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Efficient uncoupled stochastic analysis with non-proportional damping

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ABSTRACT

The use of normal modes of vibration in the analysis of structures with non-proportional damping reduces the size of the resulting set of governing equations, but does not decouple them. A common practice consists in decoupling the equations by disregarding the off-diagonal elements in the modal damping matrix. Recently, an approximation based on an asymptotic expansion of the modal transfer matrix has been proposed in a deterministic framework to partially account for off-diagonal terms, but still with a set of uncoupled equations. This paper aims at extending this method in a stochastic context. First the mathematical background is introduced and the method is illustrated with a simple example. Then its relevance is demonstrated within the context of the structural analysis of a large and realistic structure.

1. Introduction

The dynamic analysis of structures having a large number of degrees of freedom (DOFs) is particularly efficient in a modal basis. In classical structural mechanics, the structural damping matrix is often constructed with the assumption of proportionality, i.e. as a linear combination of mass and stiffness matrices. Consequently, the modal damping matrix and thus the modal transfer matrix are diagonal. With this assumption, the equations of motion are decoupled: an n-DOF linear system is substituted by a set of m independent 1-DOF linear systems [1]. The necessary inversion of a full transfer matrix – in a frequency domain context – is therefore avoided.

Although the structural damping in a structure is commonly supposed to be proportional, there exist some cases where this assumption cannot be stated. For instance, viscous dampers (shock absorbers) or aerodynamic damping usually come along with their own damping models. Consideration of these models yields a damping matrix that does not necessarily offer proportionality. In these cases, the modal projection still enables to reduce the size of the system, as usually $m \ll n$, but not to decouple the modal equations anymore.

The decoupling approximation, as proposed by Lord Rayleigh [2], consists in neglecting the off-diagonal elements of the modal damping matrix. This approach is motivated by the smallness of the off-diagonal elements compared with the diagonal ones. Recently, some paradoxical results have been highlighted about this practiced assumption. Indeed, Morzfled et al. [3]

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show that the relative smallness between off-diagonal and diagonal elements is not sufficient to ensure small decoupling errors.

Other methods have been proposed to deal with non-proportional damping. The complex modal analysis, originally proposed by Foss [4], is an extension of the classical modal analysis to complex modes in a state space. This method allows to decouple the equations of motion in the state space of the system and therefore to perform deterministic or stochastic analyses in time or frequency domains without assumption about damping matrix. Nevertheless, engineers and practitioners, especially in civil engineering, have never considered this approach as suitable, because of the difficult physical meaning of complex modes. In an alternative approach, Ibrahimbegovic [5] proposes a simple numerical algorithm considering off-diagonal damping forces as pseudo-forces applied to the uncoupled system. To avoid full transfer matrix inversion, Denoël and Degée [6] propose an asymptotic expansion of the transfer matrix assuming the relative smallness of off-diagonal elements with regards to diagonal ones.

Actually, it appears that these methods to deal with non-proportional damping, especially these simplified methods, have not been formally practiced in a stochastic context. However, in such a context, the use of simplified methods is just as appealing. Indeed, in a frequency domain approach, be it deterministic or stochastic, the analysis consists in a sequence of computations, i.e. matrix inversions, performed for a sequence of frequencies. As it requires to be repeated a large amount of times, any saving would naturally reduce the total computation burden.

Motivated by these observations, this paper essentially aims at the extension of the approximation developed in [6] to a stochastic context, which is actually not a trivial discussion, as shown next. The grounds of the deterministic method are briefly summarized in Section 2. Then Section 3 highlights the necessary modifications to make that method applicable in a stochastic context. Finally illustrative and realistic examples are given in Section 4.

2. Deterministic analysis

The equation of motion of an n-DOF linear system is

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{f} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are, respectively, the mass, damping and stiffness matrices, $\mathbf{f}(t)$ is the vector of external forces, $\mathbf{x}(t)$ is the vector of nodal displacements and the dot denotes the time derivative. The structural response of a given linear dynamic system can be computed using a restricted number m of normal modes of vibrations ($m \ll n$) [7]. These modes are gathered in a matrix $\mathbf{\Phi}$ and are normalized to have unit generalized masses as

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}=\mathbf{I}, \quad \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}=\mathbf{\Omega} \tag{2}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Omega}$ is a diagonal matrix containing generalized stiffnesses equal, in this case, to the squared circular frequencies. Using the modal superposition principle, the equation of motion (1) is written as

$$\ddot{\mathbf{q}}+\mathbf{D}\dot{\mathbf{q}}+\mathbf{\Omega}\mathbf{q}=\mathbf{g} \tag{3}$$

where $\mathbf{q}(t)$ ($\mathbf{x}=\mathbf{\Phi}\mathbf{q}$) is the vector of modal coordinates, $\mathbf{g}(t)=\mathbf{\Phi}^{\mathrm{T}}\mathbf{f}(t)$ is the vector of generalized forces and $\mathbf{D}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}$ is the modal damping matrix. The frequency domain formulation of the equation of motion in the modal basis is obtained by a side-by-side Fourier Transform of (3)

$$\mathbf{Q}=\mathbf{H}\mathbf{G} \tag{4}$$

where $\mathbf{Q}(\omega)$ and $\mathbf{G}(\omega)$ are, respectively, the Fourier transforms of $\mathbf{q}(t)$ and $\mathbf{g}(t)$. The modal transfer matrix $\mathbf{H}(\omega)$ is defined by

$$\mathbf{H}=(\mathbf{\Omega}-\omega^2\mathbf{I}+i\omega\mathbf{D})^{-1} \tag{5}$$

For convenience in the following developments, the damping matrix $\mathbf{D}$ is decomposed into two terms [3] as

$$\mathbf{D}=\mathbf{D}_d+\mathbf{D}_o \tag{6}$$

where $\mathbf{D}_d$ and $\mathbf{D}_o$ are built as $D_{d,ij}=D_{ij}\delta_{ij}$ (diagonal elements) and $D_{o,ij}=D_{ij}(1-\delta_{ij})$ (off-diagonal elements) where δ_{ij} denotes the Kronecker-delta function. Substitution of (6) into (5) yields

$$\mathbf{H}=(\mathbf{\Omega}-\omega^2\mathbf{I}+i\omega(\mathbf{D}_d+\mathbf{D}_o))^{-1}=(\mathbf{I}+\mathbf{J}_d^{-1}\mathbf{J}_o)^{-1}\mathbf{H}_d \tag{7}$$

where $\mathbf{J}_d(\omega)=\mathbf{\Omega}-\omega^2\mathbf{I}+i\omega\mathbf{D}_d$, $\mathbf{J}_o(\omega)=i\omega\mathbf{D}_o$, and $\mathbf{H}_d(\omega)=\mathbf{J}_d^{-1}$. In structural engineering, a common method to deal with non-proportional damping consists in simply neglecting the off-diagonal terms gathered in $\mathbf{D}_o$. This method is usually referred to as the *decoupling approximation* and the resulting approximate response $\mathbf{Q}_d$ is given by

$$\mathbf{Q}_d=\mathbf{H}_d\mathbf{G}. \tag{8}$$

This method, proposed by Lord Rayleigh [2], is motivated by the smallness of the off-diagonal elements compared with the diagonal ones. Recently, some paradoxical results have been highlighted about this practiced assumption. Indeed, Morzfled [3] shows that "small off-diagonal elements in $\mathbf{D}$ are not sufficient to ensure small decoupling errors". Diagonal dominance of $\mathbf{D}$, quantified by the smallness of the index of diagonality, as defined in [8] as

$$\rho(\mathbf{D})=\sigma(\mathbf{D}_d^{-1}\mathbf{D}_o) \tag{9}$$

where σ is the spectral radius, is not sufficient to ensure accuracy of the decoupling approximation. The interest of this index is to provide a scalar estimation of the smallness of $\mathbf{D}_o$ compared with $\mathbf{D}_d$, globally. The concept of diagonal dominance is used in a different way in [6] where an efficient method is proposed to deal with non-proportional damping. This method results in adding a series of correction terms to the approximate response (8). More precisely, it is shown in [6] that if the index of diagonality (9) is small, the index $\rho(\mathbf{J}) = \sigma(\mathbf{J}_d^{-1}\mathbf{J}_o)$ is also small. This allows to keep only the first two terms of the series expansion of $(\mathbf{I}+\mathbf{J}_d^{-1}\mathbf{J}_o)^{-1}$ in order the obtain a sufficiently accurate estimation of the response in many deterministic applications. The truncation to the first two terms of this asymptotic expansion, substituted in (7) leads to an approximate expression $\mathbf{H}_{c_1}$ of the transfer matrix expressed as

$$\mathbf{H}_{c_1} = (\mathbf{I}-\mathbf{J}_d^{-1}\mathbf{J}_o)\mathbf{H}_d = (\mathbf{I}-i\omega\mathbf{H}_d\mathbf{D}_o)\mathbf{H}_d \tag{10}$$

Eq. (10) shows the advantage of the method where no full matrix inversion is needed in comparison to (7). Consequently, the modal coordinates are approximated by

$$\mathbf{Q}_c = (\mathbf{I}-\mathbf{J}_d^{-1}\mathbf{J}_o)\mathbf{H}_d\mathbf{G} = \mathbf{Q}_d + \Delta\mathbf{Q} \tag{11}$$

where $\Delta\mathbf{Q}(\omega) = \mathbf{H}_d(-i\omega\mathbf{D}_o\mathbf{Q}_d)$ is the first correction term (to be added to the usual decoupling approximation result) computed by solving the same uncoupled modal equations as in (8) under the virtual modal forces $(-i\omega\mathbf{D}_o\mathbf{Q}_d)$. For large structures with many coupled modes, the inversion of the transfer matrix for each angular frequency is time consuming. With this approach, the inversion is bypassed and the accuracy improvement is achieved without any time-consuming operation, because only a diagonal matrix is inverted. More interestingly perhaps, this method allows to partially take into account the non-proportional damping but at inexpensive computational and conceptual costs, as it just requires the solution of uncoupled modal equations.

This paper aims at extending this method in a stochastic framework.

3. Stochastic analysis

When the loading is a random process (as for example when considering the wind actions), a stochastic analysis is usually performed [9]. For linear systems subjected to Gaussian loadings, such an analysis aims at computing the probabilistic properties of any stationary response of the structure (nodal/modal coordinates, internal forces, etc.) using their power spectral densities (psd). The psd matrix of the modal coordinates $\mathbf{S}^{(\mathbf{q})}(\omega)$ is obtained by pre- and post-multiplication by the modal transfer matrix of the psd matrix of the generalized Gaussian forces $\mathbf{S}^{(\mathbf{g})}(\omega)$

$$\mathbf{S}^{(\mathbf{q})} = \mathbf{H}\mathbf{S}^{(\mathbf{g})}\mathbf{H}^* \tag{12}$$

where the superscript * denotes the conjugate transpose operator. The psd matrix of the modal displacements in the uncoupled system is obtained by

$$\mathbf{S}^{(\mathbf{q}_d)} = \mathbf{H}_d\mathbf{S}^{(\mathbf{g})}\mathbf{H}_d^* \tag{13}$$

Using approximation (10) of the transfer matrix without any alteration, in (12) would lead to correction terms of the first and second orders in $\rho(\mathbf{J})$. A reason why approximation (10) works fine in solving the deterministic problem as described by (4) is that the transfer function appears as a single factor in the computation of the response. At the opposite, in the stochastic context, Eq. (12) shows that the response requires the product of two factors involving the transfer function. For consistency, it is thus now necessary to introduce the second order approximation of the transfer function

$$\mathbf{H}_{c_2} = (\mathbf{I}-\mathbf{J}_d^{-1}\mathbf{J}_o + (\mathbf{J}_d^{-1}\mathbf{J}_o)^2)\mathbf{H}_d = (\mathbf{I}-i\omega\mathbf{H}_d\mathbf{D}_o - \omega^2(\mathbf{H}_d\mathbf{D}_o)^2)\mathbf{H}_d \tag{14}$$

Its substitution into (12) provides a consistent approximation of the psd matrix of the response

$$\mathbf{S}^{(\mathbf{q})} \simeq (\mathbf{I}-\mathbf{J}_d^{-1}\mathbf{J}_o + (\mathbf{J}_d^{-1}\mathbf{J}_o)^2)\mathbf{H}_d\mathbf{S}^{(\mathbf{g})}\mathbf{H}_d^*(\mathbf{I}-\mathbf{J}_o^*\mathbf{J}_d^{-*} + (\mathbf{J}_o^*\mathbf{J}_d^{-*})^2) \tag{15}$$

Taking into account (13) and truncating expansion (15) to the second order, the psd matrix (12) is approximated by

$$\mathbf{S}^{(\mathbf{q}_{c_2})} = \mathbf{S}^{(\mathbf{q}_d)} + \Delta\mathbf{S}^{(\mathbf{q}_1)} + \Delta\mathbf{S}^{(\mathbf{q}_2)} \tag{16}$$

where the first-order correction term is

$$\Delta\mathbf{S}^{(\mathbf{q}_1)}(\omega) = -(\mathbf{H}_d\mathbf{J}_o\mathbf{S}^{(\mathbf{q}_d)} + \mathbf{S}^{(\mathbf{q}_d)}\mathbf{J}_o^*\mathbf{H}_d^*) \tag{17}$$

and the second-order correction term is

$$\Delta\mathbf{S}^{(\mathbf{q}_2)}(\omega) = (\mathbf{H}_d\mathbf{J}_o)^2\mathbf{S}^{(\mathbf{q}_d)} + \mathbf{H}_d\mathbf{J}_o\mathbf{S}^{(\mathbf{q}_d)}\mathbf{J}_o^*\mathbf{H}_d^* + \mathbf{S}^{(\mathbf{q}_d)}(\mathbf{J}_o^*\mathbf{H}_d^*)^2 \tag{18}$$

$$\Delta\mathbf{S}^{(\mathbf{q}_2)}(\omega) = -(\mathbf{H}_d\mathbf{J}_o\Delta\mathbf{S}^{(\mathbf{q}_1)} + \Delta\mathbf{S}^{(\mathbf{q}_1)}\mathbf{J}_o^*\mathbf{H}_d^*) - \mathbf{H}_d\mathbf{J}_o\mathbf{S}^{(\mathbf{q}_d)}\mathbf{J}_o^*\mathbf{H}_d^* \tag{19}$$

Eq. (16) introduces two correction terms improving the approximation of $\mathbf{S}^{(\mathbf{q})}(\omega)$ that would arise in applying the decoupling approximation. These correction terms are obtained readily. Indeed, as previously, only the diagonal matrix $\mathbf{H}_d$ requires matrix inversion; no full matrix inversion is required to obtain $\Delta\mathbf{S}^{(\mathbf{q}_1)}$ and $\Delta\mathbf{S}^{(\mathbf{q}_2)}$. If $\rho(\mathbf{D})$ is of order ϵ (a small parameter), so is the product $\mathbf{H}_d\mathbf{J}_o$ and the first correction term $\Delta\mathbf{S}^{(\mathbf{q}_1)}$ is also of order ϵ. For similar reasons, the second correction term $\Delta\mathbf{S}^{(\mathbf{q}_2)}$ is of order ϵ^2.

More correction terms are expressed in a general recurrence relation

$$\Delta \mathbf{S}^{(\mathbf{q}_{n+1})}(\omega) = -(\mathbf{H}_d \mathbf{J}_o \Delta \mathbf{S}^{(\mathbf{q}_n)} + \Delta \mathbf{S}^{(\mathbf{q}_n)} \mathbf{J}_o^* \mathbf{H}_d^*) - \mathbf{H}_d \mathbf{J}_o \Delta \mathbf{S}^{(\mathbf{q}_{n-1})} \mathbf{J}_o^* \mathbf{H}_d^*$$

for $n > 2$, even if in this paper only the first two correction terms are considered.

Although it is expectedly *globally* smaller than the first correction $\Delta \mathbf{S}^{(\mathbf{q}_1)}$, $\Delta \mathbf{S}^{(\mathbf{q}_2)}$ is not systematically negligible compared to $\Delta \mathbf{S}^{(\mathbf{q}_1)}$, because not necessarily the same elements of $\mathbf{S}^{(\mathbf{q}_d)}$ are concerned with the successive correction terms. This is demonstrated by splitting up the psd matrix of the generalized forces into its diagonal and off-diagonal elements

$$\mathbf{S}^{(\mathbf{g})} = \mathbf{S}_d^{(\mathbf{g})} + \mathbf{S}_o^{(\mathbf{g})} \tag{20}$$

i.e. into the unilateral spectral generalized forces $\mathbf{S}_d^{(\mathbf{g})}(\omega)$ and the cross-spectral generalized forces $\mathbf{S}_o^{(\mathbf{g})}(\omega)$. Insertion of (20) into (13) leads to

$$\mathbf{S}^{(\mathbf{q}_d)} = \mathbf{H}_d \mathbf{S}_d^{(\mathbf{g})} \mathbf{H}_d^* + \mathbf{H}_d \mathbf{S}_o^{(\mathbf{g})} \mathbf{H}_d^* = \mathbf{S}_d^{(\mathbf{q}_d)} + \mathbf{S}_o^{(\mathbf{q}_d)} \tag{21}$$

If the coherence between generalized forces is negligible, (21) reduces to $\mathbf{S}^{(\mathbf{q}_d)} \simeq \mathbf{S}_d^{(\mathbf{q}_d)}$ and is therefore fairly diagonal. Consequently, $\Delta \mathbf{S}^{(\mathbf{q}_1)}$ becomes a zero-diagonal matrix, as seen from (17), whereas $\Delta \mathbf{S}^{(\mathbf{q}_2)}$ remains a full matrix. Thus, when the generalized forces are almost uncorrelated, the first correction term $\Delta \mathbf{S}^{(\mathbf{q}_1)}$ brings no significant correction to the variance of modal coordinates.

In the following, the covariance matrix $\boldsymbol{\Sigma}$ of the modal coordinates is calculated by integration (along the circular frequencies) of the corresponding psd matrix

$$\boldsymbol{\Sigma}^{(\beta)} = \int_{-\infty}^{+\infty} \mathbf{S}^{(\beta)} \, d\omega \tag{22}$$

where β denotes one of the aforementioned methods, namely (i) the exact approach ($\beta \equiv \mathbf{q}$), (ii) the decoupling approximation ($\beta \equiv \mathbf{q}_d$), (iii) the proposed approximation with both correction terms ($\beta \equiv \mathbf{q}_{c_2}$) or (iv) the proposed approximation with just the first correction $\Delta \mathbf{S}^{(\mathbf{q}_1)}$ ($\beta \equiv \mathbf{q}_{c_1}$). In this latter case, the psd matrix of modal coordinates is naturally defined as

$$\mathbf{S}^{(\mathbf{q}_{c_1})} = \mathbf{S}^{(\mathbf{q}_d)} + \Delta \mathbf{S}^{(\mathbf{q}_1)} \tag{23}$$

4. Illustrations

4.1. Illustration 1

The influences of the index of diagonality and the correction terms are illustrated on a classical 2-DOF dynamical system, as represented in Fig. 1.

The mass, damping and stiffness matrices are expressed as

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \mathbf{C} = 2\sqrt{km}\begin{bmatrix} \xi+\zeta & -\zeta \\ -\zeta & \xi\sqrt{\delta\mu}+\zeta \end{bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 1+\varepsilon & -\varepsilon \\ -\varepsilon & \delta+\varepsilon \end{bmatrix} \tag{24}$$

where m, k, ξ, ζ, δ, μ and ε are parameters of the studied system. The equation of motion of this 2-DOF system is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{25}$$

where $\mathbf{x} = [x_1, x_2]^T$ is the vector of nodal displacements, $W_1(t)$ and $W_2(t)$ are uncorrelated Gaussian white noises with zero mean and spectral intensities D_1 and D_2 ($E[W_i W_j] = D_i \delta_{ij}$, $i,j = 1,2$). If $\zeta = \varepsilon = 0$, the masses are not physically connected and their motions are independent. Also if $\delta = 0$, the system is equivalent to a main structure (m_1,x_1) damped by a secondary system. The system has two normal modes and the solution is carried out in the modal basis. With an example below, the accuracy of the different methods is assessed with the relative errors between the approximated covariance matrices $\boldsymbol{\Sigma}^{(\mathbf{q}_d)}$, $\boldsymbol{\Sigma}^{(\mathbf{q}_{c_1})}$ or $\boldsymbol{\Sigma}^{(\mathbf{q}_{c_2})}$ and the exact covariance matrix $\boldsymbol{\Sigma}^{(\mathbf{q})}$. A negative relative error means an underestimation of the approximated covariance matrix compared to $\boldsymbol{\Sigma}^{(\mathbf{q})}$. It is clear that any discrepancy in the estimation of the modal response is immediately reported to the structural response. For conciseness, these latter ones are therefore not reported in this document.

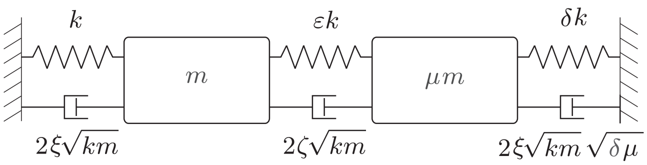

Fig. 1. 2-DOF dynamical system.

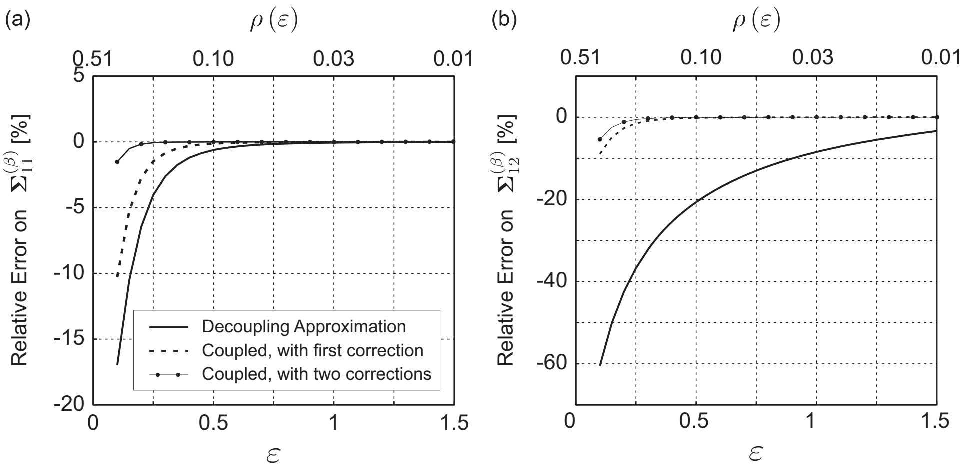

Fig. 2. Relative error on (a) the covariance of the first modal coordinate and (b) on the covariance between both modal amplitudes. Results are presented for the decoupling approximation $\boldsymbol{\Sigma}^{(\mathbf{q}_d)}$, the proposed expansion with one correction $\boldsymbol{\Sigma}^{(\mathbf{q}_{c_1})}$ and the proposed expansion with two correction terms $\boldsymbol{\Sigma}^{(\mathbf{q}_{c_2})}$. Relative errors are expressed by comparison with the exact result $\boldsymbol{\Sigma}^{(\mathbf{q})}$.

The numerical values of the parameters are set to $m=1$, $k=1$, $\delta=1$, $\mu=0.8$, $\xi=0.05$ and $\zeta=0.05$. The white noise intensities D_1 and D_2 are, respectively, equal to 5 and 10. In order to study various coupling conditions, the parameter ε takes on values in the range [0.1; 1.5] which corresponds to a range of [0.01; 0.51] for the index of diagonality.

The relative errors on the covariance of the first modal coordinate are presented in Fig. 2a, while those on the covariance between both modal coordinates are represented in Fig. 2b. For a small coupling stiffness ε, the importance of off-diagonal elements is unacceptable as illustrated by a discrepancy as large as 15 percent (resp. 60 percent) on the variance (resp. covariance) of modal amplitudes. Furthermore, as explained before, the correction term $\Delta\mathbf{S}^{(\mathbf{q}_1)}$ is not sufficient to properly correct this inaccuracy in the estimation of the variance of the response (Fig. 2a). On the contrary, the use of the second correction term in the estimation of the covariance (Fig. 2b) is clearly not mandatory.

Apart from providing a simple illustration of the concepts of the method, this example also shows that a moderate index of diagonality may result in significant errors when using the classical decoupling approximation.

4.2. *Illustration 2*

The wind velocity is usually supposed to be a Gaussian random variable characterized by a mean value U and its power spectral density S_u [10]. The effect of the mean loading is easily determined by a simple static (or quasi-static) analysis, while a stochastic analysis is required to evaluate the effects of turbulence. In a unidimensional turbulence model, the wind velocity and the applied forces are related by the squared relative velocity $V(t)$ between the wind $U+u(t)$ and the structure $\dot{x}(t)$

$$f_{tot}=\tfrac{1}{2}\rho C_d A V^2=\tfrac{1}{2}\rho C_d A(U+u-\dot{x})^2 \tag{26}$$

where ρ is the air density, $\mathbf{C}_d$ is the drag coefficient and A is the surface exposed to wind. Usually from (26) a simple linearized relation is adopted

$$f_{tot}\simeq C_a\left(\frac{U}{2}+u-\dot{x}\right) \tag{27}$$

where $C_a=\rho C_d A U$. The first term of (27) is the mean force (static analysis), the second term is the turbulent component of the force due to wind turbulence and the third term is the aerodynamic source of damping. Consideration of (27) into the equation of motion, and disregarding the mean force, gives

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}_s\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{C}_a(\mathbf{u}-\dot{\mathbf{x}}) \tag{28}$$

where $\mathbf{C}_s$ is the proportional structural damping matrix. Eq. (28) is rewritten as

$$\mathbf{M}\ddot{\mathbf{x}}+(\mathbf{C}_s+\mathbf{C}_a)\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{f} \tag{29}$$

where $\mathbf{f}(t)=\mathbf{C}_a\mathbf{u}(t)$ is the applied loading considering only the wind turbulence and $\mathbf{C}_a$ can be seen as a diagonal aerodynamic damping matrix. The diagonal elements of $\mathbf{C}_a$ are not necessarily equal, e.g. the mean velocity of wind or drag coefficients can vary along a structure. In this application, not only drag but also lift and moment coefficients contribute to the establishment of $\mathbf{C}_a$. This is done with equations similar to (27) as explained in [10]. As previously, (28) is projected in the modal basis

$$\ddot{\mathbf{q}}+(\mathbf{D}_s+\mathbf{D}_a)\dot{\mathbf{q}}+\boldsymbol{\Omega}\mathbf{q}=\mathbf{g}$$

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where $\mathbf{D}_s = \mathbf{\Phi}^T \mathbf{C}_s \mathbf{\Phi}$ is the structural modal damping matrix which is usually assumed to be diagonal and $\mathbf{D}_a = \mathbf{\Phi}^T \mathbf{C}_a \mathbf{\Phi}$ is the modal aerodynamic damping matrix which is no longer diagonal and generates modal coupling.

These simple concepts of aerodymanic damping, summarized for the purpose of clarity, are readily extended to the buffeting analysis of a 3-D structure with drag, lift and moment aerodynamic coefficients, and in a 3-D wind-flow, i.e. accounting for 3-D components of the wind turbulence [11]. Such a refined model was considered in the following illustration, which deals with a buffeting analysis of the Viaduct of Millau, see Fig. 3.

This seven-span cable-stayed bridge (about 2.5 km long), crossing the Tarn valley about 350 m above the bed of river, is the highest bridge ever built in Europe. This bridge will also remain famous for the launching technique used to erect the deck.

The structure is modeled with a finite element software (FineLg, [12]). The model counts 1425 nodes and 2439 beam elements with 12 DOFs. The first 40 modes are kept and have natural frequencies below 1 Hz. The structural modal damping matrix is built to ensure a structural damping ratio of 0.3 percent in these 40 modes.

The characteristics of the wind were chosen in accordance with [13] and with on-site measurements. Three zones are considered with different characteristics of mean velocity U and standard deviations σ_u, σ_v, σ_w (longitudinal, vertical and transversal turbulences) given in Table 1. Considering the aerodynamic damping, the index $\rho(\mathbf{D}) = \rho(\mathbf{D}_s + \mathbf{D}_a)$ is equal to 1.02.

Exact variances of the modal coordinates and the related correlation coefficients are represented in Fig. 4. The modal truncation is justified by the decrease of the variance of the modal coordinates as depicted in Fig. 4a. Concerning modes 17, 18 and 19, the vicinity of their natural frequencies (resp. 0.531 Hz, 0.533 Hz and 0.534 Hz) and the similarities of mode shapes induce dynamic coupling and high modal correlations as shown in Fig. 4b. Also, coupling is noticed for modes 29, 30 and 31. Additional observations about the structural behavior of this structure are reported in [14].

The covariance matrices $\mathbf{\Sigma}^{(\mathbf{q})}$, $\mathbf{\Sigma}^{(\mathbf{q}_d)}$, $\mathbf{\Sigma}^{(\mathbf{q}_{c_1})}$ and $\mathbf{\Sigma}^{(\mathbf{q}_{c_2})}$ are computed and the respective correlation matrices $\boldsymbol{\rho}^{(\mathbf{q})}$, $\boldsymbol{\rho}^{(\mathbf{q}_d)}$, $\boldsymbol{\rho}^{(\mathbf{q}_{c_1})}$ and $\boldsymbol{\rho}^{(\mathbf{q}_{c_2})}$ are deduced.

Fig. 5 depicts the relative error on the variance of the modal coordinates obtained with the different approximations. For the three approximations, the errors are the most significant for the two groups of correlated modes (17, 18, 19 and 29, 30, 31). Fig. 5a indicates that neglecting modal coupling, i.e. neglecting off-diagonal terms, induces a maximum error of +45 percent on the variance of mode 19. The proposed method does not neglect this coupling and allows to reduce this error down to +10 percent (Fig. 5c). Comparison between Fig. 5a and b shows the influence of the first correction term on the variances: the sign of the errors is changed but the order of magnitude is the same as before. With the proposed method (Fig. 5c) and an extension to the second order, the errors are significantly reduced even if they remain maximum for the groups of coupled modes (+10 percent). Fig. 5 also illustrates the alternate convergence towards the exact values.

In the event of negligible modal correlation, an SRSS combination (square root of the sum of the squares) of modal amplitudes would be used to calculate variances of structural responses. Because the proposed method reduces significantly the error on the modal amplitudes, the error in a SRSS combination will also be reduced. In this example, however, the importance of modal correlation (see Fig. 4b) requires a CQC approach (complete quadratic combination), at least in the establishment of structural responses that are concerned by the clustered modes 17, 18, 19 or 29, 30, 31.

Fig. 6 shows the errors on the correlation coefficients obtained with the decoupling approximation (in this case, the correlation of modal coordinates just results from the correlation of generalized forces) and the proposed approximation. They are expressed as an absolute difference with exact coefficients. The decoupling approximation (Fig. 6a) provides important differences (up to 0.2), especially for the groups of correlated modes. The proposed method reduces significantly these differences down to 0.06 (Fig. 6b), precisely because the correlation coming from non-proportional damping is integrated in this approximation.

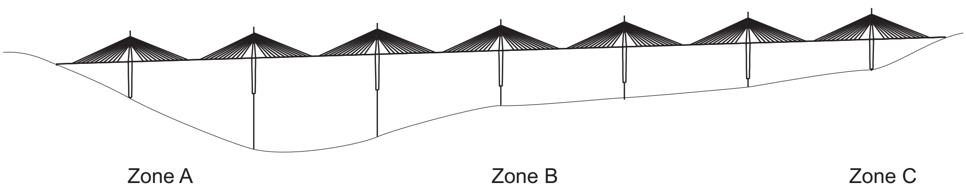

Fig. 3. Shape of the Viaduct of Millau.

Table 1
Millau Viaduct: main characteristics of the wind velocity from on-site measurements.

Zones	U (m/s)	σ_u (m/s)	σ_v (m/s)	σ_w (m/s)
Zone A	38	6.5	6.5	4.5
Zone B	34	5.5	5.5	4.0
Zone C	36	5.5	5.5	4.0

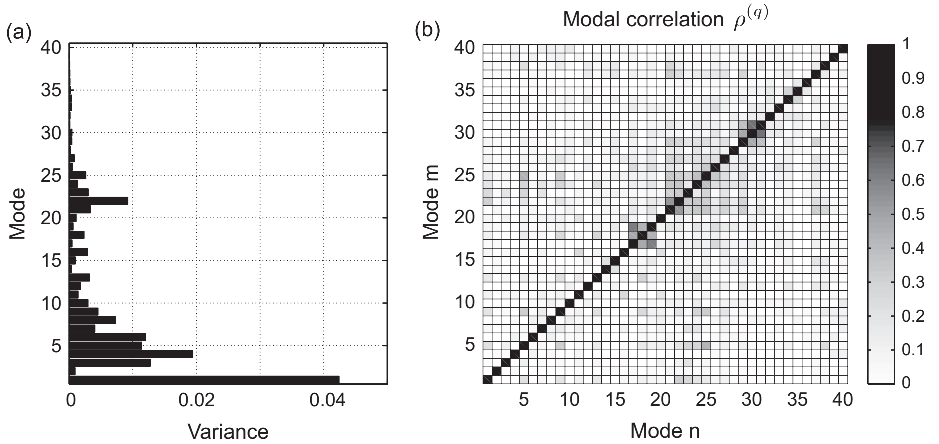

Fig. 4. (a) Exact variances of the modal coordinates and (b) exact correlation coefficients of modal coordinates.

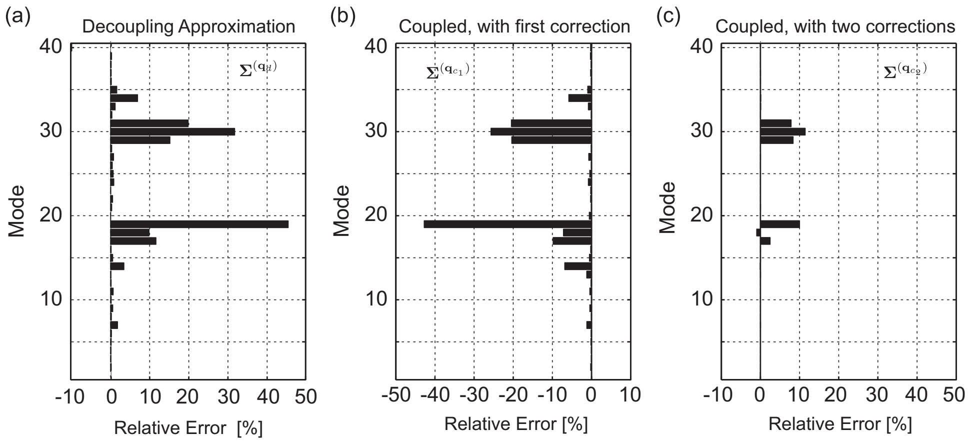

Fig. 5. Relative error on the variance of the modal coordinates for different approximations (a) $\mathbf{\Sigma}^{(\mathbf{q}_d)}$, (b) $\mathbf{\Sigma}^{(\mathbf{q}_{c_1})}$ and (c) $\mathbf{\Sigma}^{(\mathbf{q}_{c_2})}$. Relative errors are expressed with respect to $\mathbf{\Sigma}^{(\mathbf{q})}$.

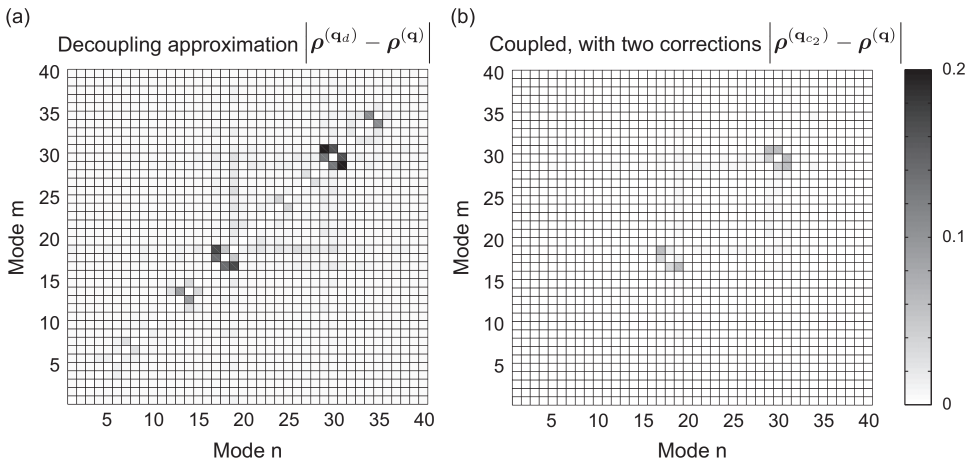

Fig. 6. Differences between exact correlation matrix of modal coordinates $\boldsymbol{\rho}^{(\mathbf{q})}$ and approximations (a) $\boldsymbol{\rho}^{(\mathbf{q}_d)}$ and (b) $\boldsymbol{\rho}^{(\mathbf{q}_{c_2})}$.

The covariance matrices of bending moments in 2058 beam elements are calculated using a CQC approach with the full covariance matrices $\mathbf{\Sigma}^{(\mathbf{q}_{c_2})}$ and $\mathbf{\Sigma}^{(\mathbf{q}_d)}$. Therefore, both errors on variances and correlation coefficients of the modal amplitudes influence the variances of the bending moments. Fig. 7 shows histograms of the relative errors on variances

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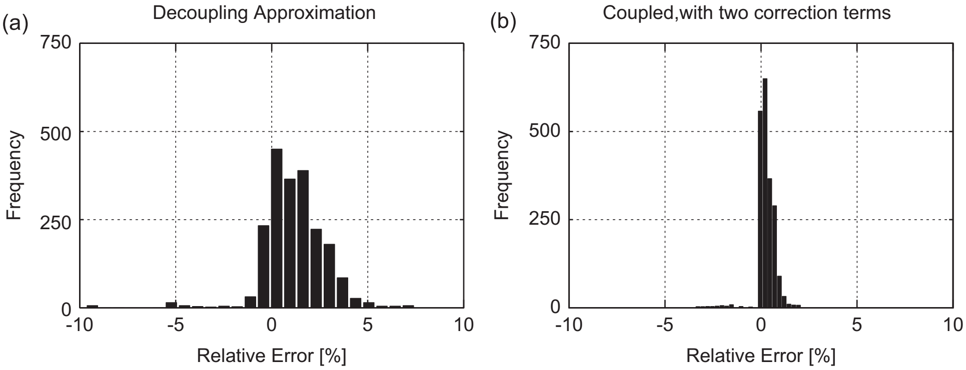

Fig. 7. Histograms of the relative error on bending moments for approximations based on (a) $\Sigma^{(\mathbf{q}_d)}$ and (b) $\Sigma^{(\mathbf{q}_{c_2})}$. Both results are obtained with a complete quadratic combination (CQC).

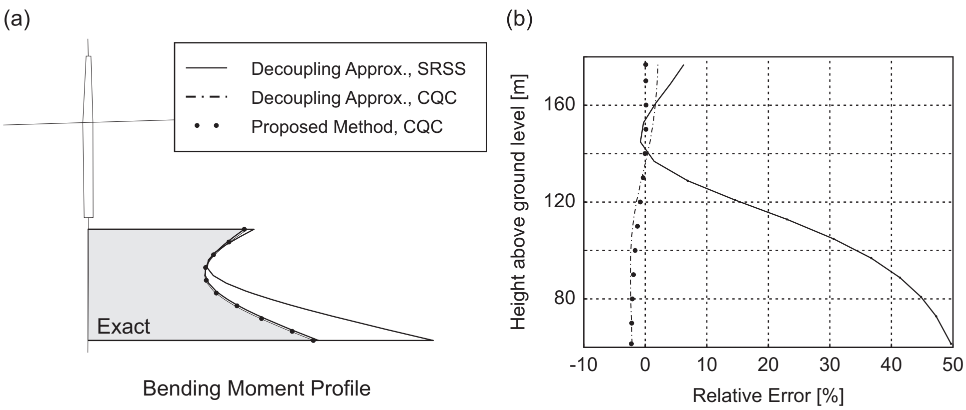

Fig. 8. (a) Bending moment diagram and (b) relative errors on bending moment for different approximations in the third pile of Millau Viaduct.

of bending moments with the two approximations. The proposed method provides a better estimation of the covariance of modal coordinates and therefore of the variances of the bending moment, as shown in Fig. 7b. For instance, more than 2000 estimated errors are less than 1 percent (against only 1000 errors for the decoupled approximation) and the most important error is -3.4 percent (against -9.7 percent for the decoupled approximation). The error is homogeneously reduced in the structure with the proposed method in spite of an index of diagonality close to 1. These results emphasize not only the importance of modal coupling resulting from non-proportional damping, but also the pertinence of the proposed method.

As shown in Fig. 8, the error on bending moment in the third pile (from the left in Fig. 3) with the proposed method does not exceed 3 percent. The bending moment in this pile depends on the correlated modes, especially 17, 18 and 19, see [14]. In this figure, the proposed method is compared with the widely used decoupling SRSS approximation (neither coupling nor correlation is taken into account) and with decoupling CQC approximation (no modal coupling is considered, but the modal correlation is taken into account). Fig. 8b shows that the common assumption consisting in neglecting coupling and correlation induces unacceptable errors (around 50 percent). The proposed method compares favorably with the other CQC approach, although both allow for a significant reduction of the discrepancy pertaining to the SRSS combination. The reasons why the decoupling CQC approximation provides acceptable results are that the errors due to coupling (overestimation, see Fig. 5a) and correlation (underestimation by 20 percent, see Fig. 6a) compensate for this particular application. This could not be the case for any other application. However, the proposed approach seeks to reduce them both and successively with the first two corrections to the covariance of modal coordinates. This makes it indisputably more robust and reliable. This example illustrates clearly the validity and the interest of the method.

5. Conclusions

Large structures submitted to dynamic loads are usually analyzed in a modal analysis frame. Indeed, in comparison to a nodal analysis and under some assumptions, it allows one to decouple the equations off motion, to reduce the size of the model and most of all end up with a tangible analysis method. A usual assumption consists in considering the structural damping as proportional. In this case the modal transfer matrix is diagonal, the inversion operation is trivial and

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straightforward. It further allows a simple understanding of the structural behavior. For these reasons, the design of large structures under dynamic loads usually hinges on a modal analysis, and this concept is well accepted by practitioners.

Wind loads, dampers or shock absorbers generate non-proportional damping and do not allow to formally solve coupled equations of motion in a modal analysis. This paper introduces an approximation based on a second-order asymptotic expansion of the modal transfer matrix in a Gaussian stochastic framework. The developed method proposes to solve uncoupled equations of motion, as usually, and to partially account for the response due to non-proportional damping with correction terms. The extension of the method from a deterministic to a stochastic framework highlights the importance of two correction terms.

The efficiency of the proposed method has been illustrated with the design under wind loads of the Millau Viaduct. For this structure, non-proportional damping has a significant influence on the response. The comparison with results obtained with the decoupling approximation (consisting in simply neglecting off-diagonal terms) indicates that the proposed method decreases significantly the errors on the covariance matrix of modal coordinates. Consequently, the method decreases also the error on the estimation of any subsequent structural response, e.g. internal bending moments.

In the proposed method, the solution may be undertaken as a sequence of uncoupled analyses, as usually performed. For convenience, each of the proposed correction terms may be related to a virtual stochastic loading. Eventually these virtual loadings are applied to the uncoupled model (i.e. in the usual modal basis), in order to progressively refine the response in a simple, efficient and rapid manner.

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