Computer Assisted Mechanics and Engineering Sciences, 8: 261–270, 2001.

On a general decomposition of the error of an approximate stress field in elasticity

Jean-François Debongnie
Méthodes de Fabrication, Université de Liège, Chemin des Chevreuils, 1 – Sart Tilman, 4000 Liège, Belgium

Pierre Beckers
Infographie, Université de Liège, Chemin des Chevreuils, 1 – Sart Tilman, 4000 Liège, Belgium

(Received June 30, 2000)

The errors of finite element approximations are analysed in a general frame, which is completely independent from the way through which the approximate solution was obtained. It is found that the error always admits decomposition in two terms, namely the equilibrium error and the compatibility error, which are orthogonal. Each of these admits upper and lower bounds that can be computed in a post-processing scheme.

1. Introduction

Most a posteriori error evaluations were developed for the standard case of displacement elements. As is well known, the errors then appear as equilibrium defects and the question is thus reduced to measuring disequilibrium.

A first way consists in isolating local defects, which are of two types, namely interior and interface defects. The problem is then to define a suitable norm. At this stage, it has to be noted that the often cited combination of a L^2-norm for interior defects and an interface L^2-norm for interface defects leads to too strong a norm, which fails to converge to zero where the mesh is refined, as it can be seen on a lot of counterexamples. This drawback is partially solved by the Gago procedure [9] where the two added norms are beforehand multiplied by a suited factor depending on the mesh size. But this correction does not take due account of the fact that there is compensation between both kinds of defects. As pointed in [2] and implicitly admitted by Diez et al. [5], disequilibrium has in fact to be measured by its maximum work, that is to say, by a norm in the dual space.

A second classical way consists in generating a so-called "better" stress field and to compare it to the initial one, so as to obtain their energetic distance. It is assumed that this distance may be considered as a suitable error measure. However, if one excepts the case of Ladevèze's approach [10–12] in which the connected field is a statically admissible one, the quality of a stress field is generally defined from heuristics only, so that the method is quite questionable.

Concerning equilibrium elements, it is clear that the errors are incompatibilities of the stress field. The corresponding first approach has been reported by Zhong [15]. Recently this kind of approach has also been presented in [13].

But now, what about finite elements that are nor of the displacement nor of the equilibrium type? Such elements are numerous, including mixed and hybrid elements and also non-conforming displacement elements that pass the patch test. Here, disequilibrium and incompatibility are both present, so that the previous approaches do not apply.

We therefore found useful to take the problem in a more general form, analysing the error of any stress approximation, independently of the way by which it is obtained. In this frame, it is possible to define a general decomposition of the error in an equilibrium error and a compatibility error, which are orthogonal. Both errors admit a double expression, from which naturally follow upper and lower bound approximations. The upper bound coincides with the Ladevèze estimator for the equilibrium error and with the dual-Ladevèze estimator for the compatibility error. The lower bound approach is related to norms in the dual space and, for the equilibrium error, it is in perfect accordance with the views of Diez et al. Such a treatment of the compatibility error was, to our knowledge, never proposed.

2. General notations

Let us consider a bounded domain Ω, with boundary Γ. To any *displacement* field u defined in Ω is associated a *strain* field ε by a differential operator ∂,

$$\varepsilon = \partial u. \tag{1}$$

This operator verifies the following integration by part property,

$$\int_\Omega \sigma^T \partial u \, \mathrm{d}\Omega = \int_\Gamma u^T L^T \sigma \, \mathrm{d}\Gamma - \int_\Omega u^T \partial^T \sigma \, \mathrm{d}\Omega, \tag{2}$$

where the conjugate operator ∂^T and a surface operator L^T appear.

Stress-strain relations, known in elasticity as Hooke's law, are of the form

$$\sigma = H\varepsilon, \tag{3}$$

where H is a bounded uniformly positive definite matrix.

Displacements are submitted to the condition that on a non-zero measure part Γ_1 of the boundary, their values are given,

$$u = \bar{u} \qquad \text{on } \Gamma_1 . \tag{4}$$

Finally, the equilibrium equations, in their strong form, are

$$\partial^T \sigma + f = 0 \qquad \text{in } \Omega \tag{5}$$

and

$$L^T \sigma = t \qquad \text{on } \Gamma_2 = \Gamma - \Gamma_1 . \tag{6}$$

This set of equations of course requires some conditions on the different fields, which will be explicitly given in the following.

3. The stress space

Let S be the space of square-integrable stress fields, equipped with the scalar product

$$(\sigma, \tau) = \int_\Omega \sigma^T H^{-1} \tau \, \mathrm{d}\Omega. \tag{7}$$

It is a variant of the classical L^2 space. The corresponding norm will be called *energetic norm.*

Among all possible stress fields, a special mention is due to stresses that derive from a displacement field by the relation

$$\sigma = H\partial u.$$

In order to obtain a bounded energy, one must have

$$\int_\Omega (\partial u)^T H \partial u \,\mathrm{d}\Omega < \infty . \tag{8}$$

This condition defines *finite energy displacements*, whose space will be noted V. A particular subspace $V_o \subset V$ is composed of those displacements that verify

$$u = 0 \qquad \text{on } \Gamma_1 . \tag{9}$$

Such displacements will be called *homogeneous*. It now follows from Korn's inequality that the quantity

$$\|u\|_{V_0} = \left(\int_\Omega (\partial u)^T H \partial u \,\mathrm{d}\Omega \right)^{\frac{1}{2}} \tag{10}$$

is a suitable norm on V_0, and that V_0, equipped with this norm, is a complete space. Consequently, its image in S,

$$C_0 = \{\sigma = H\partial u,\ u \in V_0\} = H\partial V_0 , \tag{11}$$

is one to one and C_0 is a closed subspace of S. In the following, such stress fields deriving from a homogeneous displacements will be called *homogeneous compatible stress fields.*

Let us now turn to the orthogonal complement of C_0. It is the subspace E_0 of stress fields σ verifying

$$(\sigma, H\partial\nu) = \int_\Omega \sigma^T \partial \nu \,\mathrm{d}\Omega = 0, \tag{12}$$

for any displacement $\nu \in V_0$. This condition is the weak form of the equilibrium equations

$$\begin{aligned} \partial^T \sigma &= 0 \qquad \text{in } \Omega, \\ L^T \sigma &= 0 \qquad \text{on } \Gamma_2 . \end{aligned} \tag{13}$$

Consequently, E_0 is the *space of self-stresses.* C_0 being closed, one has

$$C_0 \perp E_0 \qquad \text{and} \qquad E_0 \perp C_0 . \tag{14}$$

4. The displacement approach

The prescribed boundary value $\bar{u}$ of the displacement on Γ_1 may be viewed as the trace on Γ_1 of a finite energy displacement field, which may also be noted $\bar{u}$ without any risk of confusion. The true displacement field therefore lies in the linear manifold $\bar{u}+V_0$. This reduces the elastic problem to the determination of a displacement field $u \in \bar{u} + V_0$ corresponding to stresses that are in equilibrium. The condition for this is that, for each $\nu \in V_0$,

$$\int_\Omega (\partial u)^T H \partial \nu \,\mathrm{d}\Omega = \int_\Omega f^T \nu \,\mathrm{d}\Omega + \int_{\Gamma_2} t^T \nu \,\mathrm{d}\Gamma . \tag{15}$$

The solution of this variational problem exists and is unique, from Korn's inequality. Stresses are then computed by

$$\sigma = H\partial u. \tag{16}$$

This is the displacement approach.

5. The Equilibrium Approach

Here, equilibrium is supposed to hold *a priori*. In other words, any candidate for a solution has to be a stress field σ verifying

$$\int_\Omega \sigma^T \partial\nu \, \mathrm{d}\Omega = \int_\Omega f^T \nu \mathrm{d}\Omega + \int_{\Gamma_2} t^T \nu \, \mathrm{d}\Gamma, \tag{17}$$

for any $\nu \in V_0$. Now, it is clear that a particular solution $\bar\sigma$ of this equation always exists, a trivial example being the exact solution of the elastic problem. The general solution of equation (17) is thus given by the linear manifold $\bar\sigma + E_0$. The equilibrium method consists to find an element of this manifold that verifies the so-called *compatibility condition*

$$\sigma \in H\partial\bar u + C_0\,, \tag{18}$$

which ensures that σ derives from a finite energy displacement verifying the boundary condition (3). Equivalently,

$$\sigma - H\partial\bar u \in C_0\,,$$

and since C_0 is the orthogonal complement of E_0, this is to say that for each self-stress field τ,

$$(\sigma - H\partial\bar u, \tau) = 0.$$

Explicitly, this condition writes

$$\int_\Omega \sigma^T H^{-1} \tau \, \mathrm{d}\Omega - \int_\Omega (\partial\bar u)^T \tau \, \mathrm{d}\Omega = 0$$

and integration by parts of the second term leads to

$$-\int_\Omega (\partial\bar u)^T \tau \, \mathrm{d}\Omega = -\int_{\Gamma_1} \bar u^T L^T \tau \, \mathrm{d}\Gamma - \int_{\Gamma_2} \bar u^T L^T \tau \, \mathrm{d}\Gamma + \int_\Omega \bar u^T \partial^T \sigma \, \mathrm{d}\Omega, = -\int_{\Gamma_1} \bar u^T L^T \tau \, \mathrm{d}\Gamma,$$

since τ is a self-stress field. The final result is

$$\int_\Omega \sigma^T H^{-1} \tau \, \mathrm{d}\Omega - \int_{\Gamma_1} \bar u^T L^T \tau \, \mathrm{d}\Gamma = 0, \tag{19}$$

which is the well-known complementary energy principle.

At this stage, one may conclude by the following characterisation of the exact solution of the elastic problem. It is the only stress field that simultaneously lies in the two manifolds $H\partial\bar u + C_0$ and $\bar\sigma + E_0$.

6. The Error of an Arbitrary Stress Field

6.1. A decomposition of the error

Let us now consider an *arbitrary* stress field $\theta \in S$. If σ is the true stress field, i.e., the solution of the elastic problem, their difference

$$\eta = \theta - \sigma \tag{20}$$

may be called the *stress error*.

But we know that any stress field may be decomposed in two terms, one being in C_0, and the other in E_0. We therefore write

$$\eta = \eta_C + \eta_E \tag{21}$$

with

$$\eta_C \in C_0\,, \qquad \eta_E \in E_0\,.$$

It is clear that

$$\|\eta\|^2 = \|\eta_C\|^2 + \|\eta_E\|^2. \tag{22}$$

To interpret this decomposition, note first that for any stress field τ, the norm may be computed by

$$\|\tau\| = \sup_{\substack{\rho \in S \\ \rho \neq 0}} \frac{(\tau, \rho)}{\|\rho\|}\,.$$

In the present case, however, the scanning of S may be limited, because

$$\begin{aligned} (\eta_C, \rho) &= 0 \qquad \text{if} \quad \rho \in E_0\,, \\ (\eta_E, \rho) &= 0 \qquad \text{if} \quad \rho \in C_0\,, \end{aligned}$$

so that

$$\|\eta_C\| = \sup_{\substack{\rho \in C_0 \\ \rho \neq 0}} \frac{(\eta_C, \rho)}{\|\rho\|} = \sup_{\substack{\rho \in C_0 \\ \rho \neq 0}} \frac{(\eta, \rho)}{\|\rho\|} \tag{23}$$

and

$$\|\eta_E\| = \sup_{\substack{\rho \in E_0 \\ \rho \neq 0}} \frac{(\eta_E, \rho)}{\|\rho\|} = \sup_{\substack{\rho \in E_0 \\ \rho \neq 0}} \frac{(\eta, \rho)}{\|\rho\|}\,.$$

6.2. The equilibrium error

Let us first examine η_C. A displacement field $\nu \in V_0$ is associated with any $\rho \in C_0$, such that $\rho = H\partial\nu$. So,

$$(\eta, H\partial\nu) = \int_\Omega \theta^T \partial\nu \,\mathrm{d}\Omega - \int_\Omega \sigma^T \partial\nu \,\mathrm{d}\Omega.$$

But from equilibrium,

$$\int_\Omega \sigma^T \partial\nu \,\mathrm{d}\Omega = \int_\Omega f^T \nu \,\mathrm{d}\Omega + \int_{\Gamma_2} t^T \nu \,\mathrm{d}\Gamma,$$

so that

$$(\eta, H\partial\nu) = \int_\Omega \theta^T \partial\nu \,\mathrm{d}\Omega - \int_\Omega f^T \nu \,\mathrm{d}\Omega - \int_{\Gamma_2} t^T \nu \,\mathrm{d}\Gamma = des(\nu). \tag{24}$$

We will call this linear functional the *disequilibrium functional* because it vanishes if and only if θ is in equilibrium with the loads. Finally, one obtains

$$\|\eta_C\| = \sup_{\substack{\nu \in V_0 \\ \nu \neq 0}} \frac{des(\nu)}{\|\nu\|_{V_0}} = \|des\|_{V_0'}\,, \tag{25}$$

that is *the norm of the disequilibrium functional in the dual* V_0' *of* V_0. For this reason, η_C may be interpreted as the *equilibrium error*.

An equivalent expression of $\|\eta_C\|$ may be obtained by making use of the Riesz representation theorem, which says that for any bounded linear functional $F(v)$ on a Hilbert space R, there exists a unique element $w \in R$ such that

$$(w, \nu)_R = F(\nu), \tag{26}$$

for every $\nu \in R$ and

$$\|w\|_R = \|F\|_{R'} \, . \tag{27}$$

In the present case, this is to say that there exists a homogeneous displacement field $w_C \in V_0$ such that for every $\nu \in V_0$

$$(w_C, \nu)_{V_0} = \int_\Omega (\partial w_C)^T H \partial \nu \, \mathrm{d}\Omega = des(\nu) \tag{28}$$

and

$$\|\eta_C\| = \|des\|_{V_0'} = \|w_C\|_{V_0} \, . \tag{29}$$

This property, which was mentioned in [5], reduces the evaluation of the equilibrium error to a new variational problem.

6.3. The compatibility error

Turning now to the second term η_E, note that the true solution is of the form

$$\sigma = H\partial\bar{u} + \rho, \qquad \rho \in C_0 \, ,$$

so that for each $\tau \in E_0$,

$$(\sigma, \tau) = (H\partial\bar{u}, \tau) = \int_{\Gamma_1} \bar{u}^T L^T \tau \, \mathrm{d}\Gamma.$$

One has thus

$$(\eta, \tau) = (\theta, \tau) - (\sigma, \tau) = \int_\Omega \theta^T H^{-1} \tau \, \mathrm{d}\Omega - \int_{\Gamma_1} \bar{u}^T L^T \tau \, \mathrm{d}\Gamma = inc(\tau). \tag{30}$$

This linear functional will be called the *incompatibility functional* because it vanishes if and only if θ verifies compatibility. The result is thus

$$\|\eta_E\| = \sup_{\substack{\tau \in E_0 \\ \tau \neq 0}} \frac{inc(\tau)}{\|\tau\|} = \|inc\|_{E_0'} \, , \tag{31}$$

that is *the norm of the incompatibility functional in the dual* E_0' *of* E_0. It is thus legitimate to interpret η_E as the *compatibility error.*

A variational definition of η_E is also possible. In fact, it is the stress field verifying for every $\tau \in E_0$,

$$(\eta_E, \tau) = inc(\tau). \tag{32}$$

6.4. Comments on these results

It is thus found that in the general case, the stress error is composed of an equilibrium error η_C, which is a homogeneous compatible field and a compatibility error η_E which is a self-stress field. In the finite element frame, if strictly compatible elements are used, the compatibility error vanishes and η_C is the only term to be investigated. Conversely, with equilibrium elements, $des(\nu)$ identically vanishes, and the only error is η_E. But a lot of finite element models are nor of displacement nor of equilibrium type. It is the case of mixed and hybrid models, of non-conforming elements passing the patch test, and so on. With such elements, *both errors coexist*, a fact that is generally overlooked, as most error measures are related on equilibrium only.

7. THE ERRORS AS DISTANCES TO SOME MANIFOLDS

A useful geometrical interpretation may be given of the two components of the error.

7.1. Equilibrium error

Any solution of the equilibrium equations is of the form

$$\rho = \sigma + \tau$$

where σ is the true solution and $\tau \in E_0$. One has thus

$$\theta - \rho = \theta - \sigma - \tau = \eta - \tau = \eta_C + \eta_E - \tau$$

with $\eta_C \in C_0$ and $(\eta_E - \tau) \in E_0$, so that

$$\|\theta - \rho\|^2 = \|\eta_C\|^2 + \|\eta_E - \tau\|^2 \geq \|\eta_C\|^2,$$

the equality holding when $\tau = \eta_E$, that is, for $\rho = \sigma + \eta_E$.

As it is clear that $\sigma + E_0 = \bar{\sigma} + E_0$, where $\bar{\sigma}$ is the above mentioned particular solution of the equilibrium equations, one obtains

$$\|\eta_C\| = \inf_{\rho \in \bar{\sigma} + E_0} \|\theta - \rho\|. \tag{33}$$

So, *the equilibrium error of a field θ is its distance to the linear manifold of stress fields that are in equilibrium.*

7.2. Compatibility error

A similar analysis holds for the compatibility error. It starts from the fact that any compatible stress field is of the form

$$\lambda = \sigma + H\partial\nu$$

where σ is the true solution and $\nu \in V_0$. From this follows

$$\theta - \lambda = \theta - \sigma - H\partial\nu = \eta - H\partial\nu = \eta_E + \eta_C - H\partial\nu$$

with $\eta_E \in E_0$ and $(\eta_C - H\partial\nu) \in C_0$, so that

$$\|\theta - \lambda\|^2 = \|\eta_E\|^2 + \|\eta_C - H\partial\nu\|^2 \geq \|\eta_E\|^2,$$

the equality holding if $H\partial\nu = \eta_C$, that is, $\lambda = \sigma + \eta_C$.

Owing to the fact that $\sigma + C_0 = H\partial\bar{u} + C_0$, where $\bar{u}$ is the particular displacement field that complies with the boundary conditions on Γ_1, one obtains

$$\|\eta_E\| = \inf_{\nu \in \bar{u} + V_0} \|\theta - H\partial\nu\|. \tag{34}$$

The compatibility error of a field θ is thus its distance to the linear manifold of compatible stress fields.

8. Upper and lower bounds of the errors

8.1. Necessity of approximating the errors

One can hardly imagine to exactly computing the errors η_C and η_E because this would imply the *exact* solution of a problem which is at least even difficult as the initial problem, for which an approximate solution was the only possible way. The error evaluation shall thus also be conceived as an approximate one. But precisely, the preceding results implicitly contain a suited methodology to obtain upper and lower bounds of the errors.

8.2. Lower bounds of the equilibrium error

It directly follows from (25) that for any non zero homogeneous displacement ν, one has

$$\frac{\|des(\nu)|}{\|\nu\|_{V_0}} \leq \|\eta_C\|. \tag{35}$$

The left hand side thus constitutes a *lower bound* of the equilibrium error. This interesting conclusion has however to be tempered by the following remark. Let us suppose that the approximate stress field θ is obtained from a Rayleigh-Ritz approximation of the displacement type. This is to say that a subspace $V_{h0} \subset V_0$ has been chosen, and the approximate solution is a displacement $u_h \in u_0 + V_{h0}$ that verifies

$$\int_\Omega (\partial u_h)^T H \partial \nu_h \, \mathrm{d}\Omega = \int_\Omega f^T \nu_h \, \mathrm{d}\Omega + \int_{\Gamma_2} t^T \nu_h \, \mathrm{d}\Gamma$$

for every $\nu_h \in V_{h0}$. The stress field is now

$$\theta = H \partial u_h \,.$$

It is clear that under these conditions,

$$des(\nu_h) = 0,$$

for any $\nu_h \in V_{h0}$. So, to obtain a nonzero lower bound, it is necessary to introduce *new homogeneous displacement fields that are not elements of* V_{h0}. Such fields may be obtained by h-type or p-type enrichment.

Choosing test displacement fields with a local support leads to *local error measures* which may be very instructive in what concerns the error distribution.

A refined version of the method consists to choose some subspace $W_{h0} \subset V_0$ and to determine displacements $w_h \in W_{h0}$ such that

$$(w_h, \nu_h)_V = des(\nu_h) \tag{36}$$

for every $\nu_h \in W_{h0}$. It is in fact a Rayleigh-Ritz approximation of the displacement w_C defined by (28). The lower bound is then $\|w_h\|_{V_0}$. This procedure was proposed by Diez, Egozcue and Huerta [5], on local finite element refinements. But obtaining a lower bound of the global error from local ones is somewhat difficult because with overlapping patches, coupling terms appear and with non-overlapping patches, too low a global error is obtained.

8.3. Upper bounds of the equilibrium error

As the equilibrium error of a field θ is from (33) the distance between this field and the equilibrated manifold $\bar{\sigma} + E_0$, it is clear that any equilibrated stress field τ verifies

$$\|\eta_C\| \leq \|\theta - \tau\| \tag{37}$$

and thus leads to an *upper bound* of the equilibrium error. As used to control the quality of displacement formulations, this property was the basis of the so-called *dual analysis* methods. Recall that in the *conventional* dual analysis, as defined and used by Fraeijs de Veubeke and his coworkers [1, 4, 6–8, 14], an equilibrium model of the same structure is obtained by a Rayleigh- Ritz process and then compared to the displacement solution. In counterpart to the fact that a second analysis is required at each step, both θ and τ converge to the true solution when the mesh is refined, so that the upper bound (37) effectively converges to zero. A *non- conventional* dual analysis was developed by Ladevèze [10–12], in which an equilibrated stress field is generated from the displacement analysis results, through a post-processing scheme. The upper bound property is maintained, but it is no more guaranteed that the equilibrated fields converge to the true solution, so that the obtained error measure does not necessarily converge to zero when the mesh is refined. In other words, the non-conventional dual analysis certainly detects bad meshes, but it is unable to see that convergence actually occurs.

What is obtained here is the fact that for *any* stress field, conventional and non- conventional dual analysis continues to hold and lead to an evaluation of the equilibrium part of the error.

8.4. Lower bounds of the compatibility error

Concerning the compatibility error, lower bounds may be obtained from Eq. (33). *To any self-stress field τ is associated a lower bound of the form*

$$\frac{|inc(\tau)|}{\|\tau\|} \leq \|\eta_E\|. \tag{38}$$

Here also, in the case where θ is obtained from a Rayleigh-Ritz scheme of the equilibrium type, with stress variations in some subspace $E_{h_0} \subset E_0$, it is necessary, in order to obtain nontrivial bounds, to introduce new self-stress fields which are not elements of E_{h_0}.

Local self-stress fields will lead to local error measures. In the same way as for the equilibrium error, the best choice of a lower bound from a given subspace $F_{h_0} \subset E_0$ is obtained by solving the variational problem which consists to find $\rho_h \in F_{h_0}$ such that

$$(\rho_h\,,\,\tau_h) = inc(\tau_h) \tag{39}$$

for every $\tau_h \in F_{h_0}$. The lower bound is then $\|\rho_h\|$. It is the dual form of the method of Diez et al., and the same comments completely hold.

8.5. Upper bounds of the compatibility error

The compatibility error of a field θ is from (34) its distance to the linear manifold of compatible stress fields [13]. Therefore, any compatible displacement field $\nu \in \bar{u} + V_0$ leads to the inequality

$$\|\eta_E\| \leq \|\theta - H\partial\nu\|. \tag{40}$$

This opens the way of dual analysis methods, the auxiliary field being now a compatible one. Both types, conventional and unconventional, are possible. The latter one could be named a *dual Ladevèze method.* Our discussion concerning the equilibrium error remains valid by symmetry and will not be repeated.

8.6. Combining upper and lower bounds of the errors

Upper and lower bounds of the errors have distinct properties that make them complementary. *Upper bounds* are always on the safe side and it is clear that unconventional dual analyses constitute a good way to obtain a *global* error measurement. Assuming for instance that a great error is found, a mesh refinement is necessary. But in most cases, the errors are not uniformly distributed, and the refinements may be restricted in some particular zones. Here, *lower bound* techniques may be useful, as they are particularly suited to give *local* error measures. In contrary, obtaining global errors from a lower bound technique is not easy, as it requires the solution of a greater problem than the initial one, a difficulty that can be circumvented only by abandoning the lower bound property [5].

It thus seems that a proper combination of upper and lower bound techniques could be a very effective tool in an adaptive mesh procedure.

9. Conclusions

Any stress field, whatever is the way to obtain it, can be affected by two and only two types of errors, namely, the equilibrium error and the compatibility error. Displacement models only lead to an equilibrium error, and equilibrium models, only to a compatibility error. But mixed elements, hybrid elements and also the numerous non-conforming displacement elements that pass the patch test exhibit both errors. This fact seems to have been generally overlooked, perhaps from the fact that a general analysis such as the preceding one was not available.

For each type of error, upper and lower bounds can be constructed, and seem to be complementary as the first one is a direct way to obtain a global error, and the second one, best suited to local error evaluations.

Note finally that our analysis includes older results, such as the conventional dual analysis, the Ladevèze method and the results of Diez et al., and gives their dual versions for the compatibility error.

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