Coordinated motion design on Lie groups

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***Abstract*—The present paper proposes a unified geometric framework for coordinated motion on Lie groups. It first gives a general problem formulation and analyzes ensuing conditions for coordinated motion. Then, it introduces a precise method to design control laws in fully actuated and underactuated settings with simple integrator dynamics. It thereby shows that coordination can be studied in a systematic way once the Lie group geometry of the configuration space is well characterized. Applying the proposed general methodology to particular examples allows to retrieve control laws that have been proposed in the literature on intuitive grounds. A link with Brockett's double bracket flows is also made. The concepts are illustrated on $SO(3)$, $SE(2)$ and $SE(3)$.**

I. Introduction

Recently, many efforts have been devoted to the design and analysis of control laws that coordinate swarms of identical autonomous agents — e.g. oscillator synchronization [1], [2], flocking mechanisms [3], [4], vehicle formations [5], [6], [7], [8], [9], spacecraft formations [10], [11], [12], [13], [14], [15], mechanical system networks [16], [17], [18] and mobile sensor networks [19], [20], [21], [22], [23]. For systems on vector spaces, so-called *consensus algorithms* are shown to be efficient and robust [24], [25], [26], [27], [3], [28], and allow to address many relevant engineering issues and tasks [24], [5], [29]. However, in many applications, the agents to coordinate evolve on nonlinear manifolds: oscillators evolve on the circle $S^1 \cong SO(2)$, satellite attitudes on $SO(3)$ and vehicles move in $SE(2)$ or $SE(3)$; these particular manifolds share the geometric structure of a *Lie group*. Coordination on nonlinear manifolds is inherently more difficult than on vector spaces. The goal of the present paper is to propose a unified geometric framework for coordinated motion on Lie groups, from a geometric definition of "coordination" to a geometric derivation of control laws for coordination like those proposed in [20], [21], [19], [30], [31], [32], in fully actuated and underactuated settings with simple integrator dynamics. The objective is to reach a state where the *motion* of the agents is coordinated, while the values of their relative positions are a priori left arbitrary; definitions of "coordinated motion" and "relative positions" on a Lie group are the subject of Section II.

Symmetries: The key point for the developments in this paper is *invariance* (or *symmetry*) in the behavior of the swarm of agents with respect to their absolute position *on the Lie group*: only *relative* positions (*on the Lie group*) matter. For instance, the configuration of a rigid body in the 3-dimensional physical world is given by an orientation and a position vector in $\mathbb{R}^3$, whose combination corresponds to a position on Lie group $SE(3)$. For rigid body coordination, it is then natural to write control laws that can be interpreted as internal forces in the swarm, rather than forces depending on an external reference frame which would privilegiate some arbitrary choice of orientation and origin. Independence with respect to reference frame corresponds to invariance with respect to applying to all agents the same Lie group translation on $SE(3)$.

The symmetries determine how to define meaningful quantities for the swarm, like "relative positions" on the Lie group, and what the dynamics of the coupled agents can be. *Coordinated motion* — in short *coordination*— is defined as all situations where relative positions on the Lie group are fixed. Feedback control laws that asymptotically enforce coordination must be designed on the basis of error measurements involving appropriately invariant quantities (e.g. *relative* agent positions on the Lie group).

Previous work: Results about synchronization ("reaching a common point") and coordinated motion ("moving in an organized way") on vector spaces are becoming well established [28], [24], [27], [26]. Because a vector space can be identified with its tangent plane, both synchronization and coordinated motion can be seen as consensus problems on the same vector space: the former is a *position* consensus while the latter is a *velocity* consensus. Note that considering the motion of agents *with the Lie group structure of* $\mathbb{R}^n$ implies that only *position vectors* in $\mathbb{R}^n$ and associated *translational motion* are covered. In contrast, as soon as *orientation/rotation* of the vehicles or of the formation moving in a vector space is considered, the configuration space becomes the non-trivial Lie group $SE(n)$. In general, when the configuration space is a Lie group, synchronization and coordinated motion are fundamentally different. The geometric viewpoint for dynamical systems on Lie groups is very well studied; see basic results in [33], [34] for simplified dynamics like those considered in the present paper, and [35], [36], [37], [34] for a geometric theory of *mechanical* systems on Lie groups. General results for *synchronization* on compact Lie groups are proposed in [38], which points to related examples in the literature. But to the best of the authors' knowledge, a unified geometric viewpoint for *coordinated motion* — in short *coordination* — on Lie groups is still lacking. Close to the present paper in its geometric flavor, [39] builds invariant *observers* for systems with Lie group symmetries; observer design can be seen as two-agent leader-follower synchronization on Lie groups.

In applications, the ubiquitous example of motion on Lie groups is a rigid body in $\mathbb{R}^n$. When translational motion is

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discarded, the configuration space reduces to the compact Lie group $SO(n)$ characterizing the body's orientation; an element of $SO(n)$ can be represented by the $n \times n$ rotation matrix between a frame attached to the rigid body and a hypothetical fixed reference frame. The standard example of this type is satellite attitude control, where *synchronization*, i.e. obtaining equal orientations, has recently attracted much attention [10], [11], [13], [40], [12], [41], [42], [18], [43], [31], [15], with and without external reference tracking; note that synchronization is a very special case of coordination. Considering rotations *and translations*, the configuration space of an n-dimensional rigid body becomes the non-compact Lie group $SE(n) = \mathbb{R}^n \ltimes SO(n)$. Recently, coordination has been investigated on $SE(2)$ [8], [20], [21] and $SE(3)$ [9], [19], [16], [17] in the underactuated setting of *steering control* where the linear velocity is fixed in the body's frame. Motion on $SE(n)$ with steering control is also directly linked to the evolution of a Serret-Frenet frame with curvature control, as explained in [33]. Results taking into account the full mechanical dynamics for rigid body motion are more difficult to obtain — see for instance applications of the framework of [35] for coordination on $SO(3)$ and $SE(3)$ in [18], [43] and [16], [17] respectively. Considering simplified dynamics, as in the present paper, can be useful either to build a high-level planning controller or as a preliminary step towards an integrated mechanical controller, as illustrated for synchronization on $SO(3)$ in [31] and [32], [44] respectively.

Contributions: The main goal of the present paper is to provide a unified geometric framework for coordinated motion on Lie groups, proceeding as follows. (i) Coordination on Lie groups is defined from first principles of symmetry, distinguishing three variants: *left-invariant*, *right-invariant* and *biinvariant* coordination. (ii) Expressing the conditions for coordination in the associated Lie algebra, a direct link is drawn between coordination on Lie groups and consensus in vector spaces. (iii) It is investigated how biinvariant coordination restricts compatible relative positions through a geometrically meaningful relation. These properties are independent of the dynamics. Going over to control laws, simplified first-order dynamics are assumed for individual agents, but underactuation is explicitly modeled; communication among agents is restricted to a reduced set of links that can possibly be directed and time-varying. (iv) Control laws based on standard vector space consensus algorithms are given that achieve the easier tasks of right-invariant coordination and fully actuated left-invariant coordination for any initial condition on general Lie groups. (v) A general method is proposed to design control laws that achieve biinvariant coordination of fully actuated agents when communication links are undirected and fixed; extension to more general communication settings can be made along the lines of [21]. Biinvariant coordination is a rather academic problem, but (vi) the proposed design method is shown to apply to the practically most relevant problem of left-invariant coordination of underactuated agents. The proposed controller architecture consists of two steps, adding to the consensus algorithm a position controller derived from geometric Lyapunov functions. The position controllers are directly linked to the double bracket flows of [45] for gradient systems on adjoint orbits.

The power of the geometry is illustrated on $SO(3)$, $SE(2)$ and $SE(3)$ by analyzing the meaning of the geometric conditions for coordination, and by designing corresponding control laws with the proposed general methodology. The obtained controllers have been previously proposed in the literature, but were derived on the basis of intuitive arguments for particular applications. In that sense, the novelty of the present paper is not in the expression of the control laws but in showing that they can be derived in a unifying and systematic manner with the proper geometric setting.

The present paper focuses on the achievement of coordinated *motion* only, in the sense that the objective is for the swarm to move and *conserve* relative positions on the Lie group; the actual *values* of the relative positions on the Lie group, as long as they are compatible with the coordinated motion, are not controlled. However, applications often require to stabilize particular relative positions on the Lie group which are more efficient than others e.g. for sensing, power consumption or at least collision avoidance. The focus of the present work — motion with fixed relative positions on the Lie group — can be viewed as "orthogonal" to driving the agents towards particular relative positions on the Lie group. Therefore it is expected that the results of the present work can be combined with appropriately invariant relative position control algorithms on the Lie group (as e.g. from [38]), in order to both reach a particular configuration of relative positions on the Lie group and stabilize a coordinated motion of the resulting configuration. A corresponding result is proposed in [20] for steering control of planar vehicles (Lie group $SE(2)$); remaining issues concerning a general theory for this combination are discussed in [46].

Table of contents: The paper is organized as follows. Section II examines the geometric properties of coordination on Lie groups (contributions (i), (ii) and (iii)). Section III presents the control setting and basic control laws for right-invariant coordination and fully actuated left-invariant coordination (contribution (iv)). Sections IV and V present control law design methods respectively for biinvariant coordination (contribution (v)) and for underactuated left-invariant coordination (contribution (vi)). Examples are treated at the end of Sections II, IV and V.

II. The geometry of coordination

This section proposes definitions for coordination on Lie groups by starting from basic symmetry principles. It establishes conditions on velocities for coordination and examines implications. Except that the symmetries must be compatible, these developments are independent of the dynamics considered for the control problem. Notations are adapted from [34].

A. Relative positions and coordination

Consider N "agents" evolving on a Lie group G, with $g_k(t) \in G$ denoting the position of agent k at time t. Let

g_k^{-1} denote the group inverse of g_k, $L_h : g \mapsto hg$ denote left multiplication, and $R_h : g \mapsto gh$ right multiplication on G.

Definition 1: The *left-invariant relative position on* G of agent j with respect to agent k is $\lambda_{jk} = g_k^{-1} g_j$. The *right-invariant relative position on* G of j with respect to k is $\rho_{jk} = g_j\, g_k^{-1}$.

Indeed, λ_{jk} (resp. ρ_{jk}) is invariant under left (resp. right) multiplication: $(hg_k)^{-1}(hg_j) = g_k^{-1} g_j \;\forall h \in G$. Left-/right-invariant relative positions are the *joint invariants* associated to left-/right-invariant action of G on $G \times G ... \times G$ (N copies). In the following, “relative positions” always refer to relative positions on G unless otherwise specified.

The two definitions of relative position lead to two types of coordination; a third type is defined by combining them.

Definition 2: Left-invariant coordination (LIC) means constant left-invariant relative positions $\lambda_{jk}(t) = g_k^{-1} g_j$ — resp. *right-invariant coordination* (RIC) means constant right-invariant relative positions $\rho_{jk} = g_j g_k^{-1}$ — for all pairs of agents j, k. *Biinvariant coordination* (BIC) means simultaneous LIC and RIC: $g_k^{-1} g_j$ and $g_j g_k^{-1}$ are constant for all j, k.

The present paper thus associates *coordination* to fixed relative positions. In contrast, *synchronization* is the situation where all agents are at the same point on G: $g_k(t) = g_j(t)$ $\forall j, k$; this is a very particular case of biinvariant coordination.

B. Velocities and coordination

Denote by $\mathfrak{g}$ the Lie algebra of G, i.e. its tangent space at identity e. This paper always considers $\mathfrak{g}$ endowed with the Euclidean metric. Denote by $[\ ,\ ]$ the Lie bracket on $\mathfrak{g}$. Let $L_{h*} : TG_g \to TG_{hg}$ and $R_{h*} : TG_g \to TG_{gh}$ be the maps on tangent spaces induced by L_h and R_h respectively. Let $Ad_g = R_{g^{-1}*} L_{g*} : \mathfrak{g} \to \mathfrak{g}$ denote the *adjoint representation*.

Definition 4: Left-invariant velocity $\xi_k^l \in \mathfrak{g}$ and right-invariant velocity $\xi_k^r \in \mathfrak{g}$ of agent k are defined by $\xi_k^l(\tau) = L_{g^{-1}(\tau)*}(\frac{d}{dt} g_k(t)|_{t=\tau})$ and $\xi_k^r(\tau) = R_{g^{-1}(\tau)*}(\frac{d}{dt} g_k(t)|_{t=\tau})$.

Indeed, $g_k(t)$ and $L_h g_k(t)$ (resp. $R_h g_k(t)$) have the same left-invariant (resp. right-invariant) velocity $\xi_k^l(t)$ (resp. $\xi_k^r(t)$), for any fixed $h \in G$. Note the important equality

$$\xi_k^r = Ad_{g_k} \xi_k^l \;. \tag{1}$$

The *adjoint orbit* of $\xi \in \mathfrak{g}$ is set $O_\xi = \{Ad_g\, \xi : g \in G\} \subseteq \mathfrak{g}$.

Proposition 1: Left-invariant coordination corresponds to equal right-invariant velocities $\xi_j^r = \xi_k^r \;\forall j, k$. Right-invariant coordination corresponds to equal left-invariant velocities $\xi_j^l = \xi_k^l \;\forall j, k$.

Proof: For λ_{jk}, $\frac{d}{dt}(g_k^{-1} g_j) = L_{g_k^{-1}*} \frac{d}{dt} g_j + R_{g_j*} \frac{d}{dt} g_k^{-1}$. But if $\frac{d}{dt} g_k = L_{g_k*} \xi_k^l$, then $\frac{d}{dt} g_k^{-1} = -L_{g_k^{-1}*} Ad_{g_k} \xi_k^l$. Thus $\frac{d}{dt}(g_k^{-1} g_j) = L_{g_k^{-1} g_j *} \xi_j^l - L_{g_k^{-1}*} R_{g_j*} Ad_{g_k} \xi_k^l = L_{g_k^{-1} g_j *} Ad_{g_j}^{-1}(Ad_{g_j} \xi_j^l - Ad_{g_k} \xi_k^l)$. Since $L_{g_k^{-1} g_j *}$ and $Ad_{g_j^{-1}}$ are invertible, $\frac{d}{dt}(\lambda_{jk}) = 0$ is equivalent to $Ad_{g_j}\, \xi_j^l = Ad_{g_k}\, \xi_k^l$ or equivalently $\xi_j^r = \xi_k^r$. The proof for right-invariant coordination is strictly analogous. △

Proposition 1 shows that coordination on the Lie group G is equivalent to consensus in the vector space $\mathfrak{g}$. Consensus in vector spaces is well-studied, see [28], [24], [25], [47], [4], [27], [26]. Biinvariant coordination requires *simultaneous* consensus on ξ_k^l and ξ_k^r; but the latter are not independent, they are linked through (1) which depends on the agents' positions.

Proposition 2: Biinvariant coordination on a Lie group G is equivalent to the following condition in the Lie algebra $\mathfrak{g}$:

$$\forall k = 1...N, \quad \xi_k^l = \xi^l \in \bigcap_{i,j} \ker(Ad_{\lambda_{ij}} - Id) \;\text{ or equivalently}$$

$$\xi_k^r = \xi^r \in \bigcap_{i,j} \ker(Ad_{\rho_{ij}} - Id)$$

Proof: RIC requires $\xi_k^l = \xi_j^l \;\forall j, k$; denote the common value of the ξ_k^l by ξ^l. Then LIC requires $Ad_{g_k}\, \xi^l = Ad_{g_j}\, \xi^l \Leftrightarrow \xi^l = Ad_{\lambda_{jk}}\, \xi^l \;\forall j, k$. The proof with ξ^r is similar. △

Proposition 2 shows that biinvariant coordination puts no constraints on the relative positions when the group is Abelian, since $Ad_{\lambda_{jk}} = Id \;\forall \lambda_{jk}$ in this case. In contrast, on a general Lie group, biinvariant coordination with non-zero velocity can restrict the set of possible relative positions as follows.

Proposition 3: Let $CM_\xi := \{g \in G : Ad_g\, \xi = \xi\}$.
a. For every $\xi \in \mathfrak{g}$, CM_ξ is a subgroup of G.
b. The Lie algebra of CM_ξ is the kernel of $ad_\xi = [\xi, \,]$, i.e. $\mathfrak{cm}_\xi = \{\eta \in \mathfrak{g} : [\xi, \eta] = 0\}$.

Proof: a. $Ad_e\, \xi = \xi \;\forall \xi$ since Ad_e is the identity operator. $Ad_g\, \xi = \xi$ implies $Ad_{g^{-1}}\, \xi = \xi$ by simple inversion of the relation. Moreover, if $Ad_{g_1}\, \xi = \xi$ and $Ad_{g_2}\, \xi = \xi$, then $Ad_{g_1 g_2}\, \xi = Ad_{g_1}\, Ad_{g_2}\, \xi = Ad_{g_1}\, \xi = \xi$. Thus CM_ξ satisfies all group axioms and must be a subgroup of G.
b. Let $g(t) \in CM(\xi)$ with $g(\tau) = e$ and $\frac{d}{dt} g(t)|_\tau = \eta$. Then $\eta \in \mathfrak{cm}_\xi =$ the tangent space to CM_ξ at e. For constant ξ, $Ad_g(t)\xi = \xi$ implies $\frac{d}{dt}(Ad_g(t))\xi = 0$, with the basic Lie group property $\frac{d}{dt}(Ad_g(t))|_\tau = ad_\eta$. Therefore $[\eta, \xi] = 0$ is necessary. It is also sufficient since, for any η such that $[\eta, \xi] = 0$, the group exponential curve $g(t) = \exp(\eta t)$ belongs to CM_ξ. △

CM_ξ and $\mathfrak{cm}_\xi$ are called the isotropy subgroup and isotropy Lie algebra of ξ; these are classical objects in group theory [35]. From Propositions 2 and 3, one method to obtain a biinvariantly coordinated motion on G is to (1) choose ξ^l in the vector space $\mathfrak{g}$ and set $\xi_k^l = \xi^l \;\forall k$ (2) position the agents on G such that $\lambda_{jk} \in CM_{\xi^l}$ for pairs j, k corresponding to the edges of an undirected tree graph; the Lie group property of CM_{ξ^l} then ensures that $\lambda_{jk} \in CM_{\xi^l}$ for *all* pairs j, k. The same can be done with ξ^r and the ρ_{jk}. Note that a swarm at rest ($\xi_k^l = \xi_k^r = 0 \;\forall k$) is always biinvariantly coordinated.

Remark 1: In many applications involving coordinated motion, reaching a particular *configuration*, i.e. *specific values* of the relative positions, is also relevant. Specific configurations are defined as extrema of a cost function in [38]. Imposing relative positions in the (intersection of) set(s) CM_ξ for some ξ can be another way to classify specific configurations; unlike [38], it

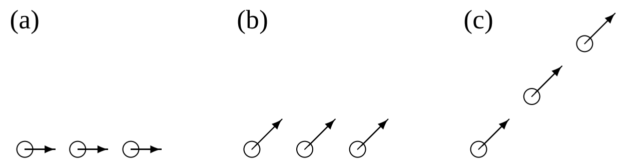

Fig. 1. Coordinated motion on $\mathbb{R}^2$ (LIC = RIC = BIC). (a) An initial situation of coordinated motion; circles represent agent positions, arrows their linear velocities. (b) The same swarm in coordinated motion with a different velocity. (c) A coordinated motion that the swarm of the first plot *cannot* reach without breaking the coordinated motion to re-position the agents.

works for non-compact Lie groups. For compact groups, there seems to be no connection between configurations characterized through CM_ξ and those defined by [38].

Remark 2: One can also first fix relative positions λ_{jk} and then characterize the set of velocities ξ compatible with biinvariant coordination. For non-Abelian groups and a sufficiently large number N of agents, this set generically reduces to $\xi = 0$.

C. Examples

The Lie group $\mathbb{R}^n$ has trivial properties; it is presented to clarify the distinction with "motion of rigid bodies in $\mathbb{R}^n$", whose configuration space is the Lie group $SE(n)$. Basic properties for the special orthogonal groups $SO(n)$ and special Euclidean groups $SE(n)$, $n \geq 2$, can be found in e.g. [33]. Left-invariant coordination for $SE(2)$ and $SE(3)$ was already formulated in Lie group notation in [8], [9].

$\mathbb{R}^n$: For $G = \mathbb{R}^n$, a point $g_k \in G$ is denoted by a position vector $r_k \in \mathbb{R}^n$.

- Group multiplication $g_k g_j$ corresponds to $r_k + r_j$, inverse g_k^{-1} to $-r_k$, and identity e to position vector 0. In particular, the group structure is decoupled in each coordinate and Abelian (i.e. group multiplication is commutative). Relative positions take the familiar form $\lambda_{jk} = \rho_{jk} = r_j - r_k \in \mathbb{R}^n$.
- The Lie algebra equals $\mathbb{R}^n$ itself, operations L_{r*} and R_{r*} reduce to the identity $\forall r \in G = \mathbb{R}^n$.
- Adjoint operator $Ad_r = Id$ for all $r \in G = \mathbb{R}^n$ and the Lie bracket is identically zero.
- LIC, RIC and BIC all collapse to the same and just require identical linear velocities in $\mathbb{R}^n$; in particular BIC implies no restrictions on relative positions. Physically, coordinated motion means a rigid formation of points in $\mathbb{R}^n$ moving with a fixed formation orientation. The direction of motion can change when varying the velocity vector, as between Fig.1 (a) and Fig.1 (b), but a rotation of the formation, as going from Fig.1 (a) to Fig.1 (c), would require breaking coordination in $\mathbb{R}^n$.

$SO(3)$: The special orthogonal group $SO(3)$ describes 3-dimensional rotations. A point g on $SO(3)$ is represented by a matrix $Q \in \mathbb{R}^{3\times 3}$ with $Q^T Q = Id$ and $\det(Q) = 1$.

- Group multiplication, inverse and identity are the corresponding matrix operations.
- The Lie algebra $\mathfrak{so}(3)$ is the set of skew-symmetric 3×3 matrices $[\omega]^\wedge$, operations $L_{Q*}\xi$ and $R_{Q*}\xi$ are represented by $Q[\omega]^\wedge$ and $[\omega]^\wedge Q$ respectively. The invertible mapping
$$\begin{pmatrix} 0 & -\omega_{(3)} & \omega_{(2)} \\ \omega_{(3)} & 0 & -\omega_{(1)} \\ -\omega_{(2)} & \omega_{(1)} & 0 \end{pmatrix} \underset{[\cdot]^\wedge}{\overset{[\cdot]^\vee}{\rightleftarrows}} \begin{pmatrix} \omega_{(1)} \\ \omega_{(2)} \\ \omega_{(3)} \end{pmatrix}$$
identifies $\mathfrak{so}(3) \ni [\omega]^\wedge$ with $\mathbb{R}^3 \ni \omega$.
- With this identification, $Ad_Q \omega = Q\omega$ and $[\omega_k, \omega_j] = [\omega_k]^\wedge \omega_j = \omega_k \times \omega_j$ (vector product).
- In the standard interpretation of Q as rigid body orientation, ω^l and ω^r are the angular velocities expressed in body frame and in inertial frame respectively.
- LIC (equal ω_k^r), RIC (equal ω_k^l) and BIC have a clear physical interpretation in this case.
- For BIC with $\omega \neq 0$, $\mathfrak{cm}_\omega = \{\lambda\omega : \lambda \in \mathbb{R}\}$ and $CM_\omega = \{\text{rotations around axis } \omega\}$. The dimension of $\mathfrak{cm}_{\xi^l}$ ($\Leftrightarrow$ of CM_{ξ^l}) is 1. Agents in BIC rotate with the same angular velocity ω_k^r in inertial space and have the same orientation up to a rotation around ω_k^r.

$SE(2)$: The special Euclidean group in the plane $SE(2)$ describes planar rigid body motions (translations and rotations). An element of $SE(2)$ can be written $g = (r, \theta) \in \mathbb{R}^2 \times S^1$ where r is a position vector in the plane and θ is orientation (or "heading").

- Group multiplication $g_1 g_2 = (r_1 + Q_{\theta_1} r_2, \theta_1 + \theta_2)$ where Q_θ is the rotation of angle θ. Identity $e = (0, 0)$ and inverse $g^{-1} = (-Q_{-\theta} r, -\theta)$.
- Lie algebra $\mathfrak{se}(2) = \mathbb{R}^2 \times \mathbb{R} \ni \xi = (v, \omega)$. Operations $L_{g*}(v, \omega) = (Q_\theta v, \omega)$ and $R_{g*}(v, \omega) = (v + \omega Q_{\pi/2} r, \omega)$.
- $Ad_g\,(v, \omega) = (Q_\theta v - \omega Q_{\pi/2} r, \omega)$ and $[(v_1, \omega_1), (v_2, \omega_2)] = (\omega_1 Q_{\pi/2} v_2 - \omega_2 Q_{\pi/2} v_1, 0)$.
- In the interpretation of rigid body motion, v^l is the linear velocity expressed in body frame, $\omega^l = \omega^r =: \omega$ is the rotation rate. For $\omega \neq 0$, v^r is *not* the body's linear velocity expressed in inertial frame; instead, $s = \frac{-Q_{\pi/2}}{\omega} v^r$ is the center of the circle drawn by the rigid body moving with $\xi^r = (v^r, \omega)$. In [20], the intuitive argument to achieve coordination is to synchronize circle centers s_k; this actually synchronizes right-invariant velocities v_k^r.
- In RIC, the agents move with the same velocity expressed in body frame (Fig.2, r). In LIC, they move like a single rigid body (or "formation"): relative orientations and relative position vectors on the plane do not change (Fig.2, l_1 and l_2). Note that any combination of translation (as on Fig.2, l_1) and rotation (as on Fig.2, l_2) of the formation composed by the agents is possible.
- In BIC, the swarm moves like a single rigid body *and* each agent has the same velocity expressed in body frame. Propositions 2 and 3 characterize $\mathfrak{cm}_{\xi^l}$ by $[\xi^l, \eta] = 0 \Leftrightarrow \omega^l v_\eta = \omega_\eta v^l$ and CM_{ξ^l} by $Ad_g\, \xi^l = \xi^l \Leftrightarrow (Q_\theta - Id) v^l = \omega^l Q_{\pi/2} r$. This leads to three different cases:
  (o) $\omega^l = v^l = 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \mathfrak{se}(2)$ and $CM_{\xi^l} = SE(2)$.
  (i) $\omega^l = 0$, $v^l \neq 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \{(v, 0) : v \in \mathbb{R}^2\}$ and $CM_{\xi^l} = \{(r, 0) : r \in \mathbb{R}^2\}$.
  (ii) $\omega^l \neq 0$, any $v^l \Rightarrow \mathfrak{cm}_{\xi^l} = \{(\frac{\omega}{\omega^l} v^l, \omega) : \omega \in \mathbb{R}\}$.

Define $C \subset \mathbb{R}^2$, the circle of radius $\frac{\|v^l\|_2}{|\omega^l|}$ containing the origin, tangent to v^l at the origin and such that v^l and ω^l imply rotation in the same direction. Then solving $Ad_g \xi = \xi$ for g and making a few calculations shows that $CM_{\xi^l} = \{(r,\, \theta) : r \in C$ and $Q_\theta v^l$ tangent to C at $r\}$. This is consistent with an intuitive analysis of possibilities for circular motion with unitary linear velocity and fixed relative position vectors and orientations in the plane.

The dimension of $\mathfrak{cm}_{\xi^l}$ ($\Leftrightarrow$ of CM_{ξ^l}) is (o) 3, (i) 2 or (ii) 1. In case (o), the configuration is arbitrary but at rest. In case (i), the agents have the same orientation and move on parallel straight lines (Fig.2, t_1). In case (ii), they move on the same circle and have the same orientation with respect to their local radius (Fig.2, t_2). Unlike for LIC, combinations of translations (t_1) and rotations (t_2) of the formation composed by the agents would *not* correspond to BIC.

$SE(3)$*:* This group describes 3-dimensional rigid body motions (translations and rotations). An element of $SE(3)$ can be written $g = (r, Q) \in \mathbb{R}^3 \times SO(3)$, with r a position vector in $\mathbb{R}^3$ and Q a rotation matrix describing orientation.

- $g_1 g_2 = (r_1 + Q_1 r_2, Q_1 Q_2)$, identity $e = (0, Id)$ and inverse $g^{-1} = (-Q^T r, Q^T)$.
- Lie algebra $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathfrak{so}(3) \ni \xi = (v, [\omega]^\wedge)$ is identified with $\mathbb{R}^3 \times \mathbb{R}^3 \ni (v, \omega)$ with the same mapping as for $SO(3)$. Operations $L_{g*}(v, [\omega]^\wedge) = (Qv, Q[\omega]^\wedge)$ and $R_{g*}(v, [\omega]^\wedge) = (\omega \times r + v, [\omega]^\wedge Q)$. As for $SO(3)$, symbol "$\times$" denotes vector product.
- $Ad_g\,(v, \omega) = (Qv + r \times (Q\omega), Q\omega)$ and $[(v_1, \omega_1), (v_2, \omega_2)] = (\omega_1 \times v_2 - \omega_2 \times v_1, \omega_1 \times \omega_2)$.
- In the interpretation of rigid body motion, left-invariant velocities v^l and ω^l are the body's linear and angular velocity respectively, expressed in body frame; the right-invariant ω^r is the angular velocity expressed in inertial frame; for $\omega^l \neq 0$, a physical interpretation for the right-invariant v^r is unclear.
- Similarly to $SE(2)$, the agents move in RIC with the same velocity expressed in body frame and in LIC with fixed relative orientations and relative position vectors in $\mathbb{R}^3$, like a single rigid body.
- In BIC, the swarm moves like a single rigid body *and* each agent has the same velocity expressed in body frame. Propositions 2 and 3 lead to three different cases characterizing $\mathfrak{cm}_{\xi^l}$ which requires $[\xi^l, \eta] = 0$
  $\Leftrightarrow \quad \omega^l \times \omega_\eta = 0$ and $\omega^l \times v_\eta = \omega_\eta \times v^l$;
  CM_{ξ^l} which requires $Ad_g\, \xi^l = \xi^l$
  $\Leftrightarrow \quad Q\omega^l = \omega^l$ and $(Q - \mathrm{Id})v^l = \omega^l \times r$.
  (o) $\omega^l = v^l = 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \mathfrak{se}(3)$ and $CM_{\xi^l} = SE(3)$.
  (i) $\omega^l = 0,\ v^l \neq 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \{(\beta, \alpha v^l) : \beta \in \mathbb{R}^3,\ \alpha \in \mathbb{R}\}$ and $CM_{\xi^l} = \{(r, Q) : r \in \mathbb{R}^3,\ Q$ characterizes rotation of axis $v^l\}$.
  (ii) $\omega^l \neq 0$, any $v^l \Rightarrow \mathfrak{cm}_{\xi^l} = \{(\alpha v^l + \beta \omega^l, \alpha \omega^l) : \alpha,\, \beta \in \mathbb{R}\}$ and $CM_{\xi^l} = \{(r, Q) \in SE(3)$ describing left-invariant relative positions of agents that are on the same cylinder of axis ω^l and radius $\frac{\|v^l - (v^l)\cdot(\omega^l)/\|\omega^l\|\|}{\|\omega^l\|}$, with orientations differing around axis ω^l by an angle exactly equal to their relative angular position on the cylinder $\}$.

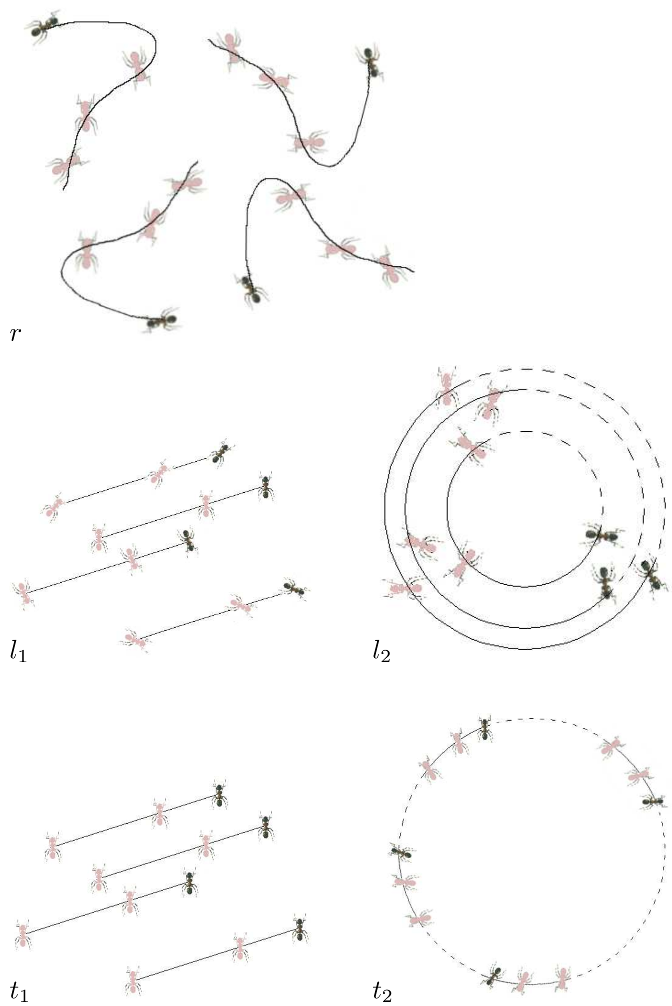

Fig. 2. Coordinated swarms (light color: intermediate planar positions and orientations in time). r: RIC with varying velocity. l_1 and l_2: LIC with $\omega_k = 0$ and $\omega_k \neq 0$ respectively; note that any combination of translation (l_1) and rotation (l_2) of the formation composed by the agents still corresponds to LIC. t_1 and t_2: BIC with $\omega_k = 0$ and $\omega_k \neq 0$ respectively; note that combinations of translations (t_1) and rotations (t_2) of the formation composed by the agents would *not* correspond to BIC.

This is again obtained by solving for g in $Ad_g \xi = \xi$ and making several basic computations; it is less obvious than for $SE(2)$ to find this result intuitively.

The dimension of $\mathfrak{cm}_{\xi^l}$ ($\Leftrightarrow$ of CM_{ξ^l}) is (o) 6, (i) 4 or (ii) 2. In case (o), the configuration is arbitrary but at rest. In case (i), the agents move on parallel straight lines and have the same orientation up to rotation around their linear velocity vector. In case (ii), for $v^l - (v^l) \cdot (\omega^l)/\|\omega^l\| \neq 0$, the agents draw helices of constant *pitch* $\omega^l \cdot v^l = \omega^r \cdot v^r$ on the cylinder; the special case $\omega^l \cdot v^l = 0$ gives circular trajectories (see figures in [9], [19]). In the degenerate situation $v^l - (v^l) \cdot (\omega^l)/\|\omega^l\| = 0$, all agents are on the rotation axis.

III. Coordination as consensus in the Lie algebra

A. Control setting

Left-invariant[1] systems on Lie groups appear naturally in many physical systems, such as rigid bodies in space and cart-like vehicles. Motivated by examples like 2-axes attitude control and steering control on $SE(2)$ or $SE(3)$, this paper considers left-invariant dynamics with affine control

$$\tfrac{d}{dt} g_k = L_{g_k *}\xi_k^l \quad \text{with} \quad \xi_k^l = a + Bu_k \quad , \; k = 1...N \, , \qquad (2)$$

where the Lie algebra $\mathfrak{g}$ is identified with $\mathbb{R}^n$, $a \in \mathbb{R}^n$ is a constant drift velocity, $B \in \mathbb{R}^{n \times m}$ has full column rank and specifies the range of the control term $u_k \in \mathbb{R}^m$; without loss of generality, the column vectors of B are assumed orthonormal. The set of all assignable ξ_k^l is denoted $\mathcal{C} = \{a + Bu : u \in \mathbb{R}^m\}$. For fully actuated agents $m = n$, (2) simplifies to $\frac{d}{dt} g_k = L_{g_k *} u_k$ without loss of generality. The following always considers $\mathfrak{g}$ endowed with the Euclidean metric. Feedback control laws must be functions of variables which are compatible with the symmetries of the problem setting, i.e. left-invariant. In terms of left-invariant variables, LIC corresponds to fixed (left-invariant) relative positions, while RIC corresponds to equal (left-invariant) velocities.

In a realistic scalable setting, full communication between all agents cannot be assumed. The information flow among agents is modeled by a restricted set of communication links; $j \rightsquigarrow k$ denotes that j sends information to k. The communication topology is associated to a graph $\mathbb{G}$. $\mathbb{G}$ is undirected if $k \rightsquigarrow j \Leftrightarrow j \rightsquigarrow k$. $\mathbb{G}$ is *uniformly connected* (see [24], [25]) if there exist an agent k and durations $\delta > 0$ and $T > 0$ such that, $\forall t$, taking the union of the links appearing for at least δ in time span $[t, t+T]$, there is a directed path $k \rightsquigarrow a \rightsquigarrow b ... \rightsquigarrow j$ from k to every other agent j.

B. Right-invariant coordination

Right-invariant coordination requires $\xi_k^l = \xi_j^l \; \forall j, k$. In the setting (2), this simply implies to agree on equal $u_k \; \forall k$; positions λ_{jk} can evolve arbitrarily. This problem is solved by the classical vector space consensus algorithm [28], [25], [47], [4], [27], [26]

$$\tfrac{d}{dt}\xi_k^l = \textstyle\sum_{j \rightsquigarrow k} \left(\xi_j^l - \xi_k^l\right) \qquad , \; k = 1...N \, , \qquad (3)$$

Using (2), it translates into $\frac{d}{dt} u_k = \sum_{j \rightsquigarrow k} \left(u_j - u_k\right)$. It exponentially achieves $\xi_k^l = \xi_j^l \; \forall j, k$ if $\mathbb{G}$ is uniformly connected. Asymptotic RIC is then ensured for any initial u_k and, of course, any relative positions λ_{jk} which actually have no influence. Agent k relies on the left-invariant velocity ξ_j^l of $j \rightsquigarrow k$.

For a time-invariant and undirected communication graph $\mathbb{G}$, (3) is a gradient descent for the disagreement cost function $V_r = \sum_k \sum_{j \rightsquigarrow k} \|\xi_k^l - \xi_j^l\|^2$, with the Euclidean metric in $\mathfrak{g}$.

[1] A right-invariant system is equivalent, simply by redefining the group multiplications.

C. Left-invariant coordination

Left-invariant coordination requires $\xi_k^r = \xi_j^r \; \forall j, k$, which suggests to use

$$\tfrac{d}{dt}\xi_k^r = \textstyle\sum_{j \rightsquigarrow k} \left(\xi_j^r - \xi_k^r\right) \qquad , \; k = 1...N \, . \qquad (4)$$

Using (1), in terms of the left-invariant variables, (4) becomes

$$\tfrac{d}{dt}\xi_k^l = \textstyle\sum_{j \rightsquigarrow k} \left(Ad_{g_k^{-1} g_j}\, \xi_j^l - \xi_k^l\right) \qquad , \; k = 1...N \qquad (5)$$

thanks to $\left(\frac{d}{dt} Ad_{g_k}\right) \xi_k^l = Ad_{g_k}[\xi_k^l, \xi_k^l] = 0$. To implement (4), agent k must know the relative position $g_k^{-1} g_j$ and velocity ξ_j^l of $j \rightsquigarrow k$.

A priori, (5) converges as (3), ensuring global exponential coordination for uniformly connected $\mathbb{G}$. However, in contrast to (3), nothing guarantees that (5) can be implemented in an underactuated setting. At equilibrium, (5) requires

$$Ad_{\lambda_{jk}}(a + Bu_j) \, = \, a + Bu_k \quad \forall j, k \, , \qquad (6)$$

which, for arbitrary relative positions of the agents, might admit no solution $(u_1, u_2, ..., u_N)$. This issue motivates the further study of underactuated LIC in Section V. Similarly, biinvariant coordination requires simultaneous consensus on left- and right-invariant velocities. At equilibrium, this means that (6) must hold with equal controls u_k, i.e.

$$Ad_{\lambda_{jk}}(a + Bu_k) \, = \, a + Bu_k \quad \forall j, k \, , \qquad (7)$$

which also puts constraints on the relative positions of the agents. For this reason, biinvariant coordination is further studied in Section IV.

The cost function $V_l = \sum_k \sum_{j \rightsquigarrow k} \|Ad_{g_k}\xi_k^l - Ad_{g_j}\xi_j^l\|^2$ associated to (4) is not left-invariant in general (it involves positions g_k), so (5) cannot be a left-invariant gradient of V_l.

Nevertheless, let $\mathcal{G}_u$ be the subclass of compact groups with unitary adjoint representation, i.e. satisfying $\|Ad_g \, \xi\| = \|\xi\|$ $\forall g \in G$ and $\forall \xi \in \mathfrak{g}$ (for instance $SO(n) \in \mathcal{G}_u$). It is possible to define a biinvariant (that is, left- and right-invariant) Riemannian metric on G if and only if $G \in \mathcal{G}_u$ [48]. Using the Euclidean metric on left-invariant velocities, as in the present paper, comes down to using a left-invariant metric, in accordance with the left-invariant setting. If $G \in \mathcal{G}_u$, then this metric is biinvariant, $V_l = \sum_k \sum_{j \rightsquigarrow k} \|\xi_k^l - Ad_{g_k^{-1} g_j}\xi_j^l\|^2$ and for fixed undirected $\mathbb{G}$, (5) is a gradient descent for V_l.

In the following, it is assumed that the agents are controllable. Obviously, controllability is sufficient for coordination as it allows the agents to reach any position from any initial condition. However, it is not always necessary, as long as positions compatible with (6) or (7) are globally reachable; in particular, for Abelian groups, all positions satisfy (6) and (7); in that case, (underactuated) LIC and BIC become trivial.

IV. Control design: fully actuated biinvariant coordination

A. Biinvariant coordination on general Lie groups

Biinvariant coordination requires to satisfy two objectives, LIC and RIC, simultaneously. In a first step, assume that the agents are given a reference right-invariant velocity ξ^r, such that LIC is ensured if each agent has velocity $\xi_k^l = Ad_{g_k}^{-1}\xi^r$ is applied $\forall k$. It remains to simultaneously achieve RIC, which, as previously shown, involves controlling relative positions. Write a general controller

$$\xi_k^l = \eta_k^l + q_k \ , \quad k = 1...N \ , \tag{8}$$

where η_k^l is a desired velocity and q_k is necessary for relative position control. Thus for the present, $\eta_k^l = Ad_{g_k}^{-1}\xi^r$. The question is how to design q_k in order to achieve BIC. For fixed undirected communication graph $\mathbb{G}$, inspired by the cost function for RIC, define

$$V_{tr}(g_1, g_2 ... g_N) = \tfrac{1}{2}\textstyle\sum_k \sum_{j \rightsquigarrow k} \|\eta_k^l - \eta_j^l\|^2$$

where $\| \ \|$ denotes Euclidean norm. V_{tr} characterizes the distance from RIC *assuming that every agent has velocity* $\xi_k^l = Ad_{g_k}^{-1}\xi^r$. Since $(\frac{d}{dt}Ad_{g_k}^{-1})\eta = -[\xi_k^l, \, Ad_{g_k}^{-1}\eta] \ \forall \eta \in \mathfrak{g}$, the time variation of V_{tr} due to motion of g_k is

$$\tfrac{d}{dt}V_{tr} = 2 \ \textstyle\sum_k \sum_{j \rightsquigarrow k} (\eta_k^l - \eta_j^l) \cdot [\eta_k^l, \, \xi_k^l] \tag{9}$$

where $\cdot$ denotes the canonical scalar product in $\mathfrak{g}$, defined with the Euclidean metric. Thus if $q_k = 0$ then $\frac{d}{dt}V_{tr} = 0$; a proper choice of q_k should allow to decrease V_{tr}. Define[2] the bracket $\langle \, , \, \rangle$ such that $\xi_1 \cdot \langle \xi_2, \xi_3 \rangle + [\xi_1, \xi_2] \cdot \xi_3 = 0 \ \forall \xi_1, \xi_2, \xi_3 \in \mathfrak{g}$. Then (9) rewrites $\frac{d}{dt}V_{tr} = 2 \sum_k \sum_{j \rightsquigarrow k} \langle \eta_k^l, \eta_k^l - \eta_j^l \rangle \cdot q_k$ and the choice

$$q_k = -\langle \eta_k^l, \ \textstyle\sum_{j \rightsquigarrow k} (\eta_k^l - \eta_j^l) \rangle \tag{10}$$

ensures that V_{tr} is non-increasing along the solutions:

$$\tfrac{d}{dt}V_{tr} = -2 \ \textstyle\sum_k \sum_{j \rightsquigarrow k} \langle \eta_k^l, \ \sum_{j \rightsquigarrow k} (\eta_k^l - \eta_j^l) \rangle^2 \ \leq 0 \, .$$

To obtain an autonomous, left-invariant algorithm for bi-invariant coordination, it remains to replace the reference velocity ξ^r by estimates on which the agents progressively agree. Since the goal is to define a common right-invariant velocity in $\mathfrak{g}$, it is natural to proceed as in Section III-C and use the consensus algorithm

$$\tfrac{d}{dt}\eta_k^r = \textstyle\sum_{j \rightsquigarrow k} (\eta_j^r - \eta_k^r) \tag{11}$$

which in terms of left-invariant velocities rewrites

$$\tfrac{d}{dt}\eta_k^l = \textstyle\sum_{j \rightsquigarrow k} (Ad_{\lambda_{jk}} \, \eta_j^l - \eta_k^l) \ - [\xi_k^l, \, \eta_k^l] \ , \ k = 1...N. \tag{12}$$

Thus the overall controller is the cascade of a consensus algorithm to agree on a desired velocity for LIC, and a position controller designed to decrease a natural distance to RIC. To implement the controller, agent k must receive from communicating agents $j \rightsquigarrow k$ their relative positions λ_{jk} and the values of their left-invariant *auxiliary variables* η_j^l.

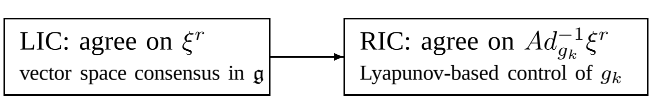

Fig. 3. Biinvariant coordination as consensus on right-invariant velocity and Lyapunov-based control to right-invariant coordination.

[2]In fact $\langle \, , \, \rangle$ expresses the effect of the Lie bracket on the dual space of $\mathfrak{g}$. It is directly related to the *coadjoint* representation of G, commonly used for mechanical systems; in general, $\langle \, , \, \rangle$ does not satisfy the Lie bracket properties.

The following result characterizes the convergence properties of controller (8),(10),(12).

Theorem 1: Consider N fully actuated agents communicating on a fixed, undirected graph $\mathbb{G}$ and evolving on Lie group G according to $\frac{d}{dt}g_k = L_{g_k *}\xi_k^l$ with controller (8),(10),(12).

(i) For any initial conditions $\eta_k^l(0)$, the $\eta_k^r(t) = Ad_{g_k}\eta_k^l(t)$ exponentially converge to $\overline{\eta^r} := \frac{1}{N}\sum_k \eta_k^r(0)$.

(ii) Define $\overline{V_{tr}}(g_1, g_2, ...g_N) := \frac{1}{2}\sum_k \sum_{j \rightsquigarrow k} \|Ad_{g_k}^{-1}\overline{\eta^r} - Ad_{g_j}^{-1}\overline{\eta^r}\|^2$. All solutions converge to the critical set of $\overline{V_{tr}}$. In particular, left-invariant coordination is asymptotically achieved for all initial conditions.

(iii) Biinvariant coordination is (at least locally) asymptotically stable.

<u>Proof:</u> Regarding convergence, (12) is strictly equivalent to (11). Therefore, (i) simply restates a well-known convergence result for consensus algorithms in vector spaces on fixed undirected graphs [26].

Since the η_k^r converge, (8),(10) is an asymptotically autonomous system; the autonomous limit system is obtained by replacing $\eta_k^l = Ad_{g_k}^{-1}\overline{\eta^r}$. From the derivation of q_k in (10), the limit system is a gradient descent for $\overline{V_{tr}}(g_1, g_2, ...g_N)$, which is smooth because the adjoint representation is smooth. According to [49], the ω-limit sets of an asymptotically autonomous system correspond to the chain recurrent sets of the limit system. From [50] the chain recurrent set of a smooth gradient system is equal to its critical points. Therefore the ω-limit set of (8),(10) is equal to the critical points of $\overline{V_{tr}}$, which proves (ii). Biinvariant coordination $\overline{V_{tr}} = 0$ is locally asymptotically stable as it is a local (and global) minimum of $\overline{V_{tr}}$, which proves (iii). △

Given $\overline{\eta^r}$, the region of attraction for BIC is a sublevel-set where $\overline{V_{tr}}$ has 0 as only critical point (in practice, as only minimum). Other local minima can involve e.g. the η_k^l evenly distributed on a circular $O_{\overline{\xi^r}}$ with $\mathbb{G}$ a ring graph (see [38]).

Extensions to varying and directed $\mathbb{G}$ can be made with additional auxiliary variables along the lines of [51], [52], [21], [19]: at a first level, consensus algorithms define a desired ξ^l *and* a desired ξ^r, which must be on the same adjoint orbit; at a second level, cost functions for individual agents ensure that they asymptotically implement the desired velocities. The consensus part is non-trivial to write in a fully left-invariant setting, because ξ^l and ξ^r must belong to the *same adjoint orbit*. The present paper proposes no explicit design of this form. For fixed undirected $\mathbb{G}$, an advantage of algorithms with "double consensus" (ξ^r and ξ^l) would be that BIC becomes the only locally stable equilibrium: interaction-related issues only depend on the performance of the consensus algorithm, for the rest the agents behave individually. It is shown in [38] how auxiliary variables can be used to build consensus algorithms that avoid spurious local minima on various spaces.

B. Biinvariant coordination on Lie groups with a biinvariant metric

When $G \in \mathcal{G}_u$, i.e. G has a biinvariant metric, the cost function $V_l = \sum_k \sum_{j \rightsquigarrow k} \|Ad_{g_k} \xi^l_k - Ad_{g_j} \xi^l_j\|^2$ can be used for left-invariant control design.

A natural idea in this context would be to combine the cost functions for LIC and RIC, writing $V_t = V_l + V_r$, and derive a gradient descent for V_t of the form $\frac{d}{dt}\xi^l_k = f(\xi^l_k, \{\xi^l_j, g_k^{-1} g_j : j \rightsquigarrow k\})$. However, simulations of the resulting control law for $SO(n)$ always converge to $\xi^l_k = 0 \ \forall k$. A possible explanation for this behavior is that the gradient controls velocities, not explicitly positions, while it was shown in Section II that BIC at non-zero velocity involves restrictions on compatible positions.

Nevertheless, the biinvariant metric allows to switch the roles of LIC and RIC in the method of Subsection IV.A, using a consensus algorithm to define a common left-invariant velocity for RIC, and a cost function to drive positions to LIC.

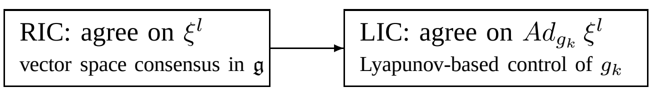

Fig. 4. Biinvariant coordination as consensus on left-invariant velocity and Lyapunov-based control to left-invariant coordination.

The RIC consensus algorithm on auxiliary variables asymptotically defines a common velocity ξ^l by

$$\tfrac{d}{dt}\eta^l_k = \textstyle\sum_{j \rightsquigarrow k} (\eta^l_j - \eta^l_k) \,, \quad k = 1...N \,. \qquad (13)$$

Then defining the cost function

$$\begin{aligned} V_{tl}(g_1, g_2 ... g_N) &= \tfrac{1}{2}\textstyle\sum_k \sum_{j \rightsquigarrow k} \|Ad_{g_k}\, \eta^l_k - Ad_{g_j}\, \eta^l_j\|^2 \\ &= \tfrac{1}{2}\textstyle\sum_k \sum_{j \rightsquigarrow k} \|\eta^l_k - Ad_{g_k^{-1} g_j}\, \eta^l_j\|^2 \end{aligned}$$

for LIC and proceeding as in the previous subsection, one obtains controller (8) with

$$q_k = \langle \eta^l_k ,\ \textstyle\sum_{j \rightsquigarrow k} (\eta^l_k - Ad_{g_k^{-1} g_j}\, \eta^l_j) \rangle \,. \qquad (14)$$

Theorem 2: Consider N fully actuated agents communicating on a connected, fixed, undirected graph $\mathbb{G}$ and evolving on $G \in \mathcal{G}_u$ according to $\frac{d}{dt} g_k = L_{g_k *} \xi^l_k$ with controller (8),(13),(14).

(i) For any initial conditions $\eta^l_k(0)$, the $\eta^l_k(t)$ exponentially converge to $\overline{\eta^l} := \frac{1}{N} \sum_k \eta^l_k(0)$.

(ii) Define $\overline{V_{tl}}(g_1, g_2, ... g_N) := \frac{1}{2}\sum_k \sum_{j \rightsquigarrow k} \|Ad_{g_k} \overline{\eta^l} - Ad_{g_j} \overline{\eta^l}\|^2$. All solutions converge to the critical set of $\overline{V_{tl}}$. In particular, right-invariant coordination is asymptotically achieved.

(iii) Biinvariant coordination is (at least locally) asymptotically stable.

<u>Proof:</u> The proof is omitted because it is similar to the one of Theorem 1. $\triangle$

The region of attraction for BIC behaves as for Theorem 1.

An advantage of Theorem 2 over Theorem 1 is that control design is directly extended to underactuated agents. Indeed, (13) defines a valid consensus velocity $\xi^l \in \mathcal{C} = \{a + Bu : u \in \mathbb{R}^m\}$ for underactuated agents provided that $\eta^l_k(0) \in \mathcal{C} \ \forall k$. The only change is that q_k, instead of the exact gradient descent in (14), is its projection onto the control range of B:

$$\xi^l_k = a + Bu_k = \eta^l_k + B\, B^T q_k \,.$$

When ξ^l is asymptotically defined with (13), the convergence argument for asymptotically autonomous systems must be extended to projections of gradient systems; a general proof of this technical issue is lacking in the present paper. It is the only reason to restrict Theorem 2 to fully actuated agents.

Brockett [45] has developed a general double-bracket form for gradient algorithms on adjoint orbits of compact semi-simple groups, using the biinvariant Killing metric. The connection with the present paper is clear: once the consensus algorithm has converged, the gradient control for agent positions involves a cost function on the adjoint orbit of $\overline{\eta^l}$ or $\overline{\eta^r}$. One example in [45] involves minimizing the distance towards a subset of $\mathfrak{g}$; a similar objective will be pursued in Section V of the present paper (but with a different class of subsets). A main difference of [45] is its focus on the evolution of variables in $\mathfrak{g}$, making abstraction of the underlying group, while the present paper actually controls positions of (possibly underactuated) agents on G. If G is a compact group and the biinvariant Killing metric coincides with the left-invariant metric of the present paper, then $\langle\, ,\, \rangle = -[\, ,\,]$ and control (10) for g_k with $\eta^r_k = \xi^r$ fixed implies that η^l_k follows the double bracket flow

$$\tfrac{d}{dt}\eta^l_k = [\eta^l_k,\, [\eta^l_k,\, \textstyle\sum_{j \rightsquigarrow k} (\eta^l_k - \eta^l_j)]] \,. \qquad (15)$$

This is the case among others for $SO(3)$.

C. Example: Biinvariant coordination in $SO(3)$

Control laws for coordination in $SO(3)$ abound in the literature — see among others papers about satellite attitude control mentioned in the Introduction. Biinvariant coordination on $SO(3)$ requires aligned rotation axes, and thus synchronizes satellite attitudes up to their phase around the rotation axis.

The compact group $SO(3)$ has a biinvariant metric, so Section IV.B applies. Algorithm (13) is used verbatim, with $\eta^l_k \in \mathbb{R}^3$ the auxiliary variable associated to angular velocity ω^l_k. As mentioned before equation (15), $\langle\, ,\, \rangle = -[\, ,\,]$ on $SO(3)$. Thus in the fully actuated case, (8),(14) lead to

$$\omega^l_k = \eta^l_k \ + \eta^l_k \times (\textstyle\sum_{j \rightsquigarrow k} Q_k^T Q_j \eta^l_j) \,, \quad k = 1...N \,. \qquad (16)$$

Theorem 2 can be strengthened as follows for specific graphs.

Proposition 4: If $\mathbb{G}$ is a tree or complete graph, then BIC is the only asymptotically stable limit set.

<u>Proof:</u> According to Theorem 2, it remains to show that BIC is the only local minimum of V_{tl}. Fixing $\eta^l_k = \omega^l \ \forall k$, critical points of V_{tl} correspond to

$$(Q_k \omega^l) \times (\textstyle\sum_{j \rightsquigarrow k} Q_j \omega^l) = 0 \quad \forall k \,. \qquad (17)$$

For the tree, start with the leaves c. Then $(Q_c \omega^l) \times (Q_p \omega^l) = 0$ where p is the parent of c. As a consequence, (17) for the parent becomes $(Q_p \omega^l) \times (Q_{pp}^T \omega^l) = 0$ where pp is the parent of p. Using this argument up to the root, all $(Q_k\omega^l)$ must be parallel. If the agents are partitioned in two anti-aligned groups, then moving those groups towards each other decreases V_{tl}; thus $V_{tl} = 0$ is the only local minimum. For the complete graph, (17) becomes $(Q_k\omega^l) \times \psi = 0 \; \forall k$, where $\psi = \sum_j Q_j\omega^l$. This implies either that all $Q_k\omega^l$ must be parallel or that $\psi = 0$. In the first case, further discussion is as for the tree. Rewriting $V_{tl} = N^2 \, \|\omega^l\|^2 - \frac{1}{2}\psi \cdot \psi$ shows that $\psi = 0$ corresponds to a maximum of V_{tl}. $\triangle$

Combining trees and cliques can yield more graphs with BIC as only asymptotically stable limit set. For others, local minima may exist. Classifying local minima of V_{tl} from graph properties is an open question.

It is straightforward to adapt (16) for underactuated agents; a popular underactuation on $SO(3)$ is to consider 2 orthogonal axes of allowed rotations $\mathbf{e}_1$ and $\mathbf{e}_2$, either controlling both rotation rates, i.e. $\omega^l_k = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, or imposing a fixed rotation rate around one axis, i.e. $\omega^l_k = \mathbf{e}_1 + u_2\mathbf{e}_2$. Both cases are controllable [33], so the Jurdjevic-Quinn theorem [53] ensures local asymptotic stability of BIC, if $\eta^l_k = \overline{\eta^l} \; \forall k$ is fixed in advance or agreed on in finite time. A formal convergence proof for the asymptotically autonomous case where the η^l_k follow (13) is currently missing.

V. Control design: underactuated left-invariant coordination

Biinvariant coordination may appear as a rather academic objective, whose motivation in applications is not clear. However, the methodology developed in Section IV for BIC control design is instrumental to achieve left-invariant coordination of underactuated agents. The latter is well motivated by practical applications. Here the role of the cost function is no longer to add a second level of coordination, but to fulfill the underactuation constraints. Unlike the academic problem setting of BIC, the present section explicitly considers the most general setting of underactuated agents as well as possibly directed and time-varying interconnection graph $\mathbb{G}$.

A. *Left-invariant coordination of underactuated agents*

The control design for underactuated LIC is decomposed in the two steps illustrated in Fig.5. Analogously to the biinvariant coordination design of Section IV.A, a feasible right-invariant velocity is determined by a consensus algorithm. The corresponding left-invariant velocity is enforced by a Lyapunov-based feedback that decreases the distance from $\mathcal{C} = \{a + Bu : u \in \mathbb{R}^m\}$ to the consensus velocity.

The consensus algorithm must enforce a feasible right-invariant velocity, that is a vector ξ^r in the set

$$O_\mathcal{C} := \{Ad_g\xi : \xi \in \mathcal{C} \text{ and } g \in G\}\,.$$

If $O_\mathcal{C}$ is convex, then it is sufficient to initialize the consensus algorithm (12) with $\eta^l_k(0) \in \mathcal{C}$. When $O_\mathcal{C}$ is not convex, the consensus algorithm must be adapted and the present paper has no general method. Strategies inspired from [38] for compact homogeneous manifolds may be helpful, as illustrated in the example below.

Now assuming a known feasible right-invariant velocity ξ^r, the design of a Lyapunov based control to left-invariant coordination proceeds similarly to Section IV.A.

Define $d(\eta, \mathcal{C})$ to be the Euclidean distance in $\mathfrak{g}$ from η to the set $\mathcal{C}$. Let $\Pi_\mathcal{C}(\eta)$ be the projection of η on $\mathcal{C}$; since $\mathcal{C}$ is convex, $\forall \eta \;\; \Pi_\mathcal{C}(\eta)$ is the unique point in $\mathcal{C}$ such that $d(\eta, \mathcal{C}) = d(\eta, \Pi_\mathcal{C}(\eta)) =: \|\eta - \Pi_\mathcal{C}(\eta)\|$. Following the same steps as in Section IV.A, define $\eta^l_k := Ad^{-1}_{g_k}\xi^r$. Writing

$$\xi^l_k = a + Bu_k = \Pi_\mathcal{C}(\eta^l_k) + Bq_k\,, \quad k = 1...N\,, \tag{18}$$

the task is to design $q_k \in \mathbb{R}^m$ such that asymptotically, g_k is driven to a point where $\eta^l_k \in \mathcal{C}$ and q_k converges to 0; this would asymptotically ensure LIC. For each individual agent k, write the cost function

$$V_k(g_k) = \tfrac{1}{2}\|Ad^{-1}_{g_k}\xi^r - \Pi_\mathcal{C}(Ad^{-1}_{g_k}\xi^r)\|^2 = \tfrac{1}{2}\|\eta^l_k - \Pi_\mathcal{C}(\eta^l_k)\|^2$$

where $\| \;\; \|$ denotes Euclidean norm. V_k characterizes the distance from η^l_k to $\mathcal{C}$, that is the distance from LIC *assuming that every agent implements* $\xi^l_k = \Pi_\mathcal{C}(Ad^{-1}_{g_k}\xi^r)$. The time variation of V_k due to motion of g_k is

$$\tfrac{d}{dt}V_k = (\eta^l_k - \Pi_\mathcal{C}(\eta^l_k)) \cdot [\eta^l_k,\, \Pi_\mathcal{C}(\eta^l_k) + Bq_k] \tag{19}$$

where $\cdot$ denotes the canonical scalar product in $\mathfrak{g}$. To go on along the lines of Section IV.A, it must hold $(\eta - \Pi_\mathcal{C}(\eta)) \cdot [\eta,\, \Pi_\mathcal{C}(\eta)] \; \leq \; 0 \;\; \forall \eta \in O_\mathcal{C}$; this condition on Lie algebra structure and control setting is satisfied for examples below. Then (19) implies $\frac{d}{dt}V_k \leq f(\eta^l_k) \cdot q_k$, where

$$f(\eta^l_k) = B^T \left\langle \eta^l_k,\, (\eta^l_k - \Pi_\mathcal{C}(\eta^l_k)) \right\rangle \tag{20}$$

when identifying $\mathfrak{g}$ with $\mathbb{R}^n$, and a natural control is

$$q_k = -f(\eta^l_k)\,, \quad k = 1...N\,. \tag{21}$$

Note that when $O_{\xi^r} \subseteq \mathcal{C}$, the position control q_k is unnecessary and vanishes, yielding simply $\xi^l_k = Ad^{-1}_{g_k}\xi^r \; \forall t$.

The overall controller is the cascade of a consensus algorithm to agree on a desired velocity for LIC, and a position controller designed from a natural Lyapunov function to reach positions compatible with underactuation constraints and so to actually achieve LIC. To implement the controller, agent k must get from other agents $j \rightsquigarrow k$ their relative positions λ_{jk} and the values of their left-invariant *auxiliary variables* η^l_j. Since agents only interact through the consensus algorithm, not through the cost function, a connected fixed undirected graph is not required: $\mathbb{G}$ can be directed and time-varying, as long as it remains uniformly connected.

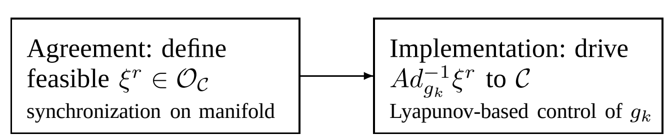

Fig. 5. Underactuated left-invariant coordination as constrained consensus on right-invariant velocity and Lyapunov-based control to left-invariant coordination.

A general characterization of the behavior of solutions of the closed-loop system is more difficult here because the position controller is not a gradient anymore. A crucial step for which the present paper proposes no explicit general solution is the design of an appropriate consensus algorithm on auxiliary variables. The other assumptions in the following result can be readily checked for any particular case.

Theorem 3: Consider N underactuated agents communicating on a uniformly connected graph $\mathbb{G}$ and evolving on Lie group G according to $\frac{d}{dt}g_k = L_{g_k *}\xi_k^l$ with controller (18),(21) where f is defined in (20), assuming that $\forall \eta \in O_{\mathcal{C}}$, it holds $(\eta - \Pi_{\mathcal{C}}(\eta))\cdot[\eta, \Pi_{\mathcal{C}}(\eta)] \leq 0$. Assume that an appropriate consensus algorithm drives the arbitrarily initiated η_k^l, $k = 1...N$, such that they exponentially agree on $Ad_{g_k}\eta_k^l \rightarrow \xi^r \in O_{\mathcal{C}}\ \forall k$, independently of the agent motions $g_k(t)$.

(i) If the agents are controllable, then LIC is locally asymptotically stable.

(ii) If, for any fixed $\eta_k^r = \xi^r$, bounded V_k implies bounded η_k^l, and $f(\eta_k^l) \rightarrow 0$ implies $g_k \rightarrow \{g : f(Ad_g^{-1}\eta_k^r) = 0\}$, then all agent trajectories on G converge to the set where $f(Ad_{g_k}^{-1}\xi^r) = 0$.

Proof: The overall system is a cascade of the exponentially stable consensus algorithm and position controller (18),(21) which is decoupled for the individual agents. Assumptions $\frac{d}{dt}V_k \leq f(\eta_k^l)\cdot q_k$ and (21) exactly mean that $V_k(g_k)$ is nonincreasing along the closed-loop solutions. Therefore, if the agents are controllable, Jurdjevic-Quinn theorem [53] implies local asymptotic stability of the local minimum $V_k = 0\ \forall k$ for the position controller. Then the overall system is the cascade of an exponentially stable system and a system for which $V_k = 0\ \forall k$ is locally asymptotically stable. Standard arguments on cascade systems (see e.g. [54], [55]) allow to conclude that $V_k = 0\ \forall k$ is locally asymptotically stable for the overall system; this proves (i).

To prove (ii), first consider the case where $\eta_k^r = \xi^r$ constant $\forall k$. Then V_k can only decrease, and since it is bounded from below it tends to a limit; therefore $\frac{d}{dt}V_k$ is integrable in time for $t \rightarrow +\infty$. For the same reason, V_k is bounded, so according to the assumption for (ii) η_k^l is bounded as well; then $\frac{d^2}{dt^2}V_k$, which is a continuous function of η_k^l, is bounded as well for the closed-loop system, such that $\frac{d}{dt}V_k$ is uniformly continuous in time for $t \rightarrow +\infty$. Barbalat's Lemma implies that $\frac{d}{dt}V_k$ converges to 0, which implies that $f(\eta_k^l)$ converges to 0, concluding the proof. Now in fact η_k^r varies, it exponentially converges to the constant $\xi^r\ \forall k$. This changes nothing to the fact that V_k tends to a finite limit and $\frac{d^2}{dt^2}V_k$ is bounded, so the same argument applies. $\triangle$

Condition $\frac{d}{dt}V_k \leq f(\eta_k^l)\cdot q_k$ is not always true when $a \neq 0$; however, it often holds in practice, as in the following example on steering control of rigid bodies. For this example, Theorem 3 is improved by showing that LIC is the *only stable* limit set. In general, possible improvements of the local stability result depend on the geometry of $O_{\mathcal{C}}$ and related consensus algorithms; particular settings of the literature feature fairly large regions of attraction (at least in simulations).

B. Example: Steering control on $SE(3)$

Left-invariant coordination on $SE(3)$ under steering control is studied in [19], [56]. The present section shows how the algorithms of [19] follow from the present general framework. Illustrations of the algorithms by numerical simulation can also be found in [19], [56].

Using the notations of Section II.C, the position and orientation of a rigid body in 3-dimensional space is written $(r_k, Q_k) =: g_k$, which is an element of the Special Euclidean group $SE(3)$; group multiplication is the usual composition law for translations and rotations, see Section II.C. Then requiring agents to "move in formation", i.e. such that the relative position and heading of agent j with respect to agent k is fixed in the reference frame of agent k, $\forall j, k$, is equivalent to requiring left-invariant coordination. Moreover, since linear and angular velocity in body frame correspond to the components (v_k^l, ω_k^l) of ξ_k^l, the problem of controlling each agent *in its own frame* with feedback involving *relative* positions and orientations of other agents only, fits the left-invariant problem setting described in Section III. The constraint of *steering control* — i.e. fixed linear velocity in agent frame $v_k^l = \mathbf{e}_1$ — implies (2) of the form

$$\xi_k^l = a + Bu_k = (\mathbf{e}_1, u_k) \quad \Rightarrow \quad \mathcal{C} = (\mathbf{e}_1, \mathbb{R}^3)\ .$$

Steering controlled agents on $SE(3)$ are controllable [33].

Following the method of Section V.A, write auxiliary variables $\eta_k^l = (\eta_{v\,k}^l, \eta_{\omega\,k}^l)$; then $\Pi_{\mathcal{C}}(\eta_k^l) = (\mathbf{e}_1, \eta_{\omega\,k}^l)$, cost function $V_k = \frac{1}{2}\|\eta_{v\,k}^l - \mathbf{e}_1\|^2$ and straightforward calculations show that (19) becomes $\frac{d}{dt}V_k = (\eta_{v\,k}^l \times \mathbf{e}_1)\cdot q_k$. This means that $(\eta - \Pi_{\mathcal{C}}(\eta))\cdot[\eta, \Pi_{\mathcal{C}}(\eta)] = 0$ and $f_{\eta_k^l} = (\eta_{v\,k}^l \times \mathbf{e}_1)$. Then (18),(21) yield the controller

$$u_k = \eta_{\omega\,k}^l + \mathbf{e}_1 \times \eta_{v\,k}^l\ , \quad k = 1...N\ . \tag{22}$$

This is the same control law as derived in [19] from intuitive arguments. If an appropriate consensus algorithm is built, then all assumptions of Theorem 3 hold, implying local asymptotic stability of 3-dimensional "motion in formation" with steering control (22); in fact, [19] slightly improves Theorem 3 by also showing that *globally*, LIC is the only stable limit set.

It remains to design a consensus algorithm for the η_k^l. For this, two cases are distinguished: linear motion $\omega^r = 0$ and helicoidal (of which a special case is circular) motion $\omega^r \neq 0$. The first case (almost) never appears from a consensus algorithm with arbitrary $\eta_k^l(0)$; it can however be imposed by $\eta_{\omega\,k}^l(0) = 0\ \forall k$, which will then remain true $\forall t \geq 0$, in order to stabilize a coordinated motion in straight line.

- If $\eta_{\omega\,k}^l = 0$ (linear motion), then $\eta_{v\,k}^l = Q_k^T\eta_{v\,k}^r$ and $O_{(\mathbf{e}_1,0)} = \{(\lambda, 0) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|\lambda\| = 1\}$. Agreement on v^r in the unit sphere can be achieved following [38], just achieving consensus in $\mathbb{R}^3$ and normalizing; in fact normalizing is not even necessary, as it would just change the gain in (22). This leads to

$$\tfrac{d}{dt}\eta_{v\,k}^l = \textstyle\sum_{j \rightsquigarrow k}(Q_k^T Q_j \eta_{v\,j}^l - \eta_{v\,k}^l) - u_k \times \eta_{v\,k}^l \tag{23}$$

for $k = 1...N$, again as in [19].

- If $\eta_{\omega\,k}^l \neq 0$, then $\eta_{\omega\,k}^l = Q_k^T\eta_{\omega\,k}^r$ and $\eta_{v\,k}^l = Q_k^T\eta_{v\,k}^r - (Q_k^T r_k) \times (Q_k^T\eta_{\omega\,k}^r)$, and $O_{\mathcal{C}} = \{(\gamma + \beta \times \alpha,\, \alpha) : \alpha, \beta, \gamma \in \mathbb{R}^3 \text{ and } \|\gamma\| \leq 1\}$. Designing a consensus algorithm, that *both* achieves agreement on $\xi^r \in O_{\mathcal{C}}$ and can be written with left-invariant variables, appears to be difficult. Similarly to the first case, suitable algorithms can be built if the overall dimension of the variables used for the consensus algorithm is enlarged with respect to the dimension of the configuration space. The consensus algorithm proposed in [19] replaces η_k^l by three components $\alpha_k = \eta_{\omega\, k}^l \in \mathbb{R}^3$, $\beta_k \in \mathbb{R}^3$ and $\gamma_k \in \mathbb{R}^3$ associated with the vectors α, β, γ used to describe $O_{\mathcal{C}}$ above; then $\eta_k^l = (\eta_{v\, k}^l,\, \eta_{\omega\, k}^l) = (\gamma_k + \beta_k \times \alpha_k,\, \alpha_k)$. The advantage of this embedding $\eta_k^l \to (\alpha_k, \beta_k, \gamma_k)$ is that left-invariant consensus algorithms can be decoupled for the α_k, the β_k and the γ_k. With the notations of the present paper, the corresponding consensus algorithm proposed in [19] is

$$\begin{aligned} \tfrac{d}{dt}\alpha_k &= \textstyle\sum_{j \rightsquigarrow k} (Q_k^T Q_j \alpha_j - \alpha_k) \,-\, u_k \times \alpha_k \\ \tfrac{d}{dt}\beta_k &= \textstyle\sum_{j \rightsquigarrow k} (Q_k^T Q_j \beta_j - \beta_k + Q_k^T (r_j - r_k)) \\ &\quad - u_k \times \beta_k - \mathbf{e}_1 \\ \tfrac{d}{dt}\gamma_k &= \textstyle\sum_{j \rightsquigarrow k} (Q_k^T Q_j \gamma_j - \gamma_k) \,-\, u_k \times \gamma_k \,, \end{aligned}$$

$k = 1 ... N$. Comparing left-invariant relative position $g_k^{-1} g_j = (Q_k^T (r_j - r_k),\, Q_k^T Q_j)$ with the terms and factors appearing in this consensus algorithm, one observes that the latter is indeed left-invariant. It can be verified (see [19]) that this algorithm indeed synchronizes the $\eta_k^r = Ad_{g_k}(\gamma_k + \beta_k \times \alpha_k,\, \alpha_k)$.

Remark 3: LIC in linear motion, i.e. with $\eta_{\omega\, k}^l = 0\ \forall k$, under steering control requires to align vectors $Q_k \mathbf{e}_1$ for all agents. This is in fact equivalent to BIC on $SO(3)$ with $\eta_k^l = \omega^l = \mathbf{e}_1$ $\forall k$. The present section thus illustrates the method for BIC on $SO(3)$ for uniformly connected $\mathbb{G}$ (instead of fixed undirected $\mathbb{G}$ as in Section IV).

Remark 4: LIC under steering control on $SE(2)$ is treated in [21], [20], where numerical simulations of the resulting algorithms can also be found. As for $SE(3)$, control algorithms obtained intuitively, with several simplifications due to the lower dimension, can be recovered with the general method of the present paper.

In fact, the group structure and control setting of steering control on $SE(2)$ are such that $\forall g \in SE(2)$ and $\forall$ steering controls $u \in \mathbb{R}$, one has

$$\begin{aligned} \xi^r &= Ad_g \xi^l = Ad_g(a + Bu) = \alpha(g,u) + Bu \\ &\qquad\qquad \text{with} \quad \alpha(g,u) \perp Bu\,. \end{aligned} \tag{24}$$

On $SE(2)$ explicitly, $a + Bu = (\mathbf{e}_1, u) \in \mathbb{R}^2 \times \mathbb{R}$ and $Ad_g(\mathbf{e}_1, u) = (Q_\theta \mathbf{e}_1 - u Q_{\pi/2} r,\, u)$, so $\alpha(g,u) = (Q_\theta \mathbf{e}_1 - u Q_{\pi/2} r,\, 0)$ and $Bu = (0,\, u)$. Then LIC automatically implies equal u_k, thus RIC, meaning that *underactuated LIC is equivalent to BIC* and imposes the same constraints on relative positions λ_{jk}. This is the case for any group and control setting satisfying (24).

For steering control on $SE(3)$, LIC is slightly different from BIC because $Ad_g(\mathbf{e}_1, u) = (Q\mathbf{e}_1 + r \times (Qu), Qu)$, so (24) would require $(Qu) \cdot (Q\mathbf{e}_1) = u \cdot \mathbf{e}_1 = 0$ which is not true in general. Therefore, for LIC under steering control the $\omega_k^l = u_k$ can differ by arbitrary rotations around $\mathbf{e}_1$, while BIC would require equal ω_k^l.

VI. Conclusion

This paper proposes a geometric framework for coordination on general Lie groups and methods for the design of controllers driving a swarm of underactuated, simple integrator agents towards coordination. It shows how the general framework provides control laws for coordination of rigid bodies, on $SO(3)$, $SE(2)$ and $SE(3)$, and allows to easily handle different settings. Formal convergence results are local, but authors working on particular applications have always observed fairly large regions of attraction (at least in simulations).

Following the numerous results about coordination on particular Lie groups, various directions are still open to extend the general framework of the present paper. A first case often encountered in practice is to stabilize *specific relative positions* of the agents ("formation control"). In [20], [21] for instance, the steering controlled agents on $SE(2)$ are not only coordinated on a circle, but regular distribution of the agents on the circle is also stabilized; in the present paper, relative positions of the agents are asymptotically fixed but arbitrary. The requirement of synchronization (most prominently, "attitude synchronization" on $SO(3)$) also fits in this category. A second important extension would be to consider *more complex dynamics*, like those of mechanical systems.

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