MATHEMATISCHE NACHRICHTEN

www.mn-journal.org

Founded in 1948 by Erhard Schmidt

WILEY

REPRINT

Math. Nachr. **285**, No. 17–18, 2157 – 2172 (2012) / **DOI** 10.1002/mana.201100291

Some characterizations of generalized Hölder spaces

Damien Kreit* and **Samuel Nicolay****

Department of Mathematics, University of Liège, Grande Traverse, 12, Bâtiment B37, B-4000 Liège, Belgium

Received 18 October 2011, revised 2 February 2012, accepted 1 June 2012
Published online 24 July 2012

Key words Generalized Hölder spaces, admissible sequences
MSC (2010) 26B35, 46B25

In this paper, we show that the main properties of the usual Hölder spaces can be transposed in the setting of the Hölder spaces defined via admissible sequences. In particular, we give several alternative definitions of these spaces.

1 Introduction

The Hölder-Zygmund space $\Lambda^\alpha\left(\mathbf{R}^d\right)$ $(\alpha > 0)$ is the space of the functions f of $L^\infty\left(\mathbf{R}^d\right)$ (one often requires the function to be also continuous) satisfying the Hölder condition

$$\sup_{|h|\leq r} \left\|\Delta_h^{[\alpha]+1} f\right\|_\infty \leq C r^\alpha, \tag{1.1}$$

for a positive constant C and any $r > 0$ (the notations used in this inequality will be given in Section 2.3). These spaces, also called Lipschitz spaces or simply Hölder spaces, provide a notion of smoothness of basic importance in areas as diverse as partial differential equations [8] or signal analysis [2]. In [11], the connections between approximation theory, calculus of finite differences, Hölder spaces, Taylor's theorem and interpolation theory are established.

Such spaces have been generalized in [9], [5]: following the notation introduced in [5], the space $\mathcal{C}_\omega^n\left(\mathbf{R}^d\right)$ $(n \in \mathbf{N})$ is defined as the space of the functions $f \in L^\infty\left(\mathbf{R}^d\right)$ such that there exists some positive constant C for which the inequality

$$\sup_{|h|\leq r} \left\|\Delta_h^n f\right\|_\infty \leq C\omega(r) \tag{1.2}$$

is satisfied for any $r > 0$, where ω is a modulus of smoothness, i.e., a non-decreasing function satisfying some natural conditions (see Section 2.4). For example, the law of the iterated logarithm [10] states that there exists a constant $C > \sqrt{2}$ such that the sample paths of the Brownian motion satisfy inequality (1.2) almost surely with $n = 1$ and

$$\omega(r) = \sqrt{r|\log|\log r||},$$

for any $r \in (0, 1)$. On the other hand, generalized Besov spaces $B_{p,q}^\sigma\left(\mathbf{R}^d\right)$ [13], where σ is a so-called admissible sequence (see Section 2.1) also provide a generalization of the usual Hölder spaces. In the classical case, i.e., when considering $B_{p,q}^\alpha\left(\mathbf{R}^d\right)$ with $\alpha > 0$, one has $\Lambda^\alpha\left(\mathbf{R}^d\right) = B_{\infty,\infty}^\alpha\left(\mathbf{R}^d\right)$; given an admissible sequence σ, it is thus natural to identify the generalized Hölder space $\Lambda^\sigma\left(\mathbf{R}^d\right)$ with $B_{\infty,\infty}^\sigma\left(\mathbf{R}^d\right)$. It can be easily shown that the spaces $\mathcal{C}_\omega^n\left(\mathbf{R}^d\right)$ are akin to the spaces $\Lambda^\sigma\left(\mathbf{R}^d\right)$.

* e-mail: D.kreit@alumni.ulg.ac.be, Phone: +32(0) 43 66 94 33, Fax: +32(0) 43 66 95 47
** Corresponding author: e-mail: S.Nicolay@ulg.ac.be, Phone: +32(0) 43 66 94 33, Fax: +32(0) 43 66 95 47

Here, we try to follow the path taken in [11], but in the general setting of the Hölder spaces defined via admissible sequences σ, in order to establish some specific properties of the spaces $B^{\sigma}_{\infty,\infty}(\mathbf{R}^d)$. The paper is organized as follows. We first give the definitions as well as some basic results. In Section 3, we look for conditions leading to embedded spaces: if for each $\alpha > 0$, $\sigma^{(\alpha)}$ is an admissible sequence, under which conditions do we have $\alpha < \beta$ implies $\Lambda^{\sigma^{(\beta)}}(\mathbf{R}^d) \subset \Lambda^{\sigma^{(\alpha)}}(\mathbf{R}^d)$? Next, we look at the relations between the spaces $\Lambda^{\sigma}(\mathbf{R}^d)$ and the spaces $C^k(\mathbf{R}^d)$ of the k-times continuously differentiable functions. In Section 5, we give alternative definitions of the generalized Hölder spaces. In most of the cases, the results holding for the usual spaces also hold for the generalized ones.

Throughout the paper, B denotes the open unit ball and we use the letter C for generic positive constant whose value may be different at each occurrence.

2 Definitions

The aim of this section is to introduce the definitions underlying the notion of generalized Hölder space as well as some notations and basic results.

2.1 Admissible sequences

Let us first recall the notion of admissible sequence [13].

Definition 2.1 A sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ of real positive numbers is called *admissible* if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,$$

for any $j \in \mathbf{N}$.

If σ is such a sequence, we set

$$\underline{\Theta}_j = \inf_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\Theta}_j = \sup_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows,

$$\underline{s} = \underline{s}(\sigma) = \lim_j \frac{\log_2 \underline{\Theta}_j}{j} \quad \text{and} \quad \overline{s} = \overline{s}(\sigma) = \lim_j \frac{\log_2 \overline{\Theta}_j}{j}.$$

Since $(\log \underline{\Theta}_j)_{j\in\mathbf{N}}$ is a subadditive sequence, such limits always exist [7].

Remark 2.2 If σ is an admissible sequence, let $\varepsilon > 0$; there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}-\varepsilon)} \leq \underline{\Theta}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\Theta}_j \leq C2^{j(\overline{s}+\varepsilon)}, \tag{2.1}$$

for any $j, k \in \mathbf{N}$.

Let us remark that if σ and γ are two admissible sequences, so are $\sigma + \gamma$, $\sigma\gamma$ (with $\underline{s}(\sigma\gamma) \geq \underline{s}(\sigma) + \underline{s}(\gamma)$ and $\overline{s}(\sigma\gamma) \leq \overline{s}(\sigma) + \overline{s}(\gamma)$) and $r\sigma$ for any $r > 0$ (with $\underline{s}(r\sigma) = \underline{s}(\sigma)$ and $\overline{s}(r\sigma) = \overline{s}(\sigma)$). Moreover, we have the following trivial result, wich will be used in the sequel.

Lemma 2.3 *If σ is an admissible sequence and $\alpha \in \mathbf{R}$, σ^{α} is an admissible sequence and*

- *if $\alpha > 0$, $\underline{s}(\sigma^{\alpha}) = \alpha\underline{s}(\sigma)$, $\overline{s}(\sigma^{\alpha}) = \alpha\overline{s}(\sigma)$,*
- *if $\alpha < 0$, $\underline{s}(\sigma^{\alpha}) = \alpha\overline{s}(\sigma)$, $\overline{s}(\sigma^{\alpha}) = \alpha\underline{s}(\sigma)$.*

We will also need some easy consequences of Remark 2.2.

Lemma 2.4 *Let σ be an admissible sequence*;

- *if $\underline{s} > 0$, there exists a positive constant C such that, for any $J \in \mathbf{N}$,*

$$\sum_{j=J}^{\infty} \sigma_j^{-1} \leq C\sigma_J^{-1},$$

- *if $n \in \mathbf{N}$ satisfies $\overline{s} < n$, there exists a positive constant C such that, for any $J \in \mathbf{N}$,*

$$\sum_{j=1}^{J} 2^{jn} \sigma_j^{-1} \leq C 2^{Jn} \sigma_J^{-1}.$$

Classical admissible sequences are the dyadic sequences.

Example 2.5 Let $\alpha \in \mathbf{R}$; the sequence $\sigma = \big(2^{-j\alpha}\big)_{j\in\mathbf{N}}$ is an admissible sequence with $\underline{s} = \overline{s} = -\alpha$.

More generally, let $\phi : [0,1] \to (0,\infty)$ be a weakly varying function, i.e., a function satisfying

$$\lim_{x\to 0} \frac{\phi(rx)}{\phi(x)} = 1,$$

for any $r > 0$. For any $\alpha \in \mathbf{R}$, the sequence $\sigma = \big(2^{-\alpha j} \phi(2^j)\big)_{j\in\mathbf{N}}$ is an admissible sequence such that $\underline{s} = \overline{s} = -\alpha$. For example, $\phi = |\log|$ is a weakly varying function.

Other examples can be found in [1], [6], [12], for instance.

2.2 Strong admissible sequences

We will sometimes need to make additional assumptions on the admissible sequences. We transpose here the concept of strong modulus of smoothness [9] to the admissible sequences.

Definition 2.6 An admissible sequence σ is a *strong admissible sequence of order* $n \in \mathbf{N}$ if there exists a constant C such that

$$\sum_{j=1}^{J} 2^{jn} \sigma_j \leq C 2^{Jn} \sigma_J \tag{2.2}$$

and

$$\sum_{j=J}^{\infty} 2^{j(n-1)} \sigma_j \leq C 2^{J(n-1)} \sigma_J, \tag{2.3}$$

for any $J \in \mathbf{N}$.

Example 2.7 Let $n \in \mathbf{N}$; for any $\alpha > 0$ chosen in $(n-1, n)$ and any $\beta \in \mathbf{R}$, the sequence $\sigma = \big(2^{-j\alpha} j^{\beta}\big)_{j\in\mathbf{N}}$ is a strong admissible sequence of order n.

Now, if $\alpha \in \mathbf{N}$, there is no $n \in \mathbf{N}$ such that the sequence defined above is strong of order n. This is easily shown by considering the cases $\alpha < n$, $\alpha > n$ and $\alpha = n$ separately.

This notion is closely related to the Boyd indices of the inverse sequence.

Lemma 2.8 *A strong admissible sequence σ of order n is such that*

$$n - 1 \leq \underline{s}\big(\sigma^{-1}\big) \leq \overline{s}\big(\sigma^{-1}\big) \leq n.$$

P r o o f. The inequalities

$$2^{kn} \sigma_k \leq \sum_{l=1}^{j+k} 2^{ln} \sigma_l \leq C 2^{(j+k)n} \sigma_{j+k}$$

hold for any $j, k \in \mathbf{N}$, which implies $\overline{s}\big(\sigma^{-1}\big) \leq n$. Moreover, one has

$$2^{(j+k)(n-1)} \sigma_{j+k} \leq \sum_{l=k}^{\infty} 2^{l(n-1)} \sigma_l \leq C 2^{k(n-1)} \sigma_k,$$

for any $j, k \in \mathbf{N}$ and therefore $n - 1 \leq \underline{s}\big(\sigma^{-1}\big)$. □

We have a partial converse result.

Lemma 2.9 *If σ is an admissible sequence satisfying*

$$n-1<\underline{s}(\sigma^{-1}) \quad \text{and} \quad \overline{s}(\sigma^{-1})<n$$

for some $n \in \mathbf{N}$, then σ is strong of order n.

Proof. Let $\varepsilon>0$ such that $\underline{s}(\sigma^{-1})-\varepsilon>n-1$; we have, using inequalities (2.1),

$$\sum_{k=J}^{\infty} 2^{k(n-1)}\sigma_k \leq 2^{J(n-1)}\sigma_J \sum_{k=1}^{\infty} 2^{k(n-1)} 2^{-k(\underline{s}(\sigma^{-1})-\varepsilon)},$$

wich implies (2.2). This is sufficient, since Lemma 2.4 implies (2.3). □

The following easy result shows that a strong admissible sequence of order n lies in between the sequences $(2^{-jn})_{j\in\mathbf{N}}$ and $(2^{-j(n-1)})_{j\in\mathbf{N}}$. It will be used in the next section to obtain embeddings between the classical Hölder spaces and the generalized ones.

Lemma 2.10 *If σ is a strong admissible sequence of order n, there exists a positive constant C such that*

$$C^{-1}2^{-jn} \leq \sigma_j \leq C2^{-j(n-1)},$$

for any $j \in \mathbf{N}$.

2.3 Generalized Hölder spaces

Given an admissible sequence σ, we introduce here the generalized Hölder space Λ^σ in such a way that Λ^σ is the space $B^{\sigma^{-1}}_{\infty,\infty}$ in "most cases", where $B^{\sigma}_{\infty,\infty}$ is a generalized Besov space, as defined in [13] for example. As usual, $[\alpha]$ will denote the greatest integer lower than α,

$$[\alpha]=\sup\{p \in \mathbf{Z} : p \leq \alpha\}$$

and $\Delta_h^n f$ will stand for the finite difference of order n: given a function f defined on $\mathbf{R}^d$ and $x,h \in \mathbf{R}^d$,

$$\Delta_h^1 f(x)=f(x+h)-f(x) \quad \text{and} \quad \Delta_h^{n+1} f(x)=\Delta_h^1\Delta_h^n f(x),$$

for any $n \in \mathbf{N}$.

Definition 2.11 Let $\alpha>0$ and σ an admissible sequence; a function $f \in L^\infty(\mathbf{R}^d)$ belongs to the space $\Lambda^{\sigma,\alpha}=\Lambda^{\sigma,\alpha}(\mathbf{R}^d)$ if there exists $C>0$ such that

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{[\alpha]+1} f\right\|_\infty \leq C\sigma_j,$$

for any $j \in \mathbf{N}$.

Of course if the sequence σ is non-decreasing, the preceding definition does not make much sense. We will show that the natural generalized Hölder spaces are the spaces $\Lambda^\sigma=\Lambda^{\sigma,\overline{s}(\sigma^{-1})}$ (see Corollary 3.3, Example 2.19 and Corollary 4.8). The application

$$|f|_{\Lambda^{\sigma,\alpha}}=\sup_j \left(\sigma_j^{-1}\sup_{|h|\leq 2^{-j}}\left\|\Delta_h^{[\alpha]+1} f\right\|_{L^\infty}\right)$$

defines a semi-norm on $\Lambda^{\sigma,\alpha}$ and therefore $\|f\|_{\Lambda^{\sigma,\alpha}}=\|f\|_{L^\infty}+|f|_{\Lambda^{\sigma,\alpha}}$ is a norm on this space. Using a classical argument to build a subsequence with the desired properties, we have the following result.

Theorem 2.12 *Let $\alpha>0$ and let σ be an admissible sequence; the space $\Lambda^{\sigma,\alpha}$ equipped with the norm $\|\cdot\|_{\Lambda^{\sigma,\alpha}}$ is a Banach space.*

Remark 2.13 If σ is an admissible sequence such that $\overline{s} < 0$, then $\Lambda^\sigma = B^{\sigma^{-1}}_{\infty,\infty}$ [13]. However, if $\overline{s} = 0$, it is well known that $B^{\sigma^{-1}}_{\infty,\infty} \not\hookrightarrow L^\infty$ [4].

The following result can be obtained effortless; it allows to introduce a slightly different definition of the spaces $\Lambda^{\sigma,\alpha}$.

Lemma 2.14 *Let $\alpha > 0$, $p \in \mathbf{Z}$ and let σ be an admissible sequence; a function $f \in L^\infty(\mathbf{R}^d)$ satisfies*

$$\sup_{|h|\le r} \left\| \Delta_h^{[\alpha]+1} f \right\|_\infty \le C\sigma_{[\log_2 1/r]+p} \tag{2.4}$$

for some positive constant C and any r if and only if $f \in \Lambda^{\sigma,\alpha}$.

2.4 Modulus of smoothness

In [5], [9], the notion of modulus of smoothness is used to obtain a generalisation of the classical Hölder spaces. The idea is to replace the modulus $|h|^\alpha$ appearing in the right-hand side of inequality (1.1) by a more general function $\omega(|h|)$. These so-obtained spaces are a particular case of the spaces $\Lambda^{\sigma,\alpha}$.

Definition 2.15 A non-decreasing function $\omega : (0,\infty) \to (0,\infty)$ is a modulus of smoothness if there exists a constant C such that

$$\omega(2x) \le C\omega(x),$$

for any $x \in (0,\infty)$. One then extends ω by setting $\omega(0) = 0$.

The link between the moduli of smoothness and the admissible sequences is given by the following easy result.

Lemma 2.16 *Let σ be an admissible sequence; there exists a modulus of smoothness ω such that $\sigma = \omega(2^{-j})_{j\in\mathbf{N}}$ if and only if σ is a decreasing sequence. Moreover, such a function ω is continuous at the origin if and only if σ converges to* 0.

Remark 2.17 Indeed, since we are only interested in the asymptotic behavior of σ, one could ask for the conditions defining an admissible sequence to hold only for $j \ge J$, for some $J \in \mathbf{N}$. In the same way, since only the behavior of ω near the origin really matters, we could define a modulus of continuity using germ functions.

This lemma together with Lemma 2.14 imply that one can replace a modulus of smoothness by a corresponding admissible sequence for the definition of the generalized Hölder spaces.

Proposition 2.18 *Let ω be a modulus of smoothness; a function $f \in L^\infty(\mathbf{R}^d)$ satisfies*

$$\sup_{|h|\le r} \left\| \Delta_h^{[\alpha]+1} f(x) \right\|_\infty \le C\omega(r) \tag{2.5}$$

for some positive constant C and any r if and only if $f \in \Lambda^{\sigma,\alpha}$, where σ is the sequence defined by $\sigma_j = \omega(2^{-j})$.

As a direct consequence, the usual Hölder spaces Λ^α $(\alpha > 0)$ are of the form Λ^σ.

Example 2.19 Let $\alpha > 0$ and $\sigma = (2^{-j\alpha})_{j\in\mathbf{N}}$. The space Λ^σ is the usual Hölder space Λ^α.

Lemmata 2.10 and 2.8 imply the following result.

Corollary 2.20 *If σ is a strong admissible sequence of order N, we have*

$$\Lambda^N \hookrightarrow \Lambda^\sigma \hookrightarrow \Lambda^{N-1}.$$

3 Embeddings between Hölder spaces

The usual Hölder spaces are embedded, i.e., $0 < \alpha < \beta$ implies $\Lambda^\beta \hookrightarrow \Lambda^\alpha$. A question that naturally arises is therefore the following: What are the conditions to impose to two admissible sequences σ and γ in order to obtain $\Lambda^\gamma \hookrightarrow \Lambda^\sigma$. Indeed, we will obtain results concerning the spaces $B^\sigma_{p,\infty}$, since we are going to deal with L^p-norms. In this section, p will designate an element of $[1,\infty]$.

3.1 Preliminary results

The main result of this section is the following.

Proposition 3.1 *Let σ be an admissible sequence and let f be an element of $L^p(\mathbf{R}^d)$ satisfying*

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_p \leq C\sigma_j,$$

with $n \in \mathbf{N}\setminus\{1\}$, for some constant C and any $j \in \mathbf{N}$. We have the three following cases:

- *if* $2^{n-1}\underline{\Theta}_1 > 1$,

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \leq C\big(\sigma_j + 2^{-j(n-1)}\big), \tag{3.1}$$

- *if* $2^{n-1}\underline{\Theta}_1 < 1$,

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \leq C\Big(\big(2^{n-1}\underline{\Theta}_1\big)^{-j}\sigma_j + 2^{-j(n-1)}\Big), \tag{3.2}$$

- *if* $2^{n-1}\underline{\Theta}_1 = 1$,

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \leq C\big(j\sigma_j + 2^{-j(n-1)}\big), \tag{3.3}$$

for any $j \in \mathbf{N}$.

Proof. For $J \in \mathbf{N}$, let us consider the classical telescopic sum

$$\begin{aligned}
&\sum_{j=0}^{J} 2^{-j(n-1)}\left(2^{n-1}\sup_{|h|\leq 2^{-J}}\left\|\Delta_{2^j h}^{n-1} f\right\|_p - \sup_{|h|\leq 2^{-J}}\left\|\Delta_{2(2^j h)}^{n-1} f\right\|_p\right)\\
&= 2^{n-1}\sup_{|h|\leq 2^{-J}}\left\|\Delta_h^{n-1} f\right\|_p - 2^{-J(n-1)}\sup_{|h|\leq 2^{-J}}\left\|\Delta_{2^{J+1}h}^{n-1} f\right\|_p.
\end{aligned}$$

The inequality

$$\left\|\Delta_h^m f\right\|_p \leq \frac{m}{2}\left\|\Delta_h^{m+1} f\right\|_p + 2^{-m}\left\|\Delta_{2h}^m f\right\|_p$$

holding for any $m \in \mathbf{N}$ and any $h \in \mathbf{R}^d$ implies

$$\begin{aligned}
&2^{n-1}\sup_{|h|\leq 2^{-J}}\left\|\Delta_h^{n-1} f\right\|_p\\
&\leq C\sum_{j=0}^{J} 2^{-j(n-1)}\sup_{|h|\leq 2^{-(J-j)}}\left\|\Delta_h^n f\right\|_p + 2^{-J(n-1)}\sup_{|h|\leq 2^{-J}}\left\|\Delta_{2^{J+1}h}^{n-1} f\right\|_p\\
&\leq C\left(\sum_{j=0}^{J}\big(2^{-(n-1)}\underline{\Theta}_1^{-1}\big)^j\right)\sigma_J + 2^{-J(n-1)}C,
\end{aligned}$$

which is sufficient to conclude. □

The sequences defined by the right-hand side of the inequalities (3.1), (3.2) and (3.3) are still admissible sequences. The previous result can be stated in terms of Boyd index.

Corollary 3.2 *Let σ be an admissible sequence and let f be an element of $L^p(\mathbf{R}^d)$ satisfying*

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_p \leq C\sigma_j,$$

with $n \in \mathbf{N}\setminus\{1\}$. We have the two following cases.

- *if* $\overline{s}(\sigma^{-1}) < n-1$, *we have*

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \leq C\sigma_j + C2^{-j(n-1)},$$

- *if* $\overline{s}(\sigma^{-1}) \geq n-1$, *for any* $\varepsilon > 0$, *there exists a constant* C_ε *such that*

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \leq C_\varepsilon 2^{j(\overline{s}(\sigma^{-1})-(n-1)+\varepsilon)}\sigma_j + C2^{-j(n-1)},$$

for any $j \in \mathbf{N}$.

Proof. For any $\varepsilon > 0$, inequalities (2.1) imply

$$\begin{aligned} &2^{n-1} \sup_{|h|\leq 2^{-j}} \left\|\Delta_h^{n-1} f\right\|_p \\ &\leq C\sum_{j=0}^{J} 2^{-j(n-1)} \sup_{|h|\leq 2^{-(J-j)}} \left\|\Delta_h^{n} f\right\|_p + 2^{-J(n-1)} \sup_{|h|\leq 2^{-J}} \left\|\Delta_{2^{J+1}h}^{n-1} f\right\|_p \\ &\leq C_\varepsilon \sigma_J \sum_{j=0}^{J} 2^{j(\overline{s}(\sigma^{-1})-(n-1)+\varepsilon)} + C2^{-j(n-1)}, \end{aligned}$$

which leads to the conclusion. □

The order $\overline{s}(\sigma^{-1})$ in the definition of the generalized Hölder spaces Λ^σ is optimum.

Corollary 3.3 *Let* σ *be an admissible sequence*; *for any* $\varepsilon > 0$, *we have* $\Lambda^\sigma = \Lambda^{\sigma,\overline{s}(\sigma^{-1})+\varepsilon}$.

3.2 Decreasing Hölder spaces

We are now able to give sufficient conditions on two admissible sequences for the corresponding spaces to be embedded.

Definition 3.4 If for any $\alpha > 0$, $\sigma^{(\alpha)}$ is an admissible sequence, the application

$$\sigma^{(\cdot)} : \alpha > 0 \mapsto \sigma^{(\alpha)}$$

is called a *family of admissible sequences* if the two following conditions are satisfied:

- $0 < \alpha < \beta$ implies $\overline{s}\big((\sigma^{(\alpha)})^{-1}\big) < \overline{s}\big((\sigma^{(\beta)})^{-1}\big)$.
- for any $n \in \mathbf{N}$, there exists $\alpha > 0$ such that $\overline{s}\big((\sigma^{(\alpha)})^{-1}\big) = n$.

Definition 3.5 A family of admissible sequences $\sigma^{(\cdot)}$ is called *decreasing* if $0 < \alpha < \beta$ implies $\Lambda^{\sigma^{(\beta)}} \hookrightarrow \Lambda^{\sigma^{(\alpha)}}$.

Definition 3.6 Let $\sigma^{(\cdot)}$ be a decreasing family of admissible sequences; the generalized Hölder exponent associated to a function $f \in L^\infty(\mathbf{R}^d)$ is defined as

$$H(f;\sigma^{(\cdot)}) = \sup\left\{\alpha > 0 : f \in \Lambda^{\sigma^{(\alpha)}}\right\}.$$

Let us now search for conditions under which a family of admissible sequences is decreasing. If $\sigma^{(\cdot)}$ is a family of admissible sequences, for $\alpha > 0$, we set $t_\alpha = \overline{s}\big((\sigma^{(\alpha)})^{-1}\big)$ and

$$\underline{\Theta}_j^{(\alpha)} = \inf_{k\in\mathbf{N}} \frac{\sigma_{j+k}^{(\alpha)}}{\sigma_k^{(\alpha)}}.$$

Proposition 3.1 gives the following result.

Corollary 3.7 *A family of admissible sequences $\sigma^{(\cdot)}$ is decreasing if for every $\beta > 0$, the following conditions are satisfied*:

- *If $t_\beta \notin \mathbf{N}$, for any $\alpha > 0$ such that $t_\alpha \in [[t_\beta], t_\beta)$, there exist a constant C and an index J such that*

$$\sigma_j^{(\beta)} \leq C\sigma_j^{(\alpha)},$$

for any $j \geq J$.

- *If $t_\beta \in \mathbf{N}$, there exists $\varepsilon > 0$ such that for any $\alpha \in (\beta - \varepsilon, \beta)$, there exist a constant C and an index J for which*

$$2^{-jt_\beta} \leq C\sigma_j^{(\alpha)},$$

for any $j \geq J$.

- *If $t_\beta \in \mathbf{N}$, there exist $\varepsilon > 0$ such that for any $\alpha \in (\beta - \varepsilon, \beta)$, there exist a constant C and an index J for which*
  - *if $2^{t_\beta}\underline{\Theta}_1^{(\beta)} > 1$,*

$$\sigma_j^{(\beta)} \leq C\sigma_j^{(\alpha)}, \quad for\ any \quad j \geq J,$$

  - *if $2^{t_\beta}\underline{\Theta}_1^{(\beta)} < 1$,*

$$\sigma_j^{(\beta)}\left(2^{t_\beta}\underline{\Theta}_1^{(\beta)}\right)^{-j} \leq C\sigma_j^{(\alpha)}, \quad for\ any \quad j \geq J,$$

  - *if $2^{t_\beta}\underline{\Theta}_1^{(\beta)} = 1$,*

$$j\sigma_j^{(\beta)} \leq C\sigma_j^{(\alpha)}, \quad for\ any \quad j \geq J.$$

Let us give an example.

Example 3.8 Let $\alpha > 0$; if β is a non-negative function defined on $(0, \infty)$, the family $\sigma^{(\alpha)} = \left(2^{-j\alpha} j^{\beta(\alpha)}\right)_{j\in\mathbf{N}}$ satisfies the hypothesis of Corollary 3.7 and is thus a decreasing family of admissible sequences.

Let us emphasize that the conditions stated in Corollary 3.7 are not sufficient.

Remark 3.9 Let $\sigma^{(\cdot)}$ be the family of admissible sequences defined by

$$\sigma_j^{(\alpha)} = \begin{cases} 2^{-j\alpha} & \text{if} \quad \alpha \in (0,\infty)\backslash\mathbf{N}, \\ 2^{-j\alpha} j^{-1} & \text{if} \quad \alpha \in \mathbf{N}. \end{cases}$$

It is easy to check that this family is a decreasing family for which the third condition of Corollary 3.7 is not satisfied.

4 Generalized Hölder spaces and smoothness

Here, we investigate the relationship between the Hölder spaces and the regularity of their elements, i.e., the possible embeddings between the spaces $\Lambda^\sigma\left(\mathbf{R}^d\right)$ and the spaces $C^k\left(\mathbf{R}^d\right)$ of the k-times continuously differentiable functions.

The space of the infinitely differentiable functions with compact support included in B will be written $C_c^\infty(B)$. In this section, ρ will denote a radial function of $C_c^\infty(B)$ such that $\rho(x) \in [0, 1]$ for any $x \in \mathbf{R}^d$ and $\|\rho\|_1 = 1$. Moreover, for any function f, one sets

$$f_\delta = \frac{1}{\delta^d} f\left(\frac{\cdot}{\delta}\right),$$

for any $\delta \neq 0$.

4.1 Preliminaries

We start by adapting two classical lemmata (see e.g., [11]). For the sake of simplicity, we only gives the results for $\mathbf{R}$, although they are still valid in $\mathbf{R}^d$, with proofs that are much the same.

Lemma 4.1 *Let $n \in \mathbf{N}$, let σ be an admissible sequence and $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ such that*

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_\infty \leq C\sigma_j, \tag{4.1}$$

for a constant $C > 0$ and any $j \in \mathbf{N}$. There exists $\phi \in C_c^\infty(\mathbf{R})$ such that

$$\sup_{0<h\leq 2^{-j}} \|f * \phi_h - f\|_\infty \leq C\sigma_j,$$

for a constant $C > 0$ and any $j \in \mathbf{N}$.

P r o o f. We may add 1, 2 or 3 to n if necessary so that we can assume that n is equal to $2m$, where m is an odd integer. Let us define

$$\gamma = \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \rho_{2j-n} \tag{4.2}$$

and $\phi = \gamma / \int \gamma\, dx$. For $\delta > 0$, one has

$$\begin{aligned}
&f * \phi_\delta(x) - f(x)\\
&= \int f(x-\delta t)\phi(t)\, dt - f(x)\\
&= \frac{1}{\int \gamma\, dt} \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \int f(x-\delta t)\rho_{2j-n}(t)\, dt - f(x)\\
&= \frac{1}{2\int \gamma\, dt} \left(\sum_{\substack{j=0\\ j\neq m}}^{n} (-1)^j \binom{n}{j} \int f(x-\delta(2j-n)t)\rho(t)\, dt - 2\Big(\int \gamma\, dt\Big) f(x) \right)\\
&= \frac{1}{2\int \gamma\, dt} \int \Delta_{\delta t}^n f(x)\rho(t)\, dt,
\end{aligned}$$

which is sufficient to conclude. □

Lemma 4.2 *Let σ be an admissible sequence and $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ a function for which there exists ρ such that*

$$\|f * \rho_{2^{-j}} - f\|_\infty \leq C\sigma_j,$$

for a constant $C > 0$ and any $j \in \mathbf{N}$. For any $k \in \mathbf{N}$, there exists $C > 0$ such that

$$\left\|D^k(f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}})\right\|_\infty \leq C\, 2^{jk}\sigma_j,$$

for any $j \in \mathbf{N}$.

P r o o f. For $\delta > 0$, let us write

$$\begin{aligned}
&f * \rho_\delta - f * \rho_{2\delta}\\
&= \rho_\delta * (f * \rho_\delta - f * \rho_{2\delta}) + \rho_\delta * (f - f * \rho_\delta) - \rho_{2\delta} * (f - f * \rho_\delta).
\end{aligned} \tag{4.3}$$

For the first term of the right-hand side, one gets

$$\begin{aligned}
\left|D^k\left(\rho_\delta * (f * \rho_\delta - f * \rho_{2\delta})\right)\right| &\leq \left\|D^k \rho_\delta\right\|_1 \|f * \rho_\delta - f * \rho_{2\delta}\|_\infty\\
&\leq C\delta^{-k}(\|f * \rho_\delta - f\|_\infty + \|f - f * \rho_{2\delta}\|_\infty),
\end{aligned}$$

which gives the following inequality for $\delta = 2^{-j}$,

$$\left|D^k\left(\rho_{2^{-j}} * (f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}})\right)\right| \leq C\, 2^{jk}\sigma_j .$$

The two other terms of the right-hand side of equality (4.3) can be handled in the same way, leading to the desired result. □

4.2 Hölder spaces and C^k spaces

If $f \in L^\infty(\mathbf{R}^d)$ satisfies inequality (4.1), let ϕ denotes the same function as in Lemma 4.1 and let us set

$$f_1 = f * \phi_{2^{-1}} \quad \text{and} \quad f_{j+1} = f * (\phi_{2^{-j-1}} - \phi_{2^{-j}}) \tag{4.4}$$

for any $j \in \mathbf{N}$. Since there exists a constant $C > 0$ such that $\|f_j\|_\infty \leq C\sigma_j$ for any $j \in \mathbf{N}$,

$$\sum_{j=1}^{k} \|f_j\|_\infty \leq C \sum_{j=1}^{k} \sigma_j ,$$

for any $k \in \mathbf{N}$. Therefore, if the series $\sum_j \sigma_j$ converges, we have $f = \sum_{j=1}^\infty f_j$ in L^∞. We then have the following result.

Proposition 4.3 *Let σ an admissible sequence satisfying*

$$\sum_{j=1}^{\infty} 2^{jk}\sigma_j < \infty,$$

for some $k \in \mathbf{N}$. If the function $f \in L^\infty(\mathbf{R}^d)$ satisfies

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_\infty \leq C\sigma_j ,$$

for some $n \in \mathbf{N}$ and any $j \in \mathbf{N}$, then f is equal almost everywhere to a bounded element of $C^k(\mathbf{R}^d)$.

Proof. The series $\sum_j f_j$ converges uniformly to a function that is equal almost everywhere to f and

$$\|D^\nu f_j\|_\infty \leq C 2^{jk}\sigma_j ,$$

for any $j \in \mathbf{N}$ and any multi-index ν such that $|\nu| \leq k$. We thus get that the series $\sum_j D^\nu f_j$ converges uniformly, which ends the proof. □

The following result can also be deduced from Corollary 2.20 and the properties of the spaces Λ^α with $\alpha > 0$ (see Example 4.6).

Corollary 4.4 *If σ is a strong admissible sequence of order n, $f \in \Lambda^\sigma$ implies that f is equal almost everywhere to a bounded element of $C^{n-1}(\mathbf{R}^d)$.*

Let $\lceil\cdot\rceil$ denote the ceil function, i.e., $\lceil\alpha\rceil = \lim_{\beta\to 1^-}[\alpha+\beta]$. If $\sigma^{(\cdot)}$ is a family of admissible sequences, we set $\underline{t}_\alpha = \underline{s}\big((\sigma^{(\alpha)})^{-1}\big)$.

Corollary 4.5 *Let $\sigma^{(\cdot)}$ be a decreasing family of admissible sequences; if $\underline{t}_\alpha > 0$ then $f \in \Lambda^{\sigma^{(\alpha)}}$ implies that f is equal almost everywhere to a bounded element of $C^{\lceil \underline{t}_\alpha \rceil - 1}(\mathbf{R}^d)$.*

Example 4.6 Let $\alpha > 0$ and let f be an element of Λ^α. If $\alpha \notin \mathbf{N}$, f is equal almost everywhere to a bounded element of $C^{[\alpha]}(\mathbf{R}^d)$; otherwise, f is equal almost everywhere to a bounded element of $C^{\alpha-1}(\mathbf{R}^d)$. It is well known that these embeddings are optimum, since, for example, the function $|\cdot|$ belongs to Λ^1.

Let us now show that too small orders lead to trivial spaces.

Proposition 4.7 *If σ is an admissible sequence satisfying*

$$\sum_{j=1}^{\infty} 2^{j(n+1)}\sigma_j < \infty,$$

for some non-negative integer n, the space $\Lambda^{\sigma,n}$ is the space of the functions that are constant almost everywhere.

P r o o f. We know that f is equal almost everywhere to a function $g \in C^{n+1}(\mathbf{R}^d)$. Moreover, since

$$\lim_j \frac{\left|\Delta^{n+1}_{2^{-j}e_i} g(x)\right|}{2^{-j(n+1)}} = \left|D^{n+1}_{x_i} g(x)\right|$$

and

$$\frac{\left\|\Delta^{n+1}_{2^{-j}e_i} f(x)\right\|_\infty}{2^{-j(n+1)}} \leq C 2^{j(n+1)}\sigma_j \longrightarrow 0,$$

as j tends to ∞, we get $D^{n+1}_{x_i} g = 0$. Finally, since g is bounded, g is a constant function. □

Corollary 4.8 *Let σ an admissible sequence; if the non-negative integer n satisfies $n+1 < \underline{s}(\sigma^{-1})$, the space $\Lambda^{\sigma,n}$ is the space of the functions that are constant almost everywhere.*

5 Alternative definitions of the Hölder spaces

In this section, we give two alternative definitions of the spaces $\Lambda^{\sigma,\alpha}$. We first use approximations of a function f by polynomials or C_c^∞ functions. The rate at wich f can be approximated is bounded to the spaces $\Lambda^{\sigma,\alpha}$. If some additional assumptions are made about the admissible sequences, these spaces can also be characterized using derivatives or Taylor's expansion. Such results are well known for the classical Hölder spaces Λ^α, with $\alpha > 0$ (see e.g., [11]).

5.1 A characterization via a convolution product

When looking at how good an element of Λ^σ can be approximated by a smooth function, one gets the following result: a function $f \in L^\infty(\mathbf{R}^d)$ belongs to the classical Hölder space Λ^α $(\alpha > 0)$ if and only if there exist a function $\phi \in C_c^\infty(\mathbf{R}^d)$ and a positive constant C such that

$$\|f - f * \phi_\delta\|_\infty \leq C\delta^\alpha,$$

for any $\delta > 0$. This result is still valid in the general setting of the admissible sequences, although some weak additional hypotheses have to be made.

Proposition 5.1 *Let σ an admissible sequence and $f \in L^\infty(\mathbf{R}^d)$. If $f \in \Lambda^{\sigma,\alpha}$ $(\alpha > 0)$, there exists a function $\phi \in C_c^\infty(\mathbf{R}^d)$ such that*

$$\|f - f * \phi_{2^{-j}}\|_\infty \leq C\sigma_j, \tag{5.1}$$

for a constant $C > 0$ and any $j \in \mathbf{N}$. Conversely, if inequality (5.1) holds for a function $\phi \in C_c^\infty(\mathbf{R}^d)$ and if the admissible sequence satisfies

$$\sum_{j=1}^{J} 2^{j([\alpha]+1)}\sigma_j \leq C 2^{J[\alpha]+1}\sigma_J \tag{5.2}$$

and

$$\sum_{j=J+1}^{\infty} \sigma_j \leq C\sigma_J, \tag{5.3}$$

for a constant $C > 0$ and any $J \in \mathbf{N}$, then f belongs to $\Lambda^{\sigma,\alpha}$.

P r o o f. Lemma 4.1 implies the first part of the result. Let us show the converse part. If f_j is defined by equality (4.4), we have $\Delta_h^{[\alpha]+1} f = \sum_{j=1}^{\infty} \Delta_h^{[\alpha]+1} f_j$ in $L^\infty(\mathbf{R}^d)$ and thus, if we set $n = [\alpha]+1$,

$$\begin{aligned}\|\Delta_h^n f\|_\infty &\leq \sum_{j=1}^{J} \|\Delta_h^n f_j\|_\infty + \sum_{j=J+1}^{\infty} \|\Delta_h^n f_j\|_\infty \\ &\leq \sum_{j=1}^{J} |h|^n \|D^n f_j\|_\infty + \sum_{j=J+1}^{\infty} 2^n \|f_j\|_\infty \\ &\leq C|h|^n \sum_{j=1}^{J} 2^{jn}\sigma_j + C \sum_{j=J+1}^{\infty} \sigma_j,\end{aligned}$$

for any $J \in \mathbf{N}$. Using the hypothesis on the admissible sequence, we get

$$\|\Delta_h^n f\|_\infty \leq C\big(1 + |h|^n 2^{Jn}\big)\sigma_J,$$

which is sufficient to conclude. □

Lemma 2.4 gives the following result.

Corollary 5.2 *Let σ an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$; a function $f \in L^\infty(\mathbf{R}^d)$ belongs to Λ^σ if and only if inequality* (5.1) *is satisfied.*

Remark 5.3 Let $\alpha > 0$, let σ be an admissible sequence such that inequalities (5.2) and (5.3) are satisfied and define

$$\Phi = \Bigg\{\phi \in C_c^\infty(\mathbf{R}^d) \ : \ \phi = \frac{2\gamma}{\binom{2m}{m}}, \text{ where } \gamma \text{ satisfies equality (4.2) for some } m \\ \text{and } \sup_{|\nu|=[\alpha]+1} \|D^\nu \phi\|_1 \leq 2^{[\alpha]+1}\Bigg\}.$$

When looking at how the characterization with the convolution has been obtained, it is easy to check that the semi-norm defined by

$$|f| = \inf_{\phi\in\Phi} \sup_{j\in\mathbf{N}} \big\{\sigma_j^{-1} \|f * \phi_{2^{-j}} - f\|_\infty\big\}$$

is such that the norm $\|\cdot\| = \|\cdot\|_\infty + |\cdot|$ is equivalent to $\|\cdot\|_{\Lambda^{\sigma,\alpha}}$.

5.2 A polynomial characterization

The definition of the classical Hölder spaces is often given in terms of approximation by polynomials: $f \in L^\infty(\mathbf{R}^d)$ belongs to Λ^α $(\alpha > 0)$ if there exist a constant $C > 0$ and a polynomial P of degree less than α such that

$$\|f - P\|_{L^\infty(hB+x_0)} \leq C|h|^\alpha,$$

for any $x_0 \in \mathbf{R}^d$. We generalize here this result. We will denote the set of polynomials of degree at most n by $\mathbf{P}_n$. We will also need the following classical result [3]: Let $n \in \mathbf{N}$ and $f \in L^\infty(\mathbf{R}^d)$; there exists a constant $C > 0$ depending on n and d such that, for any $x_0 \in \mathbf{R}^d$ and $r > 0$, the following inequality is satisfied,

$$\inf_{P\in\mathbf{P}_{n-1}} \|f - P\|_{L^\infty(rB+x_0)} \leq C \sup_{|h|\leq r} \|\Delta_h^n f\|_{L^\infty(rB+x_0)}. \tag{5.4}$$

We have the following easy result.

Lemma 5.4 *Let $n \in \mathbf{N}$ and $f \in L^\infty(\mathbf{R}^d)$; there exists a constant $C > 0$ such that, for any $x_0 \in \mathbf{R}^d$ and any polynomial $P \in \mathbf{P}_{n-1}$,*

$$\sup_{|h|\leq r} \|\Delta_h^n f\|_{L^\infty(rB+x_0)} \leq C\|f-P\|_{L^\infty((n+1)rB+x_0)}.$$

P r o o f. One directly gets

$$\begin{aligned}\sup_{|h|\leq r} \left\|\Delta_h^n f\right\|_{L^\infty(rB+x_0)} &= \sup_{|h|\leq r} \left\|\Delta_h^n (f-P)\right\|_{L^\infty(rB+x_0)} \\ &\leq 2^n \|f-P\|_{L^\infty((n+1)rB+x_0)},\end{aligned}$$

which is the wanted result. □

Proposition 5.5 *Let σ an admissible sequence, $\alpha > 0$ and $f \in L^\infty(\mathbf{R}^d)$; f belongs to $\Lambda^{\sigma,\alpha}$ if and only if there exists a constant $C > 0$ such that*

$$\inf_{P\in\mathbf{P}_{[\alpha]}} \|f-P\|_{L^\infty(2^{-j}B+x_0)} \leq C\,\sigma_j, \tag{5.5}$$

for any $x_0 \in \mathbf{R}^d$ and any $j \in \mathbf{N}$.

P r o o f. From inequality (5.4), we get that $f \in \Lambda^{\sigma,\alpha}$ implies inequality (5.5). Now, let $f \in L^\infty(\mathbf{R}^d)$ and suppose that inequality (5.5) is satisfied. We set $n = [\alpha]+1$ and $k = [\log_2(n+1)]+1$. There exists $J \in \mathbf{N}$ such that for any polynomial $P \in \mathbf{P}_{[\alpha]}$, we have

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_{L^\infty(2^{-j}B+x_0)} \leq C\|f-P\|_{L^\infty(2^{k-j}B+x_0)},$$

for any $j \geq J$. Therefore, inequality (5.5) implies the existence of a constant $C' > 0$ for which

$$\sup_{|h|\leq 2^{-j}} \left\|\Delta_h^n f\right\|_{L^\infty(2^{-j}B+x_0)} \leq C'\sigma_{j-k} \leq C''\sigma_j,$$

whenever $j \geq J$. This is sufficient to conclude, since C'' does not depend on j or x_0. □

Remark 5.6 Let $\alpha > 0$ and let σ be an admissible sequence. The semi-norm defined by

$$|f| = \left\| \sup_{j\in\mathbf{N}} \left\{ \sigma_j^{-1} \inf_{P\in\mathbf{P}_{[\alpha]}} \|f-P\|_{L^\infty(2^{-j}B+\cdot)} \right\} \right\|_\infty$$

is such that the norm $\|\cdot\| = \|\cdot\|_\infty + |\cdot|$ is equivalent to $\|\cdot\|_{\Lambda^{\sigma,\alpha}}$.

5.3 A characterization using derivatives

The usual Hölder spaces Λ^α can be defined without using finite differences, polynomials or convolutions. For $\alpha \in (0,1)$, the usual Hölder space Λ^α is the space of the functions $f \in L^\infty(\mathbf{R}^d)$ such that

$$\sup_{|h|\leq 2^{-j}} \|f(\cdot+h) - f(\cdot)\|_\infty \leq C2^{-j\alpha},$$

for some constant $C > 0$ and any $j \in \mathbf{N}$, while to belong to Λ^1, f must satisfy

$$\sup_{|h|\leq 2^{-j}} \|f(\cdot+h) + f(\cdot-h) - 2f(\cdot)\|_\infty \leq C2^{-j}.$$

Now, if $\alpha \in (k, k+1]$ for some integer $k \geq 1$, a function of $L^\infty(\mathbf{R}^d)$ is an element of Λ^α if it is equal almost everywhere to a function f with the following properties: $f \in C^k(\mathbf{R}^d)$ and f as well as $D_j f$ belong to $\Lambda^{\alpha-1}$ for any $j \in \{1,\dots,d\}$. Such a definition is still valid in the general case, as far as the admissible sequence satisfies some additional requirements.

Proposition 5.7 *Let σ be an admissible sequence and let k, n be two positive integers such that $k < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < n$. Any element of Λ^σ is equal almost everywhere to a function $f \in C^k(\mathbf{R}^d)$ satisfying $D^\nu f \in L^\infty(\mathbf{R}^d)$ for any multi-index ν such that $|\nu| \leq k$ and*

$$\sup_{|h| \leq 2^{-j}} \left\| \Delta_h^{n-|\nu|} D^\nu f \right\|_\infty \leq C \sigma_j 2^{j|\nu|}, \tag{5.6}$$

for any $j \in \mathbf{N}$ and $|\nu| \leq k$. Conversely, if $f \in L^\infty(\mathbf{R}^d) \cap C^k(\mathbf{R}^d)$ satisfies inequality (5.6) *for $|\nu| = k$, then $f \in \Lambda^\sigma$.*

P r o o f. Let $f \in \Lambda^\sigma$ and f_j be defined as before. There exists a function g that is equal to f almost everywhere such that

$$\sum_{j=1}^{\infty} D^\nu f_j = D^\nu g,$$

with uniform convergence, for any ν such that $|\nu| \leq k$. Now, Lemma 4.2 implies

$$\sum_{j=1}^{\infty} \|D^\nu f_j\|_\infty \leq C \sum_{j=1}^{\infty} 2^{jk} \sigma_j < \infty.$$

Let ν be a multi-index such that $|\nu| \leq k$ and $h \in \mathbf{R}^d$, $J \in \mathbf{N}$ such that $|h| \leq 2^{-J}$. Using the mean value theorem, we get

$$\begin{aligned}
\left\| \Delta_h^{n-|\nu|} D^\nu f \right\|_\infty &\leq \sum_{j=1}^{J} \left\| \Delta_h^{n-|\nu|} D^\nu f_j \right\|_\infty + \sum_{j=J+1}^{\infty} \left\| \Delta_h^{n-|\nu|} D^\nu f_j \right\|_\infty \\
&\leq \sum_{j=1}^{J} |h|^{n-|\nu|} \sup_{|\alpha|=n-|\nu|} \|D^{\alpha+\nu} f_j\|_\infty \\
&\quad + C \sum_{j=J+1}^{\infty} |h|^{k-|\nu|} \sup_{|\alpha|=k-|\nu|} \|D^{\alpha+\nu} f_j\|_\infty \\
&\leq C \sum_{j=1}^{J} |h|^{n-|\nu|} 2^{nj} \sigma_j + C \sum_{j=J+1}^{\infty} |h|^{k-|\nu|} 2^{kj} \sigma_j \\
&\leq C 2^{J|\nu|} \sigma_J .
\end{aligned}$$

Let us now show the converse result. If h satisfies $|h| \leq 2^{-j}$, we have, using the mean value theorem and inequality (5.6),

$$\left\| \Delta_h^n f \right\|_\infty \leq C |h|^k \sup_{|\nu|=k} \left\| \Delta_h^{n-k} D^\nu f \right\|_\infty \leq C\, 2^{-jk} 2^{jk} \sigma_j ,$$

which is sufficient to conclude. □

The same demonstration can be used to show the following results.

Proposition 5.8 *Let σ be a strong admissible sequence of order $n \in \mathbf{N}$. Any element of Λ^σ is equal almost everywhere to a function $f \in C^n(\mathbf{R}^d)$ satisfying $D^\nu f \in L^\infty(\mathbf{R}^d)$ for any multi-index ν such that $|\nu| \leq n$ and*

$$\sup_{|h| \leq 2^{-j}} \left\| \Delta_h^{n-|\nu|} D^\nu f \right\|_\infty \leq C\, 2^{j|\nu|} \sigma_j , \tag{5.7}$$

for any $j \in \mathbf{N}$ and $|\nu| \leq n$. Conversely, if $f \in L^\infty(\mathbf{R}^d) \cap C^n(\mathbf{R}^d)$ satisfies inequality (5.7) *for $|\nu| = n$, then $f \in \Lambda^\sigma$.*

Lemma 5.9 *Let k be a positive integer, let σ be an admissible sequence and $f \in L^\infty\big(\mathbf{R}^d\big) \cap C^k\big(\mathbf{R}^d\big)$. If there exist a natural number $n > k$ and a constant $C > 0$ such that*

$$\sup_{|h| \le 2^{-j}} \big\|\Delta_h^{n-k} D^\nu f\big\|_\infty \le C\, 2^{jk} \sigma_j,$$

for any $j \in \mathbf{N}$ and any multi-index ν such that $|\nu| = k$, then $f \in \Lambda^{\sigma, n-1}$.

Remark 5.10 Let σ an admissible sequence and let k, n be two positive integers such that $k < \underline{s}\big(\sigma^{-1}\big) \le \overline{s}\big(\sigma^{-1}\big) < n$. The space Λ^σ is the space of the functions $f \in L^\infty\big(\mathbf{R}^d\big) \cap C^k\big(\mathbf{R}^d\big)$ such that

$$\sup_{|\nu|=k} \sup_{j \in \mathbf{N}} \frac{\sup_{|h| \le 2^{-j}} \big\|\Delta_h^{n-k} D^\nu f\big\|_\infty}{2^{jk}\sigma_j} < \infty.$$

5.4 Taylor's formula

The Taylor's expansion also provide a characterization of the usual Hölder spaces. When looking at Section 5.2, one can expect for this result to also hold in the general setting of the admissible sequences. It should be noted however that the polynomial approximating the function depends on the scale j. We thus have to make additional assumptions in order to get rid of this dependence.

We first need a lemma.

Lemma 5.11 *If $f \in C^k\big(\mathbf{R}^d\big)$, we have, for any $x, h \in \mathbf{R}^d$,*

$$f(x+h) = \sum_{|\nu| \le k} \frac{h^\nu}{|\nu|!} D^\nu f(x) + \frac{|h|^k}{k!} R_k(x,h),$$

where

$$|R_k(x,h)| \le \sum_{|\nu|=k} \sup_{|\delta| \le |h|} \big\|\Delta_\delta^1 D^\nu f\big\|_\infty.$$

Proof. Let us fix $x \in \mathbf{R}^d$ and consider the function g defined on $\mathbf{R}$ as follows, $g : t \mapsto f(x+th)$. Using integration by parts, one gets

$$\begin{aligned} g(1) - g(0) &= \int_0^1 Dg(t)\, dt \\ &= [(t-1)Dg(t)]_0^1 - \int_0^1 (t-1) D^2 g(t)\, dt \\ &= \sum_{j=1}^{k-1} \frac{1}{j!} D^j g(0) + (-1)^{k-1} \int_0^1 \frac{(t-1)^{k-1}}{(k-1)!} D^k g(t)\, dt \\ &= \sum_{j=1}^{k-1} \frac{1}{j!} D^j g(0) + (-1)^{k-1} \int_0^1 \frac{(t-1)^{k-1}}{(k-1)!} D^k g(0)\, dt \\ &\quad + (-1)^{k-1} \int_0^1 \frac{(t-1)^{k-1}}{(k-1)!} \big(D^k g(t) - D^k g(0)\big)\, dt, \end{aligned}$$

which is sufficient to conclude. □

We have the following version of Taylor's expansion theorem.

Proposition 5.12 *Let σ be a strong admissible sequence of order $n \in \mathbf{N}$. If $f \in \Lambda^\sigma$, one has, for any $x, h \in \mathbf{R}^d$,*

$$f(x+h) = \sum_{|\nu| \le n-1} \frac{h^\nu}{|\nu|!} D^\nu f(x) + \frac{|h|^{n-1}}{(n-1)!} R_{n-1}(x,h), \tag{5.8}$$

almost everywhere, where R_{n-1} satisfies

$$\sup_{x,|h|\leq 2^{-j}} |R_{n-1}(x,h)| \leq C\sigma_j 2^{j(n-1)},$$

for any $j \in \mathbf{N}$. Conversely, if $f \in L^\infty(\mathbf{R}^d) \cap C^{n-1}(\mathbf{R}^d)$ satisfies equality (5.8)*, then $f \in \Lambda^\sigma$.*

Proof. If f is an element of Λ^σ, there exists a function $g \in C^{n-1}(\mathbf{R}^d)$ that is equal almost everywhere to f. Lemma 5.11 and Proposition 5.8 allow to conclude. The converse result has been shown in Section 5.2. □

Remark 5.13 In [9], the properties of functions f for which there exist a polynomial P and a constant C such that

$$\|f - P\|_{L^\infty(hB+x)} \leq C\omega(|h|),$$

for any $x \in \mathbf{R}^d$, where ω is a modulus of smoothness, are investigated. If ω is a strong modulus of continuity, i.e., if the corresponding sequence σ is a strong admissible sequence, such functions are the elements of Λ^σ.

References

[1] A. Almeida, Wavelet bases in generalized Besov spaces, J. Math. Anal. Appl. **304**(1), 198–211 (2005).

[2] A. Arneodo and B. Audit, Edwarrd-Benedict Brodie of Brodie, Samuel Nicolay, Marie Touchon, Yves D'Aubenton-Carafa, Maxime Huvet, and Claude Thermes, Fractals and wavelets: What can we learn on transcription and replication from wavelet-based multifractal analysis of DNA sequences?, in: Encyclopedia of Complexity and Systems Science, edited by R. A. Meyers (Springer, 2009), pp. 3893–3924.

[3] Yu A. Brudnyĭ, A multidimensional analog of a theorem of Whitney, Math. USSR Sb. **11**(2), 157–170 (1970).

[4] A. M. Caetano and S. D. Moura, Local growth envelopes of spaces of generalized smoothness: the critical case, Math. Inequal. Appl. **7**(4), 573–606 (2004).

[5] M. Clausel, Quelques notions d'irrégularité uniforme et ponctuelle: le point de vue ondelettes, PhD thesis, University of Paris XII (2008).

[6] W. Farkas and H.-G. Leopold, Characterisations of function spaces of generalised smoothness, Ann. Mat. Pura Appl. IV. Ser. **185**(1), 1–62 (2006).

[7] M. Fekete, On the distribution of roots of algebraic equations with integral coefficients, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. **17**, 228–249 (1923).

[8] Q. Han and F. Lin, Elliptic Partial Differential Equations, Courant Institute of Mathematical Sciences, second edition (New-York, 2011).

[9] S. Jaffard and Y. Meyer, Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions, Mem. Am. Math. Soc. **123**(587), (1996).

[10] A. Khintchine, Über einen Satz der Wahrscheinlichkeitsrechnung, Fundam. Math. **6**, 9–20 (1924).

[11] S. G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, Exp. Math. **1**(3), 193–260 (1983).

[12] T. Kühn, H.-G. Leopold, W Sickel, and L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces II, Proc. Edinb. Math. Soc. II. Ser., **49**(2), 331–359 (2006).

[13] S. D. Moura, On some characterizations of Besov spaces of generalized smoothness, Math. Nachr. **280**(9–10), 1190–1199 (2007).

