Antiwindup regulation of saturated linear systems

F. Forni, L. Zaccarian, R. Sepulchre

Abstract—We consider the output regulation problem for linear systems subject to actuators saturation. A state-feedback unconstrained regulator is transformed into an error-feedback bounded regulator by introducing an observer which is then decomposed into the sum of unconstrained and antiwindup dynamics. The unconstrained dynamics are regulated towards a predefined reference trajectory, by a control signal which may violate the saturation bounds during transients. The antiwindup dynamics transiently store the mismatch between unconstrained and constrained dynamics. The antiwindup design also applies to a predefined error-feedback dynamic regulator for the unconstrained system, as in standard antiwindup setting. As a particular case of distinct interest, the design provides a new global asymptotic stabilizer for saturated null-controllable systems.

I. INTRODUCTION

The performance of any control methodology is limited by the finite range of the actuators. This issue is particularly relevant for regulation problems in the presence of magnitude saturated inputs. The dependence of the feedforward control action on exogenous reference or disturbance signals, combined with input saturations, may reduce or invalidate the regulation action of the controller.

Several solutions to the regulation problem for saturated linear systems can be found in the literature. Semiglobal and global solutions to the regulation problem for saturated null controllable linear systems are provided in [10], [6], [19], [3]. They rely on a low-gain approach [10], [6], [19], or on a predefined global bounded stabilizer [3]. A slow convergence rate to the desired output is the main drawback of these constructions. Approaches that combine low-gain and high-gain controllers [13], [14], [2], [24] take into account saturation bounds while improving the convergence rate.

A different perspective is provided by antiwindup methods (see for example [9], [20], [23]), which address constrained stabilization through augmentation of the closed-loop state, by way of the so-called “antiwindup dynamics”. Antiwindup methods secure a controller that performs well in the absence of saturation (the so-called “unconstrained controller”) against performance degradation in the presence of input saturation. They do so by “storing” in the antiwindup state the transient mismatch between the constrained and unconstrained trajectories.

The present paper proposes an antiwindup solution to the constrained linear regulation problem (i.e. the regulation problem for systems with input saturation $\sigma_M(\cdot)$ of magnitude $M \in \mathbb{R}_{>0}$). The approach is based on the introduction of a dynamic controller $\zeta$, a typical step in the general solution of the regulator problem via error feedback [7, Chapter 8]. Mimicking the solution proposed in [15], the controller dynamics are decomposed as the sum of the unconstrained dynamics and the antiwindup dynamics, a typical step in antiwindup approaches. This decomposition is tailored to regulation problems: the unconstrained dynamics are regulated towards the desired trajectory $\pi(\omega)$ ( $\omega$ is the state of the exogenous dynamics), by a signal $u$ which is allowed to violate the saturation constraints during transients, while the antiwindup dynamics are stabilized to zero, to ensure asymptotic regulation of the overall closed-loop system.

By initializing the antiwindup dynamics at zero, the proposed design recovers the classical antiwindup feature of preserving locally, that is, as long as the control does not saturate, the unconstrained response, while it guarantees global regulation through the stabilization of the antiwindup dynamics.

State feedback stabilization is a particular case of special interest of bounded regulation theory. Our general construction provides a new dynamical state-feedback global stabilizer for saturated null-controllable linear systems, which differs from earlier contributions in the literature ([21], [18], [22], [5], [11], [25]). The proposed stabilizer has the advantageous feature of allowing for an arbitrary state feedback in the nonsaturated region of the state-space, augmented with antiwindup dynamics that depend on a single tuning parameter.

The paper is organized as follows. In Section II, we transform a state-feedback unconstrained regulator into an error-feedback bounded regulator by introducing and decomposing an observer/dynamic controller. An antiwindup schemes is provided in Section III, for null-controllable saturated linear systems. The approach directly applies to observer based stabilizing controllers. In Section IV we extend the antiwindup design to closed loops for which a predefined error-feedback regulator is provided. The design is illustrated on a PI controller. In Section V we show how to transform a state-feedback unconstrained stabilizing controller into a saturated (global) stabilizing controller. The design is illustrated on a triple integrator. Conclusions follow.

Notation: For a given vector $s$, $s_i$ denotes the $i$th element of $s$, $|s| = \sqrt{s^T s}$, and $|s|_\infty = \max_i(|s_i|)$. For any given bound

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors.

F. Forni and R. Sepulchre are with the Department of Electrical Engineering and Computer Science, Université de Liège, 4000 Liège, Belgium, f.forni@ulg.ac.be, r.sepulchre@ulg.ac.be. L. Zaccarian is with CNRS, LAAS; Université de Toulouse, F-31077, Toulouse, France, zaccarian@laas.fr.
Then, assuming detectability of the pair \((\bar{C}, \bar{A})\), global exponential stability of the mismatch \(\xi - \dot{\xi}\) (for any input \(u\)) is enforced by the output injection

\[ v = \bar{T}(\bar{C}\xi - \epsilon) \quad (7) \]

with \(\bar{T} + \bar{T}\bar{C}\) Hurwitz. Finally, for \(M = \infty\) regulation is achieved by selecting the control input \(u\) as

\[ \alpha(\xi) = \Gamma\omega + K(\dot{x} - \Pi\omega) \quad (8) \]

where \(K\) guarantees that \((A + BK)\) is Hurwitz. Explicitly, using \(\delta x = \dot{x} - \Pi\omega\) and \(\sigma(u) = u\), we have

\[
\begin{align*}
\dot{x} &= A\dot{x} + B\sigma_M(u) + W\omega \\
0 &= CT - Q.
\end{align*}
\]

Then, assuming detectability of the pair \((\bar{C}, \bar{A})\), global exponential stability of the mismatch \(\xi - \dot{\xi}\) (for any input \(u\)) is enforced by the output injection

\[ v = \bar{T}(\bar{C}\xi - \epsilon) \quad (7) \]

with \(\bar{T} + \bar{T}\bar{C}\) Hurwitz. Finally, for \(M = \infty\) regulation is achieved by selecting the control input \(u\) as

\[ \alpha(\xi) = \Gamma\omega + K(\dot{x} - \Pi\omega) \quad (8) \]

where \(K\) guarantees that \((A + BK)\) is Hurwitz. Explicitly, using \(\delta x = \dot{x} - \Pi\omega\) and \(\sigma(u) = u\), we have

\[
\begin{align*}
\delta x &= A\delta x + W\omega + B\alpha(\dot{x}, \omega) - \Pi S\omega + v(x) + \Pi\nu(\omega) \\
&= (A + BK)\delta x + v(x) + Wv(\omega) + \Pi(\omega) - \Pi S\omega.
\end{align*}
\]

Regulation of \(x\) to \(\Pi\omega\) is ensured by the convergence of \(\xi\) to \(\xi\). In fact, (9) is input to state stable (ISS) and its state \(\delta x\) converges to zero as the input \(v\) converges to zero.

In order to account for the saturation constraint \((M < \infty)\), we decompose the observer dynamics into \(\dot{\xi} = \xi - \xi\) where \(\xi = [x^T, \omega^T] \) and \(\xi = [x^T, \omega^T] \) are, respectively, the unconstrained and the antiwindup components. Their dynamics are given by

\[
\begin{align*}
\dot{\xi} &= \bar{A}\xi + \bar{B}u + \nu \\
\xi &= \bar{A}\xi + \bar{B}(u - \sigma_M(u)) + \nu.
\end{align*}
\]

where \(\nu\) and \(\nu\) are output injections that must satisfy

\[ v = \nu - \nu \quad (11) \]

The unconstrained regulator is obtained when a solution exists to the regulator equations

\[ I \Pi S = A\Pi + B\Gamma + W \]

where \(A = A\hat{x} + B\sigma_M(u) + W\omega\) and \(\sigma_M(u) = u - \sigma_M(u)\). The controller is formulated by introducing the observer

\[ \begin{align*}
\dot{x} &= A\dot{x} + B\sigma_M(u) + W\omega + v(x) \\
\dot{\omega} &= S\omega + v(\omega),
\end{align*} \]

that is compactly rewritten as

\[
\begin{align*}
\dot{\xi} &= \bar{A}\xi + \bar{B}\sigma_M(u) + v \\
\dot{\epsilon} &= \bar{C}\xi \\
\bar{A} &= \begin{bmatrix} A & W \\ S & \bar{S} & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ \bar{B} & \bar{S} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C \quad 0 \end{bmatrix}, \quad v = \begin{bmatrix} u(x) \\ \nu(\omega) \end{bmatrix}.
\end{align*}
\]

A block diagram of the closed loop is shown in Figure 1. It is evident that when \(A\) is a Hurwitz matrix, \(v = 0\) guarantees regulation: convergence between \(\xi\) and \(\xi\) is ensured by \(v\), from any initial condition and control signal \(u\); assuming \(|\Gamma\omega| < M\) (feasible feedforward term) and initializing \(\omega = 0\), we have that \(\omega(t) \leq 0\) for \(t \geq 0\), \([x_n, \omega_n] \) converges to \((\Pi\omega, \omega)\), and the control value \(u = \alpha(\xi)\) eventually enters the saturation bounds; then, the mismatch dynamics of \(x_n - \dot{x} = x_n\) become \(\dot{x} = Ax_n\), which converges exponentially to zero.
When $A$ is not Hurwitz (or for initial values of $\omega_{aw} \neq 0$), $v_{aw}$ is used to enforce asymptotic stability of the antiwindup dynamics. One may further wish to inject a correction signal $c$ with the goal of modifying the unconstrained reference dynamics to improve the transient behavior of the closed loop in the presence of saturation. A simple design for $v_{aw}$ is based on the next lemma, which will be used in next section to achieve global regulation for null-controllable systems.

**Lemma 1:** Suppose that $A$ has no eigenvalue on the open right half plane. Consider the transformation $T$ such that $J = T^{-1}AT$ is a matrix in real Jordan form. Then, the system $\dot{x} = Ax + q$ is globally asymptotically stabilized at zero by $q = -\rho T \sigma(T^{-1}x)$, for any given $\rho > 0$.

*Proof of Lemma 1:* Consider a coordinate transformation $x = Tz$ such that $J = T^{-1}AT$ is a matrix in real Jordan form, that is, whose Jordan blocks $J^i$ have either the form

$$
\begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & \cdots & 0 & \lambda
\end{bmatrix},
$$

or

$$
\begin{bmatrix}
D & I_2 \\
I_2 & D
\end{bmatrix},
$$

where $D = \begin{bmatrix} \lambda & \theta \\ -\theta & \lambda \end{bmatrix}$.

Using the coordinate transformation, rewrite the system $\dot{x} = Ax + q$ as $\dot{z} = Jz - \rho \sigma(z)$. The stability of $z$ is analyzed by looking separately at each subsystem $\dot{z}^i = J^i z^i - \rho \sigma(z^i)$ associated to the $i$th Jordan block.

If $J^i$ has eigenvalues with negative real part, the input $-\rho \sigma(z^i)$ improves the speed of convergence to zero, preserving the stability for any given $\rho$. Suppose that $J^i$ has real eigenvalues at zero. Using $z^i = [z_1, \ldots, z_\nu]^T$, we have

$$
\dot{z}_1 = -\rho \sigma(z_1) + z_2 \\
\dot{z}_2 = -\rho \sigma(z_2) + z_3 \\
\cdots \\
\dot{z}_\nu = -\rho \sigma(z_\nu)
$$

(13)

where we neglected the index $i$ for simplicity of the exposition, and $\nu$ is the dimension of the Jordan block $J^i$.

Asymptotic stability of the system above is established by induction. As a base case consider the subsystem $\Sigma_\nu$ given by the $z_\nu$ dynamics. Take $V_\nu = \frac{1}{\rho^2} z_\nu^2$ then $\dot{V}_\nu = -\rho \sigma(z_\nu) z_\nu$, which establishes GAS (and LES) of $z_\nu = 0$. Suppose now that the subsystem $\Sigma_{\nu+1}$, whose states are $(z_{\nu+1}, \ldots, z_{\nu})$ is GAS at 0. Since $\Sigma_{\nu+1} : (z_{\nu+1}, \ldots, z_{\nu})$ is the cascade of $\Sigma_{\nu+1} : (z_{\nu+1}, \ldots, z_{\nu})$ and $\Sigma_\nu : z_\nu = -\rho \sigma(z_\nu) + z_{\nu+1}$, we prove GAS of $\Sigma_{\nu+1}$ by showing that $\Sigma_\nu$ is GAS and its solutions are forward complete [17]. To this aim, consider the function $V_j = \frac{1}{2} z_j^2$ which satisfies $\dot{V}_j = -\rho \sigma(z_j) z_{j} + z_{j+1}$. Then, for $z_{\nu+1} = 0$ $\Sigma_{\nu+1}$ is GAS at zero, while $\dot{V}_j \leq z_{j+1}^2 \leq 2V_j + z_{j+1}^2$, that is, $V_j(t)$ remains bounded for any bounded signal $z_{\nu+1}(t)$ [8, Lemma A.1], which establish forward completeness of the solutions to $\Sigma_j$ for any bounded $z_{\nu+1}(t)$.

A similar inductive argument proves the case of real Jordan blocks with imaginary eigenvalues. We show only the base case. Consider the subsystem $\dot{z}_i = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} z_i - \rho \sigma(z_i)$ where now $z_i = \begin{bmatrix} z_{i,a} & z_{i,b} \end{bmatrix}^T \in \mathbb{R}^2$. Take the function $V_i = \frac{1}{2} z_i^2$. Then, $\dot{V}_i = -\rho \sigma(z_{i,a}) z_{i,a} - \rho \sigma(z_{i,b}) z_{i,b}$ which establish GAS of the zero state.

**III. MAIN RESULT**

Output feedback global regulation of saturated null-controllable linear systems is achieved by the following design.

**Theorem 1:** Consider a minimal realization $(A,B,C)$ of (1) and suppose that all the eigenvalues of $A$ in (1) lie in the closed left-half plane. For a given $0 \leq \rho < 1$, consider an invariant set $\Omega$ such that for each $\omega(0) \in \Omega$ the solution to (2) satisfies $|\Gamma \omega(t)| \leq \rho M$ for all $t \geq 0$ (feasibility of the feedforward term). Consider gains $K$ such that $A + BK$ is an Hurwitz matrix. Finally, consider a transformation $T$ such that $J_T = T^{-1}AT$ is a matrix in real Jordan form. Then, given any $0 < M < \infty$, there exists $k_2 > 0$ sufficiently small such that, for any $k_2 > 0$ and $\gamma > 0$, the selection

$$
v_{aw} = \begin{bmatrix} -k_1 T \sigma(T^{-1} x_{aw}) \\ -k_2 \sigma(\omega_{aw}) \end{bmatrix}
$$

(14)

ensures global asymptotic stability of $A$ in (12) restricted to $\omega \in \Omega$ for the dynamics for (1),(2),(10),(7),(11),(14).

The design of Theorem 1 is an antiwindup design in the following sense: for feasible feedforward terms $|\Gamma \omega(t)| \leq \rho M$ the regulator (7), (10), (11), (14) provides a solution to the bounded regulation problem $(0 < M < \infty)$. Initializing the antiwindup dynamics at zero, the regulator preserves locally, that is, as long as $s_{\Omega}(u) = u$, the responses induced on the plant by the unconstrained controller (7), (5), (8), while it guarantees global regulation through the stabilization of the antiwindup dynamics.

**Remark 1:** The terms $-k_1 T \sigma(T^{-1} x_{aw})$ and $-k_2 \sigma(\omega_{aw})$ in (14) can be replaced by $-k_1 T \sigma(T^{-1} x_{aw})$ and $-k_2 (\gamma T^{-1} x_{aw})$ with $\gamma > 1$ to improve the convergence to zero near the origin.

**Remark 2:** Theorem 1 applies to (possible unstable) plants of the form

$$
\dot{x} = Ax + B \sigma_N(\nu) + B \psi(y) + W \omega, \quad N \in \mathbb{R}_{>0},
$$

(15)

provided that $|\psi(y)|_\infty < \delta N$, for some $0 < \delta < 1$, and that $\dot{x} = Ax + B \sigma_N(\nu)$ is a null-controllable system. In fact, the precompensation $\nu = u - \psi(y)$ transforms (15) into (1) with $M = (1 - \delta) N$.

**Proof of Theorem 1 (sketch):** We use the subvectors $v(x)$, $v_{aw}$ of dimension $n$, and $\psi(\omega)$, $\psi_{aw}$ of dimension $m$, to denote $v = \begin{bmatrix} v(x) \\ v(\omega) \end{bmatrix}$ and $v_{aw} = \begin{bmatrix} v(x) \\ v_{aw} \end{bmatrix}$.

1. The dynamics $x_e = x_{nc} - \Pi \omega_{nc}$ converges to a ball of radius $\bar{\lambda}_{v_{aw}}/|v_{aw}|$ for $\bar{\lambda} = \frac{\lambda_{\max}(P)}{1 - \delta} \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$. In fact, $v$...
converges to zero (by construction), and there exists $V = \frac{1}{2}x^TPx$, $P = PT > 0$, such that for any $0 < \delta < 1$,
\[
\dot{V} \leq -\delta|x_e|^2 \text{ if } |x_e| \geq \frac{\lambda_{\max}(P)}{1-\delta}(|v(x)| + |\eta(x)| + |\Pi v(\omega)|).
\] (16)

(ii) $\omega_{nc}$ asymptotically converges to $\omega$. Consider $\delta_\omega = \omega_{nc} - \omega = \delta \omega (\omega - \omega)$, we get
\[
\dot{\delta}_\omega = S \delta_\omega - k_2 \sigma(\omega_{aw}) + v(\omega) = \dot{\delta}_\omega - k_2 \sigma(\omega + (\omega - \omega)) + v(\omega),
\] (17)

that is, an exponentially stable linear dynamics forced by $\omega - \omega$ and $v(\omega)$, both converging asymptotically to zero (by construction).

(iii) From (i) and (ii), for any initial condition $(\xi(0), \xi_{nc}(0), \xi_{aw}(0))$, there exist $T > 0$ and (a sufficiently small) $k_1 > 0$, such that $u(t) - \sigma_M(u(t)) = 0$ for all $t \geq T$. In fact, from (16), (14), and from the fact that $v$ converges to 0, taking $k_1 < \frac{M(1-\rho)}{K|\lambda_1|}$ where $\rho M$ is the bound on $|\Gamma(\omega)|$, we get
\[
\dot{V} \leq -\delta|x_e|^2 \text{ for } |x_e| \geq \frac{\lambda_{\max}(P)}{1-\delta}|k_1|.
\]
Thus, from (i), $x_e$ converges asymptotically to a ball radius $\lambda_1 k_1$. Since $\lambda_1 k_1 < \frac{M(1-\rho)}{|\lambda_1|}$, there exists a time $T > 0$ such that
\[
|u(t)| \leq |\Gamma(\omega(t)) + [K(x_{nc}(t) - \Pi \omega_{nc}(t))]| \leq \rho M + |K|\lambda k_2 \leq M \text{ for each } t \geq T.
\] (18)

(iv) For $u - \sigma_M(u) = 0$, $\xi_{aw}$ converges asymptotically to zero. In fact, $\frac{d}{dt}||\xi_{aw}|^2 \leq \gamma_1 |\xi_{aw}|^2 + \gamma_2 |dz_M(u)|^2 + \gamma_3 |k_2|^2$, for some $\gamma_1, \gamma_2, \gamma_3 > 0$, which establish forward completeness by [8, Lemma A.1]. By forward completeness, the fact that $v$ converges to 0, and eventually $u - \sigma_M(u) = 0$, $\xi_{aw}$ is GAS. In fact, it is the cascade of two asymptotically stable systems: $\omega_{aw}$ satisfies $\frac{d}{dt} \omega_{aw} = -2k_2 \sigma(\omega_{aw})$, and $x_e$ asymptotically converges to zero by Lemma 1 (for $\omega_{aw} = 0$).

For $\omega \in \Omega$, (i)-(iv) guarantee global asymptotic convergence to the set $\mathcal{A}$. In fact, if $\xi_{aw}$ converges to zero then, from (16), also $x_e$ converges to zero. Global convergence and local exponential stability (near the origin) (1),(2),(10),(7),(11),(14) is a linear closed loop) guarantee that $\mathcal{A}$ restricted to $\omega \in \Omega$ is globally asymptotically stable.

Remark 3: Replacing the vector $v_{aw}^{(x)} = -k_1 T \sigma(T^{-1}x_{aw})$ in (14) by $v_{aw}^{(x)} = -\max(k_3|x_{nc} - \Pi \omega_{nc}|, k_1) T \sigma(T^{-1}x_{aw})$ still guarantees regulation (for $k_1, k_3$ sufficiently small) while extending the control authority of $v_{aw}$ for large feedback errors $|x_{nc} - \Pi \omega_{nc}|$. This improves the stabilization to zero of the antiwindup dynamics.

Remark 4: The vector $v_{aw}^{(\omega)} = -k_2 \sigma(\omega_{aw})$ of $v_{aw}$ in Theorem 1 asymptotically stabilizes the antiwindup subsystem $\omega_{aw}$ when $\omega_{aw}(0) \neq 0$. In fact, for $\xi_{aw}(0) = 0$, selecting $v_{aw}^{(\omega)} = 0$ still guarantees regulation, since $u - \sigma_M(u)$ does not enter directly the $\omega_{aw}$ dynamics, which remains at zero for all $t \geq 0$.

The stabilizing role of $v_{aw}^{(\omega)}$ can be generalized by adding a correction term to it in order to modify the trajectories of $\xi_{nc}$ (stored in $\xi_{aw}$, which is temporarily driven away from zero), to improve the transient performance of the overall closed loop. For example, $v_{aw}^{(\omega)} = -k_2 \sigma(\omega_{aw}) + c$, $c = -k_3 |dz_M(\omega_{nc})| \omega_{nc}$, $k_3 > 0$, enforces a dynamical shift of the exosystem solutions whenever the feedforward constraint $\rho M$ is violated. In fact, from the assumption on the feasibility of the feedforward term $\Gamma u(t)$ in Theorem 1, if $|\Gamma \omega_{nc}| > \rho M$ then $\omega_{nc}$ is certainly an inaccurate estimate of $\omega$. A fast correction of this estimate is one of the goals of the antiwindup scheme. For example, the estimation of an unknown constant disturbance in PI regulation may exceed the control authority, in which case the closed-loop dynamics would benefit from an immediate reduction of the estimated disturbance.

IV. ANTIWINDUP ON PREDEFINED REGULATORS

The error feedback antiwindup regulator of Section II and III is based on the decomposition of an observer into unconstrained and antiwindup dynamics. In this section we apply the decomposition directly to an error-feedback unconstrained dynamic regulator, thus showing the potential of the antiwindup approach on predefined unconstrained controllers.

For the plant and the exogenous system (1), (2), (3), suppose that the dynamic controller
\[
\begin{align*}
\dot{\eta} &= F \eta + G e \\
u &= H \eta,
\end{align*}
\] (19)
where $F$ has no eigenvalues with positive real parts, is an error-feedback unconstrained dynamic regulator, thus showing the potential of the antiwindup approach on predefined unconstrained controllers.

In fact, considering the coordinate transformation $(x_{nc}, x_{aw}, \eta_{nc}, \eta_{aw}) := (x + x_{aw}, x_{aw}, \eta_{nc}, \eta_{aw})$, and using $e_{nc} = C x_{nc} - Q = e + C x_{aw}$, we get
\[
\begin{align*}
\dot{x}_{nc} &= A x_{nc} + B (u - \sigma_M(u)) + v_{aw}^{(x)} \\
\dot{\eta}_{nc} &= F \eta_{nc} + G e_{nc} + v_{aw}^{(\eta)} \\
u &= H(\eta + \eta_{aw})
\end{align*}
\] (20)

which characterize a system having the structure of (10).

For the analysis of the closed-loop system, the controller $\eta$ is now replaced by $\eta_{nc}$ which drives an unconstrained plant in feedback from the error $e_{nc} = e + C x_{aw}$. Following [1], this allows the controller to take into account saturations effects when it computes the update of its state. $\eta_{nc}$ may require a transient control action very different from the actual saturated control input applied to the plant. The induced excess of saturation is stored in the antiwindup dynamics. Then, following Theorem 1, it is possible to design
Remark 5: The proof of Theorem 1 is adapted as follows. The dynamic controller (19) guarantees that $J = (A + BH)F$ is Hurwitz and that there exists matrices $\Pi$ and $\Sigma$ such that $\Pi + BHB + W - \Pi S = 0$, $C\Pi - Q = 0$, and $F\Sigma + G\Pi - GQ - \Sigma S = F\Sigma - \Sigma S = 0$ [7, Chapter 8]. Therefore, following the approach of Theorem 1, $v_{aw} = (v_{aw}^x, v_{aw}^\eta)$ stabilizes the (error) closed-loop dynamics given by $\xi_e = [x_{nc} - \Pi u_n - \Sigma \eta]$, and $\xi_{aw} = [x_{aw}^x, \eta_{aw} - \Sigma \omega]$, corresponding to:

$$
\dot{\xi}_e = J\xi_e + v_{aw} \\
\dot{\xi}_{aw} = \left[ A_{GC}^0 \right] \xi_{aw} + \left[ B_1 \right] (u - \sigma_M(u)) + v_{aw}
$$

(22)

By assumptions, Lemma 1 applies to $\dot{\xi}_{aw} = \left[ A_{GC}^0 \right] \xi_{aw} + v_{aw}$ ($F$ has no eigenvalues with positive real parts).

Example 1: Consider the speed regulation problem of the following simple motor model

$$
\dot{w} = -\tau_L + \sigma(u)
$$

(23)

where $w$ is the speed of the motor, $u$ is the input torque and $\tau_L$ is the constant load torque. Without saturation, regulation of $w$ to the reference speed $w_{ref}$ is achieved by the PI controller

$$
u = -k_p(w - w_{ref}) + \tau
$$

(24)

We consider $k_p = 2$ and $k_i = 1$.

We assume that $|\tau_L| \leq \frac{M}{2} = \frac{1}{2}$ and we propose the following anticontrol modification of the PI controller.

$$
u_{aw} = u - \sigma(u) + v_{aw}^x
$$

$$
\dot{\tau}_{aw} = -k_t w_{aw} + v_{aw}^\eta
$$

$$
\dot{\tau} = -k_t (w - w_{ref})
$$

(25)

where $v_{aw}^x = -\lambda_1 \max(1, |\tau + \tau_{aw}|) \sigma(\gamma w_{aw})$ and $v_{aw}^\eta = -\lambda_2 \sigma(\gamma w_{aw}) + c$. Because the integral part of the PI control is directly connected to the load torque estimation, we consider the correction $e = -\lambda_1 [d\tau_S(\tau_{nc}) |\tau_{nc}]$ which dynamically limits the estimation $\tau_{nc} = \tau + \tau_{aw}$ of the torque within $[-\frac{1}{2}, \frac{1}{2}]$. Finally, the setup of the gains follows Theorem 1, with $\lambda_1 = 0.5$ (sufficiently small), $\lambda_2 = 5$, $\lambda_3 = 5$, and $\gamma = 10$.

A comparison between the two controllers is in Figure 2. The effect of small saturation transients is reported in the upper row: torque load $\tau_L = 0.5$ and initial reference $w_{ref} = 1$. After 20 seconds $w_{ref}$ switches to $-1$. Large saturation transients are reported in the bottom row. For a constant load of 0.5, we consider an initial reference $w_{ref} = 4$, that switches to $-4$ after 20 seconds.

V. GLOBAL STABILIZATION WITH BOUNDED CONTROL REVISITED

Bounded regulation by error feedback entails bounded stabilization by output feedback. Therefore, Theorem 1 characterizes a global output-feedback stabilizer for saturated null-controllable linear systems. For the case when the state of the plant is available, the observer (5) is no longer needed to achieve stabilization, and a straightforward application of the proposed framework brings new insights on the classical problem of achieving global asymptotic stability with bounded state feedback [21], [18], [22], [5], [12], [11], [25]. This problem has been long studied since the early paper [16] where it was shown that global bounded stabilization is achievable if and only if the system is null controllable with bounded inputs. Moreover, all the solutions provided are necessarily nonlinear due to the result of [4] establishing that global stabilization of a triple integrator cannot be achieved by a saturated linear state feedback.

Differently from state-feedback results available in literature, global stabilization is achieved here with a dynamic state-feedback, the anticontrol dynamics. Indeed, an arbitrary nominal controller $u = Kx$ is augmented with an anticontrol dynamics which store transiently the mismatch between the unconstrained and constrained dynamics. We emphasize that the stabilization of the anticontrol dynamics is not constrained by the control directions given by the matrix $B$, which allows for a simpler global stabilizer, as show in Lemma 1. As a consequence, instead of relying on a low gain stabilizer by nesting of saturations [21] or by summing them [18], the proposed approach uses a simple low gain injection on the whole state vector dynamics. Finally, the overall closed loop is stabilized indirectly by the combination of this global stabilizer of the anticontrol dynamics and a suitable modification of the nominal controller from $u = Kx$ into $u = K(x + x_{aw})$.

The details of the construction are as follows: consider the stabilization problem of (1) for $\omega(t) = 0$ for all $t \geq 0$, and consider an anticontrol dynamics given by $\dot{x}_{aw} = Ax_{aw} + Bu + v_{aw}$. Using the coordinate transformation $(x_{nc}, x_{aw}) = (x + x_{aw}, x_{aw})$ we get the equivalent system

$$
\dot{x}_{nc} = Ax_{nc} + Bu + v_{aw}
$$

(26)

$$
\dot{x}_{aw} = Ax_{aw} + B(u - \sigma_M(u)) + v_{aw}
$$

From Theorem 1 we obtain the following result.

Corollary 1: Consider a minimal realization $(A, B, C)$ of (1) and suppose that all the eigenvalues of $A$ in (1) lie in
the closed left-half plane. Suppose \( \omega = 0 \) and consider a transformation \( T \) such that \( J_A = T^{-1}AT \) is a matrix in real Jordan form.

Then, for any given \( K \) such that \( (A + BK) \) is Hurwitz, and for any \( 0 < M < \infty \), there exists a sufficiently small gain \( k > 0 \) such that \( v_{aw} = -kT\sigma(T^{-1}x_{aw}) \) and \( u = Kx_{nc} (= K(x + x_{aw})) \) globally asymptotically stabilizes (26). Clearly, asymptotic stability of (26) entails asymptotic stabilization of (1). Corollary 1 guarantees global asymptotic stability for any arbitrary linear controller \( u = Kx \), by a simple tuning of the scalar parameter \( k \). Beyond its simplicity, a particular (antiwindup) feature of the design is that it guarantees global stability while preserving locally the performance of the predetermined linear controller. The proposed approach is illustrated on the stabilization of a triple integrator in Example 2 below.

**Example 2:** Consider the stabilization problem of a saturated triple integrator. The dynamics is given by \( \dot{x} = Ax + Bu \) with \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). The controller \( u = Kx \) locally stabilizes the system using the LQR gains \( K = [-3.2 -4.3 -2.9] \). The eigenvalues of \( (A + BK) \) have real part smaller than \(-0.5\).

Introducing \( x_{aw} \) and using the coordinate transformation \( (x_{nc}, x_{aw}) = (x + x_{aw}, x_{aw}) \) we get \( x_{nc} = Ax_{nc}+Bu_{aw}(x) \), \( x_{aw} = Ax_{aw}+B(u-\sigma(u)) + vr \). The control input is then selected as \( u = Kx_{nc} = K(x + x_{aw}) \). Finally, the antiwindup stabilizer \( v_{aw} \) is given by \( v_{aw} = -k(\gamma x_{aw}) \), with \( k = 0.5 \) and \( \gamma = 10 \).

Simulations in Figure 3 show the case \( x_0 = [0 \ 1 \ 0.5] \). Recall that, according to [4], no saturated linear controller can globally stabilize the triple integrator, so it is not surprising to see nonconverging red curves in Figure 3.

![Fig. 3. Control of a saturated triple integrator. Large initial conditions. Red line - no AW, black line - AW, dashed line - antiwindup state.](image)

**VI. CONCLUSIONS**

The paper proposes an antiwindup approach to the constrained output regulation problem: the unconstrained controller dynamics is augmented with an antiwindup state that transiently store the mismatch between the constrained and the unconstrained dynamics. Illustration on a standard PI regulator problem suggest that the proposed approach is valuable even in standard bounded regulator problems. When the state of the plant is available, the proposed approach provides also a new global asymptotic stabilizer for saturated null-controllable linear systems, as shown in the stabilization of a triple integrator. Finally, the results reported in [15] suggest to investigate extensions of the proposed antiwindup approach to classes of port-Hamiltonian systems.

**REFERENCES**


