{nc} + Bu + W\omega + v_{aw}^{(x)} \\ \dot{\eta}_{nc} &= F\eta_{nc} + Ge_{nc} + v_{aw}^{(\eta)} \\ \dot{x}_{aw} &= Ax_{aw} + B(u - \sigma_M(u)) + v_{aw}^{(x)} \\ \dot{\eta}_{aw} &= F\eta_{aw} + GCx_{aw} + v_{aw}^{(\eta)} \\ u &= H\eta_{nc} \end{aligned} \tag{21}$$

which characterize a system having the structure of (10).

For the analysis of the closed-loop system, the controller η is now replaced by η_{nc} which drives an unconstrained plant in feedback from the error $e_{nc} = e + Cx_{aw}$. Following [1], this allows the controller to take into account saturations effects when it computes the update of its state. η_{nc} may require a transient control action very different from the actual saturated control input applied to the plant. The induced excess of saturation is stored in the antiwindup dynamics. Then, following Theorem 1, it is possible to design

$v_{aw} = (v_{aw}^{(x)}, v_{aw}^{(\eta)})$ to stabilize (x_{aw}, η_{aw}) to zero, recovering regulation.

Remark 5: The proof of Theorem 1 is adapted as follows. The dynamic controller (19) guarantees that $J = \left[\begin{smallmatrix} A & BH \\ GC & F \end{smallmatrix}\right]$ is Hurwitz and that there exists matrices Π and Σ such that $A\Pi + BH\Sigma + W - \Pi S = 0$, $C\Pi - Q = 0$, and $F\Sigma + GC\Pi - GQ - \Sigma S = F\Sigma - \Sigma S = 0$, [7, Chapter 8]. Therefore, following the approach of Theorem 1, $v_{aw} = (v_{aw}^{(x)}, v_{aw}^{(\eta)})$ stabilizes the (error) closed-loop dynamics given by $\xi_e = \left[\begin{smallmatrix} x_{nc} - \Pi\omega \\ \eta_{nc} - \Sigma\omega \end{smallmatrix}\right]$ and $\xi_{aw} = \left[\begin{smallmatrix} x_{aw} \\ \eta_{aw} \end{smallmatrix}\right]$, corresponding to:

$$\begin{array}{rcl} \dot{\xi}_e &=& J\xi_e + v_{aw} \\ \dot{\xi}_{aw} &=& \left[\begin{smallmatrix} A & 0 \\ GC & F \end{smallmatrix}\right] \xi_{aw} + \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right] (u - \sigma_M(u)) + + v_{aw} \\ \dot{\omega} &=& S\omega \\ u &=& H\Sigma\omega + \left[\begin{smallmatrix} 0 & H \end{smallmatrix}\right] \xi_e \end{array} \tag{22}$$

By assumptions, Lemma 1 applies to $\dot{\xi}_{aw} = \left[\begin{smallmatrix} A & 0 \\ GC & F \end{smallmatrix}\right] \xi_{aw} + v_{aw}$ (F has no eigenvalues with positive real parts). ⌟

Example 1: Consider the speed regulation problem of the following simple motor model

$$\dot{w} = -\tau_L + \sigma(u) \tag{23}$$

where w is the speed of the motor, u is the input torque and τ_L is the constant load torque. Without saturation, regulation of w to the *reference* speed w_{ref} is achieved by the PI controller

$$\begin{array}{rcl} u &=& -k_p(w - w_{ref}) + \tau \\ \dot{\tau} &=& -k_i(w - w_{ref}). \end{array} \tag{24}$$

We consider $k_p = 2$ and $k_i = 1$.

We assume that $|\tau_L| \leq \frac{M}{2} = \frac{1}{2}$ and we propose the following antiwindup modification of the PI controller.

$$\begin{array}{rcl} \dot{w}_{aw} &=& u - \sigma(u) + v_{aw}^{(x)} \\ \dot{\tau}_{aw} &=& -k_i w_{aw} + v_{aw}^{(\eta)} \\ \dot{\tau} &=& -k_i(w + w_{aw} - w_{ref}) \\ u &=& -k_p(w + w_{aw} - w_{ref}) + \tau + \tau_{aw} \end{array} \tag{25}$$

where $v_{aw}^{(x)} = -\lambda_1 \max(1, |\tau + \tau_{aw}|)\sigma(\gamma w_{aw})$ and $v_{aw}^{(\eta)} = -\lambda_2\sigma(\gamma\tau_{aw}) + c$. Because the integral part of the PI control is directly connected to the load torque estimation, we consider the correction $c = -\lambda_3 |\mathrm{dz}_{\frac{1}{2}}(|\tau_{nc}|)\tau_{nc}$ which dynamically limits the estimation $\tau_{nc} = \tau + \tau_{aw}$ of the torque within $[-\frac{1}{2}, \frac{1}{2}]$. Finally, the setup of the gains follows Theorem 1, with $\lambda_1 = 0.5$ (sufficiently small), $\lambda_2 = 5$, $\lambda_3 = 5$, and $\gamma = 10$.

A comparison between the two controllers is in Figure 2. The effect of small saturation transients is reported in the upper row: torque load $\tau_L = 0.5$ and initial reference $w_{ref} = 1$. After 20 seconds w_{ref} switches to -1. Large saturation transients are reported in the bottom row. For a constant load of 0.5, we consider an initial reference $w_{ref} = 4$, that switches to -4 after 20 seconds. ⌟

V. Global Stabilization with Bounded Control Revisited

Bounded regulation by error feedback entails bounded stabilization by output feedback. Therefore, Theorem 1 characterizes a global output-feedback stabilizer for saturated

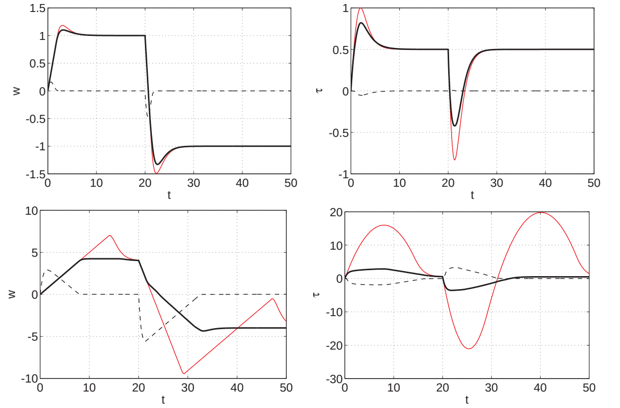

Fig. 2. UP: w_{ref} from 1 to -1, BOTTOM: w_{ref} from 4 to -4. Red line - no AW, black line - AW, dashed line - aw state. $\tau_L = 0.5$.

null-controllable linear systems. For the case when the state of the plant is available, the observer (5) is no longer needed to achieve stabilization, and a straightforward application of the proposed framework brings new insights on the classical problem of achieving global asymptotic stability with bounded state feedback [21], [18], [22], [5], [12], [11], [25]. This problem has been long studied since the early paper [16] where it was shown that global bounded stabilization is achievable if and only if the system is null controllable with bounded inputs. Moreover, all the solutions provided are necessarily nonlinear due to the result of [4] establishing that global stabilization of a triple integrator cannot be achieved by a saturated linear state feedback.

Differently from state-feedback results available in literature, global stabilization is achieved here with a dynamic state-feedback, the antiwindup dynamics. Indeed, an arbitrary nominal controller $u = Kx$ is augmented with an antiwindup dynamics which store transiently the mismatch between the unconstrained and constrained dynamics. We emphasize that the stabilization of the antiwindup dynamics is not constrained by the control directions given by the matrix B, which allows for a simpler global stabilizer, as show in Lemma 1. As a consequence, instead of relying on a low gain stabilizer by nesting of saturations [21] or by summing them [18], the proposed approach uses a simple low gain injection on the whole state vector dynamics. Finally, the overall closed loop is stabilized indirectly by the combination of this global stabilizer of the antiwindup dynamics and a suitable modification of the nominal controller from $u = Kx$ into $u = K(x + x_{aw})$.

The details of the construction are as follows: consider the stabilization problem of (1) for $\omega(t) = 0$ for all $t \geq 0$, and consider an antiwindup dynamics given by $\dot{x}_{aw} = Ax_{aw} + B(u - \sigma_M(u)) + v_{aw}$. Using the coordinate transformation $(x_{nc}, x_{aw}) = (x + x_{aw}, x_{aw})$ we get the equivalent system

$$\begin{array}{rcl} \dot{x}_{nc} &=& Ax_{nc} + Bu + v_{aw} \\ \dot{x}_{aw} &=& Ax_{aw} + B(u - \sigma_M(u)) + v_{aw}. \end{array} \tag{26}$$

From Theorem 1 we obtain the following result.

Corollary 1: Consider a minimal realization (A, B, C) of (1) and suppose that all the eigenvalues of A in (1) lie in

the closed left-half plane. Suppose $\omega = 0$ and consider a transformation T such that $J_A = T^{-1}AT$ is a matrix in real Jordan form.
Then, for any given K such that $(A + BK)$ is Hurwitz, and for any $0 < M < \infty$, there exists a sufficiently small gain $k > 0$ such that $v_{aw} = -kT\sigma(T^{-1}x_{aw})$ and $u = Kx_{nc}$ $(= K(x + x_{aw}))$ globally asymptotically stabilizes (26). ⌟
Clearly, asymptotic stability of (26) entails asymptotic stabilization of (1). Corollary 1 guarantees global asymptotic stability for any arbitrary linear controller $u = Kx$, by a simple tuning of the scalar parameter k. Beyond its simplicity, a particular (antiwindup) feature of the design is that it guarantees global stability while preserving locally the performance of the predetermined linear controller. The proposed approach is illustrated on the stabilization of a triple integrator in Example 2 below.

Example 2: Consider the stabilization problem of a saturated triple integrator. The dynamics is given by $\dot{x} = Ax + B\sigma(u)$ with $A = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$. The controller $u = Kx$ locally stabilizes the system using the LQR gains $K = \left[\begin{smallmatrix} -3.2 & -4.3 & -2.9 \end{smallmatrix}\right]$. The eigenvalues of $(A + BK)$ have real part smaller than -0.5.

Introducing x_{aw} and using the coordinate transformation $(x_{nc}, x_{aw}) = (x + x_{aw}, x_{aw})$ we get $\dot{x}_{nc} = Ax_{nc} + Bu + v_{aw}^{(x)}$, $\dot{x}_{aw} = Ax_{aw} + B(u - \sigma(u)) + v_{aw}^{(x)}$. The control input is then selected as $u = Kx_{nc} = K(x + x_{aw})$. Finally, the antiwindup stabilizer $v_{aw}^{(x)}$ is given by $v_{aw}^{(x)} = -k\sigma(\gamma x_{aw})$, with $k = 0.5$ and $\gamma = 10$.

Simulations in Figure 3 show the case $x_0 = \left[\begin{smallmatrix} 0 & 1 & 0.5 \end{smallmatrix}\right]$. Recall that, according to [4], no saturated linear controller can globally stabilize the triple integrator, so it is not surprising to see nonconverging red curves in Figure 3. ⌟

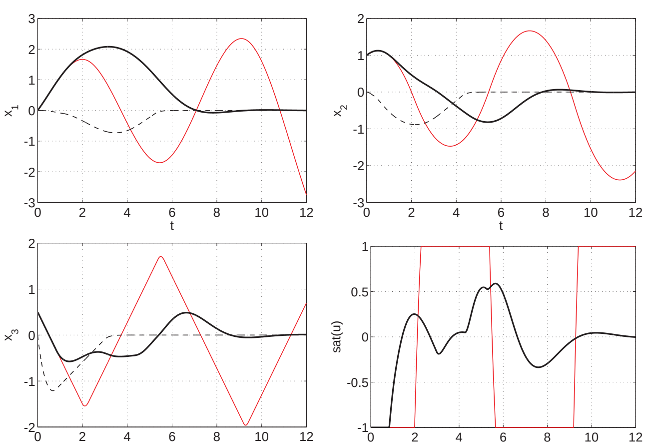

Fig. 3. Control of a saturated triple integrator. Large initial conditions. Red line - no AW, black line - AW, dashed line - antiwindup state.

VI. Conclusions

The paper proposes an antiwindup approach to the constrained output regulation problem: the unconstrained controller dynamics is augmented with an antiwindup state that transiently store the mismatch between the constrained and the unconstrained dynamics. Illustration on a standard PI regulator problem suggest that the proposed approach is valuable even in standard bounded regulator problems. When the state of the plant is available, the proposed approach provides also a new global asymptotic stabilizer for saturated null-controllable linear systems, as shown in the stabilization of a triple integrator. Finally, the results reported in [15] suggest to investigate extensions of the proposed antiwindup approach to classes of port-Hamiltonian systems.

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