Structural Properties of Bounded Languages with Respect to Multiplication by a Constant

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1) Abstract numeration systems

**Definition. [P. Lecomte, M. Rigo, 2001]** An abstract numeration system is a triple $S = (L, \Sigma, <)$ where $L$ is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of $L$ with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\text{rep}_S : \mathbb{N} \rightarrow L \quad \text{val}_S = \text{rep}_S^{-1} : L \rightarrow \mathbb{N}.$$
### Examples

1) $a^*$

<table>
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<tr>
<th>$n$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
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<tr>
<td>rep($n$)</td>
<td>$\varepsilon$</td>
<td>$a$</td>
<td>$aa$</td>
<td>$aaa$</td>
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2) $\{a, b\}^*$, $a < b$

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3) $a^*b^*$, $a < b$

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Remark. This generalizes “classical” Pisot systems like integer base systems or Fibonacci system.

\[ L = \{\varepsilon\} \cup \{1, \ldots, k - 1\}\{0, \ldots, k - 1\}^* \quad \text{or} \quad L = \{\varepsilon\} \cup 1\{0, 01\}^* \]

Definition. A set \( X \subseteq \mathbb{N} \) is \( S \)-recognizable if \( \text{rep}_S(X) \subseteq \Sigma^* \) is a regular language (accepted by a DFA).
2) Main question

If $S = (L, \Sigma, <)$ is an abstract numeration system, can we find some necessary and sufficient condition on $\lambda \in \mathbb{N}$ such that for any $S$-recognizable set $X$, the set $\lambda X$ is still $S$-recognizable?

$$X \ S\text{-rec} \quad \rightarrow \quad \lambda X \ S\text{-rec}$$
3) First results about \( S \)-recognizability

**Theorem 1.** Let \( S = (L, \Sigma, <) \) be an abstract numeration system. Any arithmetic progression is \( S \)-recognizable.

**Definition.** We denote by \( u_L(n) \) the number of words of length \( n \) belonging to \( L \).

**Theorem 2.** [Polynomial case] Let \( L \subseteq \Sigma^* \) be a regular language such that \( u_L(n) \in \Theta(n^k), \; k \in \mathbb{N} \) and \( S = (L, \Sigma, <) \). Preservation of the \( S \)-recognizability after multiplication by \( \lambda \) holds only if \( \lambda = \beta^{k+1} \) for some \( \beta \in \mathbb{N} \).
**Definition.** A language $L$ is *slender* if $u_L(n) \in O(1)$.

**Theorem 3.** [Slender case] Let $L \subseteq \Sigma^*$ be a slender regular language and $S = (L, \Sigma, <)$. A set $X \subseteq \mathbb{N}$ is $S$-recognizable if and only if $X$ is a finite union of arithmetic progressions.

**Corollary.** Let $S$ be a numeration system built on a slender language. If $X \subseteq \mathbb{N}$ is $S$-recognizable then $\lambda X$ is $S$-recognizable for all $\lambda \in \mathbb{N}$. 
**Theorem 4.** Let $\beta > 0$. For the abstract numeration system

$$S = (a^* b^*, \{ a < b \}),$$

the multiplication by $\beta^2$ preserves $S$-recognizability if and only if $\beta$ is an odd integer.
4) Bounded languages, notation

We denote by $B_\ell = a_1^* \cdots a_\ell^*$ the bounded language over the totally ordered alphabet $\Sigma_\ell = \{a_1 < \ldots < a_\ell\}$ of size $\ell \geq 1$.

We consider abstract numeration systems of the form $(B_\ell, \Sigma_\ell)$ and we denote by $\text{rep}_\ell$ and $\text{val}_\ell$ the corresponding bijections.

A set $X \subseteq \mathbb{N}$ is said to be $B_\ell$-recognizable if $\text{rep}_\ell(X)$ is a regular language over the alphabet $\Sigma_\ell$. 
In this context, multiplication by a constant $\lambda$ can be viewed as a transformation

$$f_\lambda : \mathcal{B}_\ell \rightarrow \mathcal{B}_\ell.$$ 

The question becomes then:

*Can we determine some necessary and sufficient condition under which this transformation preserves regular subsets of $\mathcal{B}_\ell$?*
Example

Let $\ell = 2$, $\Sigma_2 = \{a, b\}$ and $\lambda = 25$.

\[
\begin{array}{ccc}
8 & \times_{25} & 200 \\
\text{rep}_2 \downarrow & \downarrow \text{rep}_2 \\
a b^2 & \overset{f_{25}}{\rightarrow} & a^9 b^{10} \\
\text{N} & \times_{\lambda} & \text{N} \\
\text{rep}_\ell \downarrow & \downarrow \text{rep}_\ell \\
\mathcal{B}_\ell & \overset{f_{\lambda}}{\rightarrow} & \mathcal{B}_\ell
\end{array}
\]

Thus multiplication by $\lambda = 25$ induces a mapping $f_{\lambda}$ onto $\mathcal{B}_2$ such that for $w, w' \in \mathcal{B}_2$, $f_{\lambda}(w) = w'$ if and only if $\text{val}_2(w') = 25 \times \text{val}_2(w)$. 
5) $B_\ell$-representation of an integer

We set

$$u_\ell(n) := u_{\mathcal{B}_\ell}(n) = \#(\mathcal{B}_\ell \cap \Sigma^n)$$

and

$$v_\ell(n) := \#(\mathcal{B}_\ell \cap \Sigma_{\leq n}^{\ell}) = \sum_{i=0}^{n} u_\ell(i).$$

**Lemma 1.** For all $\ell \geq 1$ and $n \geq 0$, we have

$$u_{\ell+1}(n) = v_\ell(n) \quad (1)$$

and

$$u_\ell(n) = \binom{n + \ell - 1}{\ell - 1}. \quad (2)$$
Lemma 2. Let $S = (a_1^* \cdots a_{\ell}^*, \{a_1 < \cdots < a_\ell\})$. We have

$$\text{val}_\ell(a_1^{n_1} \cdots a_{\ell}^{n_{\ell}}) = \sum_{i=1}^{\ell} (n_i + \cdots + n_{\ell} + \ell - i).$$

Corollary. [Katona, 1966] Let $\ell \in \mathbb{N} \setminus \{0\}$. Any integer $n$ can be uniquely written as

$$n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell-1} + \cdots + \binom{z_1}{1} \quad (3)$$

with $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0.$
Example

Consider the words of length 3 in the language $a^*b^*c^*$,

$$aaa < aab < aac < abb < abc < acc < bbb < bbc < bcc < ccc.$$ 

We have $\text{val}_3(aaa) = \left(\frac{5}{3}\right) = 10$ and $\text{val}_3(acc) = 15$.

If we apply the erasing morphism $\varphi : \{a, b, c\} \to \{a, b, c\}^*$ defined by

$$\varphi(a) = \varepsilon, \varphi(b) = b, \varphi(c) = c$$

on the words of length 3, we get

$$\varepsilon < b < c < bb < bc < cc < bbb < bbc < bcc < ccc.$$ 

So we have $\text{val}_3(acc) = \text{val}_3(aaa) + \text{val}_2(cc)$ where $\text{val}_2$ is considered as a map defined on the language $b^*c^*$. 
Algorithm computing \( \text{rep}_\ell(n) \).

Let \( n \) be an integer and \( 1 \) be a positive integer.

For \( i=1, l-1, \ldots, 1 \) do
if \( n > 0 \),
find \( t \) such that \( \binom{t}{i} \leq n < \binom{t+1}{i} \)
z(i) \( \leftarrow t \)
n \( \leftarrow n - \binom{t}{i} \)
otherwise, \( z(i) \leftarrow i - 1 \)

Consider now the triangular system having \( \alpha_1, \ldots, \alpha_\ell \) as unknowns
\[
\alpha_i + \cdots + \alpha_\ell = z(\ell - i + 1) - \ell + i, \quad i = 1, \ldots, \ell.
\]
One has \( \text{rep}_\ell(n) = a_1^{\alpha_1} \cdots a_\ell^{\alpha_\ell} \).
Example

For $\ell = 3$, one gets for instance

$$12345678901234567890 = \binom{4199737}{3} + \binom{3803913}{2} + \binom{1580642}{1}$$

and solving the system

$$\begin{cases} 
  n_1 + n_2 + n_3 &= 4199737 - 2 \\
  n_2 + n_3 &= 3803913 - 1 \\
  n_3 &= 1580642
\end{cases}$$

$\Leftrightarrow (n_1, n_2, n_3) = (395823, 2223270, 1580642)$,

we have

$$\text{rep}_3(12345678901234567890) = a^{395823} b^{2223270} c^{1580642}.$$
Remark. In particular, we have $u_{B_{\ell}}(n) \in \Theta(n^{\ell-1})$.

So we have to focus only on multiplicators of the kind
\[ \lambda = \beta^\ell. \]
6) Multiplication by $\lambda = \beta^\ell$

**Theorem.** For the abstract numeration system

$$S = (a^*b^*c^*, \{a < b < c\})$$

if $\beta \in \mathbb{N} \setminus \{0, 1\}$ is such that $\beta \not\equiv \pm 1 \pmod{6}$ then the multiplication by $\beta^3$ does not preserve the $S$-recognizability.

For instance, if $\beta \equiv 2 \pmod{6}$, for $n$ large enough, we have

$$\text{rep}_3 \left[ (6k + 2)^3 \text{val}_3(a^n) \right] = a^r b^{s+(3k+1)n} c^{t+(3k+1)n}$$

where the constants $r, s, t$ are given by

$$r = 4k+6k^2, \quad s = 5k+11k^2+24k^3+18k^4, \quad t = -3k-17k^2-24k^3-18k^4.$$
Conjecture. Multiplication by $\beta^\ell$ preserves $S$-recognizability for the abstract numeration system

$$S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$$

built on the bounded language $\mathcal{B}_\ell$ over $\ell$ letters if and only if

$$\beta = \prod_{i=1}^{k} p_{\theta_i}$$

where $p_1, \ldots, p_k$ are prime numbers strictly greater than $\ell$. 
Lemma 1. For $n \in \mathbb{N}$ large enough, we have

$$|\text{rep}_\ell(\beta^\ell n)| = \beta |\text{rep}_\ell(n)| + \frac{(\beta - 1)(\ell - 1)}{2} + i$$

with $i \in \{-1, 0, \ldots, \beta - 1\}$.

Definition. For all $i \in \{-1, 0, \ldots, \beta - 1\}$ and $k \in \mathbb{N}$ large enough, we define

$$\mathcal{R}_{i,k} := \left\{ n \in \mathbb{N} : |\text{rep}_\ell(n)| = k \text{ and } |\text{rep}_\ell(\beta^\ell n)| = \beta k + \frac{(\beta - 1)(\ell - 1)}{2} + i \right\}.$$
We assume that $\beta$ satisfies the condition of the Conjecture.

**Proposition.** Let $i \in \{0, \ldots, \beta - 1\}$. There exists a constant $L \geq 0$ (depending only on $\ell$ and $\beta$) such that for all $k \geq L$, if $m = \min R_{i,k}$ and $n = \min R_{i,k+\beta^{\ell-1}}$ then

$$
\forall t \in \{2, \ldots, \ell\} : |\text{rep}_\ell(\beta^\ell m)|_{a_t} = |\text{rep}_\ell(\beta^\ell n)|_{a_t}.
$$

Furthermore, $|\text{rep}_\ell(\beta^\ell m)|_{a_1} + \beta^\ell = |\text{rep}_\ell(\beta^\ell n)|_{a_1}$. 