Prevalence of “nowhere analyticity”

Françoise Bastin
Institute of Mathematics B37, University of Liège
B-4000 Liège, Belgium
E-mail: F.Bastin@ulg.ac.be

Céline Esser
Institute of Mathematics B37, University of Liège
B-4000 Liège, Belgium
E-mail: Celine.Esser@ulg.ac.be

Samuel Nicolay
Institute of Mathematics B37, University of Liège
B-4000 Liège, Belgium
E-mail: S.Nicolay@ulg.ac.be

Abstract

This note brings a complement to the study of genericity of functions which are nowhere analytic mainly in a measure-theoretic sense. We extend this study in Gevrey classes of functions.

1 Introduction

In what follows, $C^\infty([0,1])$ denotes the linear space of the functions of class C^∞ on $[0,1]$, endowed with the sequence $(p_k)_{k\in\mathbb{N}_0}$ of semi-norms defined by

$$p_k(f) = \sup_{0\leq j\leq k} \sup_{x\in[0,1]} |f^{(j)}(x)|$$

or equivalently with the distance d defined by

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(f-g)}{1+p_k(f-g)}.$$

2010 *Mathematics Subject Classification*: 46E10, 26E05, 37C20
Key words and phrases: analytic functions, generic property, prevalence.

This space is a Fréchet space.

If f is a C^∞ function on an open interval containing x_0, its Taylor series at x_0 is denoted by

$$T(f, x_0)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n.$$

We say that f *is analytic* at x_0 if $T(f, x_0)$ converges to f on an open neighbourhood of x_0; if this is not the case, we say that f *has a singularity* at x_0. A function with a singularity at each point of an interval is called *nowhere analytic* on the interval. In case of a closed interval $[a, b]$, the convergence of the Taylor series $T(f, a)$ and $T(f, b)$ is only considered on the restriction to $[a, b]$.

If f has a singularity at x_0, then either the radius of convergence of the series is 0 (i.e. the series only converges at x_0), or the series converges in some neighbourhood of x_0 but the limit does not represent f, as small as one takes the neighbourhood of x_0. Following [B4, R1], we say that x_0 is a *Pringsheim singularity* if the radius of convergence at x_0 is 0 and a *Cauchy singularity* in the other case.

In [R3], Rudin gives explicit examples of functions with a Pringsheim singularity at each point. In [SZ], the authors obtain the property that the set of functions in $C^\infty([0, 1])$ with a Pringsheim singularity at each point of $[0, 1]$ is a *residual or comeager* subset of $C^\infty([0, 1])$ (i.e. contains a countable intersection of dense open sets of $C^\infty([0, 1])$). This implies that this set is dense in $C^\infty([0, 1])$ (by Baire's theorem) and also means that it is "generic" in the topological sense of "genericity". More general results were obtained in [B1, B2, R1] and the introduction of the paper [B1] gives a wide historical context of successive results in this direction. Let us also mention that results on "algebraic genericity" were also obtained in [B1], where it is proved that the set of functions in $C^\infty([0, 1])$ with a Pringsheim singularity at each point of $[0, 1]$ contains, except for zero, a dense linear submanifold. Concerning Cauchy singularities, Boas ([B3]) already showed in 1935 that there is no function with a Cauchy singularity at each point.

Another notion of "genericity" has also been introduced in order to generalize the concept of "almost everywhere" for Lebesgue measure to infinite dimensional spaces. Following Hunt, Sauer and Yorke ([HSY]), a Borel set B in a complete metric linear space E is said to be *shy* if there exists a Borel probability measure μ on E with compact support such that $\mu(B + x) = 0$ for any $x \in E$ (it is also known that the property on the support is automatically satisfied if E is separable). More generally, any set is called *shy* if it is contained in a shy Borel set. A set is *prevalent* if it is the complement of a shy set and a prevalent property is a property which holds on a prevalent set.

In this short note, we show (section 2) that the set of nowhere analytic functions is prevalent. This result is already mentioned in [S] but in [S], one of the arguments is the fact that the set

$$A(I, x_I) := \{f \in C^\infty([0,1]) \ : \ T(f, x_I) \text{ converges to } f \text{ on } I\} \qquad (1)$$

(where I is a closed interval of $[0,1]$ with x_I as center point) is closed in $C^\infty([0,1])$. But this is certainly not possible since the set of polynomials is included in $A(I, x_I)$ and also dense in $C^\infty([0,1])$. Concerning the prevalence of the set of functions in $C^\infty([0,1])$ with a Pringsheim singularity at each point of $[0,1]$, as far as we know, the problem is still open.

We also examine (section 3) the set of functions which are "nowhere Gevrey differentiable", using the classical definition of Gevrey classes (see the definition in the concerned section). In this case, we also obtain generic results, both in the topological and in the prevalence points of view. Since analytic functions are a particular class of Gevrey type functions, these results generalize those obtained in the analytic case. However, we kept separated sections since analytic functions are somehow more classical than Gevrey-type ones and since the result of section 2 directly brings a complement to an already mentioned one in the literature.

2 Genericity in the prevalent sense

Let us first introduce a sufficient condition for a subset to be prevalent. Let P be a finite dimensional subspace of the topological vector space E. If $f : \mathbb{R}^n \to P$ is a topological isomorphism, the measure $\mathcal{L}_P$ defined by

$$\mathcal{L}_P(B) = \mathcal{L}(f^{-1}(B \cap P))$$

for any Borel set B of E and where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^n$, is called a *Lebesgue measure on E supported by P*. Using this definition, a finite dimensional subspace $P \subset E$ is a *probe* for a subset T of E if there exists a Borel set B which contains the complement of T in E and such that

$$\mathcal{L}_P(B + e) = 0$$

for any $e \in E$. A sufficient condition for T to be prevalent is to have a probe for it.

Using this condition, it is straightforward to prove the following (which simply means that a proper linear space which is a Borel set is always shy).

Remark 2.1. *If A is a non-empty Borel subset of E such that the complement of A is a linear subspace of E, then A is prevalent.*

Proof. A probe is given by the linear span of any element a of A. Indeed, since $B = E \setminus A$ is linear, for every $e \in E$, the set

$$\{\alpha \in \mathbb{R} \ : \ \alpha a + e \ \in B\}$$

contains only one element, so has Lebesgue measure 0. □

Proposition 2.2. *The set of nowhere analytic functions on $[0,1]$ is a prevalent subset of $C^\infty([0,1])$.*

Proof. For any closed subinterval I of $[0,1]$ and x_I the center point of I, let $A(I, x_I)$ be the set given by (1). Since a function which is analytic at a point is analytic in a neighbourhood of this point, the set of nowhere analytic functions is the complement of the union of all $A(I, x_I)$ over rational subintervals $I \subset [0,1]$. Any countable union of shy sets is shy ([HSY]) and therefore, it is enough to prove that every $A(I, x_I)$ is shy. Since $A(I, x_I)$ is a proper linear subspace of $C^\infty([0,1])$, using the remark 2.1, this will be done if we show that it is a Borel set.

For any $j, n \in \mathbb{N}$, let

$$F_{n,j} = \bigcap_{x \in I} \left\{ f \in C^\infty([0,1]) : \ |T_j(f, x_I)(x) - f(x)| \leq \frac{1}{n} \right\},$$

where

$$T_j(f, x_I)(x) = \sum_{k=0}^{j} \frac{f^{(k)}(x_I)}{k!} (x - x_I)^k.$$

The definition of the topology of $C^\infty([0,1])$ and the fact that only a finite number of derivatives are involved directly imply that $F_{n,j}$ is closed in $C^\infty([0,1])$.

Using typical properties of power series, the convergence of $T(f, x_I)$ on I is equivalent to uniform convergence on I. Hence

$$A(I, x_I) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} F_{n,j},$$

which shows that $A(I, x_I)$ is a countable intersection of a countable union of closed sets, so a Borel set. □

3 About Gevrey classes

Following [CC, R2], for a real number $s > 0$ and an open subset Ω of $\mathbb{R}$, an infinitely differentiable function f in Ω is said to be *Gevrey differentiable of order s at $x_0 \in \Omega$* if there exist a compact neighbourhood I of x_0 and constants $C, h > 0$ such that

$$\sup_{x \in I} \left| f^{(n)}(x) \right| \leq C h^n (n!)^s, \quad \forall n \in \mathbb{N}_0.$$

It is clear that if a function is Gevrey differentiable of order s at x_0, it is also Gevrey differentiable of any order $s' > s$ at x_0. Remark also that the case $s = 1$ corresponds to analyticity.

Let us give an example of an element f of $C^\infty(\mathbb{R})$ such that, for any $x_0 \in \mathbb{R}$ and any $s > 0$, f is not Gevrey differentiable of order s at x_0.

Lemma 3.1. *Let λ_k, $k \in \mathbb{N}$, be a sequence of strictly positive numbers such that*

$$\lambda_k \geq (k+1)^{(k+1)^2} \quad \& \quad \lambda_{k+1} \geq 2 \sum_{j=1}^{k} \lambda_j^{2+k-j}, \qquad \forall k \in \mathbb{N}$$

and let f be the function defined on $\mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} c_k e^{i\lambda_k x} \quad \text{with } c_k = \lambda_k^{1-k}, \ k \in \mathbb{N}.$$

This function belongs to the class $C^\infty(\mathbb{R})$ and it is not Gevrey of order s at x_0, for any $x_0 \in \mathbb{R}$ and $s > 0$.

Proof. Let us first remark that such a sequence can be easily constructed (using a recurrence procedure).

Now, for every $n, k \in \mathbb{N}$, we have $c_k \lambda_k^n = \lambda_k^{1+n-k}$, hence the series

$$\sum_{k=1}^{\infty} c_k \lambda_k^n e^{i\lambda_k x}$$

is uniformly and absolutely convergent on $\mathbb{R}$. Thus $f \in C^\infty(\mathbb{R})$.

On the other hand, for every $n \in \mathbb{N},\ n \geq 2$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left|f^{(n)}(x)\right| &= \left|\sum_{k=1}^{n-1} \lambda_k^{n+1-k} e^{i\lambda_k x} \ + \ \lambda_n e^{i\lambda_n x} \ + \ \sum_{k>n} \lambda_k^{n+1-k} e^{i\lambda_k x}\right| \\
&\geq \lambda_n - \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \\
&\geq \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \\
&\geq \lambda_{n-1}^2 - \sum_{j=0}^{+\infty} \frac{1}{\lambda_j^j} \\
&\geq n^{2n^2} - e \geq \frac{1}{2} n^{2n^2}.
\end{aligned}$$

Then, given strictly positive s, C, h, we have

$$n^{2n^2} = n^{n^2} \ (n^n)^n \geq Ch^n (n^n)^s \geq Ch^n (n!)^s$$

for n large enough. So we are done. □

Now, in order to generalize the results about "nowhere analyticity", we say that a function $f \in C^\infty([0,1])$ is *nowhere Gevrey differentiable* on $[0,1]$ if f is not Gevrey differentiable of order s at x_0, for any $x_0 \in [0,1]$ and $s \geq 1$, where the compact neighbourhoods I are considered in $[0,1]$.

We are going to use the same arguments as in the analytic case to prove the following result.

Proposition 3.2. *The set of nowhere Gevrey differentiable functions is a prevalent subset of* $C^\infty([0,1])$.

Proof. Let us first note that the definition of "nowhere Gevrey differentiability" given above directly leads to the following description: the set of nowhere Gevrey differentiable functions of $C^\infty([0,1])$ is the complement of

$$\bigcup_{s \in \mathbb{N}} \bigcup_{I \subset [0,1]} B(s, I)$$

where I denotes a rational subinterval of $[0,1]$ and

$$B(s,I) = \left\{ f \in C^\infty([0,1]) : \ \exists C, h > 0 \text{ such that } \sup_{x \in I} |f^{(n)}(x)| \leq Ch^n (n!)^s \ \forall n \in \mathbb{N}_0 \right\}.$$

Hence, since in a complete metric space countable union of shy sets is shy ([HSY]), the result will be proved if we show that every $B(s, I)$ is shy.

To get this, it suffices to prove that $B(s, I)$ is a proper linear subspace of $C^\infty([0,1])$ which is also a Borel set.

It is direct to see that $B(s, I)$ is a linear subspace of $C^\infty([0,1])$ and strictly included in $C^\infty([0,1])$ (using for example the previous constructive lemma). We also have

$$B(s,I) = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}_0} \left\{ f \in C^\infty([0,1]) : \sup_{x\in I} |f^{(n)}(x)| \le m^{n+1}\ (n!)^s \right\},$$

where

$$\left\{ f \in C^\infty([0,1]) : \sup_{x\in I} |f^{(n)}(x)| \le m^{n+1}\ (n!)^s \right\}$$

is closed in $C^\infty([0,1])$. Hence $B(s, I)$ is a Borel subset of $C^\infty([0,1])$. □

Now, let us show that the generic result also holds in the topological sense.

Proposition 3.3. *The set of nowhere Gevrey differentiable functions is a residual subset of* $C^\infty([0,1])$.

Proof. We use the same definition as before for the set $B(s, I)$. So, as we already remarked previously, the set of nowhere Gevrey differentiable functions of $C^\infty([0,1])$ is the complement of

$$\bigcup_{s\in\mathbb{N}} \bigcup_{I\subset[0,1]} B(s,I)$$

where I denotes a rational subinterval of $[0,1]$. We also have

$$B(s,I) = \bigcup_{m\in\mathbb{N}} A(s,I,m)$$

where

$$A(s,I,m) = \left\{ f \in C^\infty([0,1]) : \sup_{x\in I} |f^{(n)}(x)| \le m^{n+1}\ (n!)^s,\ \forall n \in \mathbb{N}_0 \right\}.$$

To conclude, it suffices then to notice that the closed set $A(s, I, m)$ is a set with empty interior since it is included in $B(s, I)$ which is a proper linear subspace of the locally convex space $C^\infty([0,1])$. □

This last proposition can also be obtained as a special case of the following result of [B1]: For each infinite set $M \subset \mathbb{N}_0$ and each sequence $(c_n)_{n\in\mathbb{N}_0}$ of strictly positive numbers, the family

$$\left\{ f \in C^\infty([0,1]) : \exists \text{ infinitely many } n \in M \quad \text{with } |f^{(n)}(x)| > c_n\ \forall x \in [0,1] \right\}$$

is a residual subset of $C^\infty([0,1])$. Indeed, for $c_n = (n!)^n$ and $M = \mathbb{N}_0$, this last family is contained in the set of nowhere Gevrey differentiable functions, since for any $s \in \mathbb{N}, h, C > 0$, one has $(n!)^n > Ch^n(n!)^s$ for n sufficiently large.

4 Some additional results

Some generalizations can be obtained with similar techniques as the ones used in the previous sections.

Proposition 4.1. *For any sequence $(c_n)_{n\in\mathbb{N}_0}$, $c_n > 0$ $\forall n$, the set*

$$\big\{f \in C^\infty([0,1]) : \forall I \subset [0,1],\ \sup_{n\in\mathbb{N}_0} \frac{\sup_{x\in I}|f^{(n)}(x)|}{c_n} = +\infty\big\}$$

(where I denotes rational subintervals) is a prevalent subset of $C^\infty([0,1])$.

Proof. The complement of this set can be written as

$$\bigcup_{I\subset[0,1]} D_I \qquad \text{with} \qquad D_I := \big\{f \in C^\infty([0,1]) : \sup_{n\in\mathbb{N}_0} \frac{\sup_{x\in I}|f^{(n)}(x)|}{c_n} < +\infty\big\}.$$

Since in a complete metric space, a countable union of shy sets is shy, it suffices then to show that D_I is shy for each I. This is obtained as before: D_I is a linear space, strictly included in $C^\infty([0,1])$ (as shows an explicit example of [B1], Remark 2.2), and is a Borel set since it can be written as a countable union of countable intersection of closed sets

$$D_I = \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}_0} \{f \in C^\infty([0,1]) : \sup_{x\in I}|f^{(n)}(x)| \leq kc_n\}. \quad \square$$

This last proposition is a generalization of Proposition 3.2. Indeed, taking again $c_n = (n!)^n$, we see that the set mentioned in the proposition above is contained in the set of nowhere Gevrey differentiable functions.

One can also make some remarks about classes of type $C^{\{M_n\}}$ (in relation with quasi-analyticity, [R3], Chapter 19): if $(M_n)_{n\in\mathbb{N}_0}$ is a sequence of strictly positive numbers and I a subinterval of $[0,1]$, let us denote by $C^{\{M_n\}}(I)$ the linear space

$$\left\{f \in C^\infty([0,1]) :\ \exists C, h > 0 \text{ such that } \sup_{x\in I}|f^{(n)}(x)| \leq Ch^n M_n\ \forall n \in \mathbb{N}_0\right\}.$$

In fact, with $M_n = (n!)^s$, we have $B(s,I) = C^{\{M_n\}}(I)$. So, with the same computations as those used when dealing with $B(s,I)$, one gets the fact that $C^{\{M_n\}}(I)$ is shy in $C^\infty([0,1])$. As a consequence, the set of functions of $C^\infty([0,1])$ which are "nowhere in $C^{\{M_n\}}$" (that is to say, which do not belong to $C^{\{M_n\}}(I)$, for any interval I) is prevalent in $C^\infty([0,1])$.

Acknowledgements

The authors want to thank the referee for helpful comments and suggestions, which led to improvement of presentation and generalization of results.

References

[B1] L. Bernal-Gonzalez, *Lineability of sets of nowhere analytic functions,* J. Math. Anal. Appl., 340 (2008), 1284-1295

[B2] L. Bernal-Gonzalez, *Functions with successive derivatives everywhere large or small,* (Spanish, English summary) Collect. Math., 38 (1987), 117-122

[B3] R.P. Boas, *A theorem on analytic functions of a real variable,* Bull. Amer. Math. Soc., 41 (1935), 4, 233-236

[B4] R. P. Boas, *When is a C^∞ Function Analytic?,* The mathematical Intelligencer, 11 (1989), 40

[C] F.S. Cater, *Differentiable, Nowhere Analytic Functions,* Amer. Math. Monthly, 91 (1984), 618-624

[CC] S.Y. Chung and J. Chung, *There exist no gaps between Gevrey differentiable and nowhere Gevrey differentiable,* Proc. Amer. Math. Soc., 133, 3 (2005), 859-863

[HSY] B. R. Hunt, T. Sauer, et J. A. Yorke, *Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces,* Bull. Amer. Math. Soc., 27 (1992), 2, 217-238

[R1] T.I. Ramsamujh, *Nowhere Analytic C^∞ Functions,* J. Math. Anal. Appl., 160 (1991), 263-266

[R2] L. Rodino, Linear Partial Differential Operators in Gevrey Spaces, Word Scientific, London, 1993

[R3] W. Rudin, Real and Complex Analysis, McGraw-Hill, London, 1966.

[SZ] H. Salzmann, K. Zeller, *Singularitäten unendlicht oft differenzierbarer Funktionen,* Math. Z., 62 (1955), 354-367

[S] H. Shi, *Prevalence of some known typical properties,* Acta Math. Univ. Comeniane, LXX (2001), 2, 185-192