Structural Properties of Bounded Languages with Respect to Multiplication by a Constant

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1) Abstract numeration systems

**Definition [P. Lecomte, M. Rigo 2001]** An abstract numeration system is a triple $S = (L, \Sigma, <)$ where $L$ is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of $L$ with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\text{rep}_S : \mathbb{N} \to L \quad \text{val}_S = \text{rep}_S^{-1} : L \to \mathbb{N}.$$
Examples

1) $a^*$

\[
\begin{array}{c|cccccc}
\text{rep}(n) & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
n & \varepsilon & a & aa & aaa & aaaa & \cdots \\
\end{array}
\]

2) $\{a, b\}^*$, $a < b$

\[
\begin{array}{c|cccccccc}
\text{rep}(n) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
n & \varepsilon & a & b & aa & ab & ba & bb & aaa & \cdots \\
\end{array}
\]

3) $a^*b^*$, $a < b$

\[
\begin{array}{c|cccccccc}
\text{rep}(n) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\hline
n & \varepsilon & a & b & aa & ab & bb & aaa & \cdots \\
\end{array}
\]
**Definition** A set $X \subseteq \mathbb{N}$ is $S$-recognizable if $\text{rep}_S(X) \subseteq \Sigma^*$ is a regular language (accepted by a DFA).
2) Main question

If $S = (L, \Sigma, <)$ is an abstract numeration system, can we find some necessary and sufficient condition on $\lambda \in \mathbb{N}$ such that for any $S$-recognizable set $X$, the set $\lambda X$ is still $S$-recognizable?
3) First results about $S$-recognizability

**Theorem 1.** Let $S = (L, \Sigma, <)$ be an abstract numeration system. Any arithmetic progression is $S$-recognizable.

**Definition.** We denote by $u_L(n)$ the number of words of length $n$ belonging to $L$.

**Theorem 2. [Polynomial case]** Let $L \subseteq \Sigma^*$ be a regular language such that $u_L(n) \in \Theta(n^k)$, $k \in \mathbb{N}$ and $S = (L, \Sigma, <)$. Preservation of the $S$-recognizability after multiplication by $\lambda$ holds only if $\lambda = \beta^{k+1}$ for some $\beta \in \mathbb{N}$. 
Definition. A language $L$ is slender if $u_L(n) \in O(1)$.

Theorem 3. [Slender case] Let $L \subseteq \Sigma^*$ be a slender regular language and $S = (L, \Sigma, \prec)$. A set $X \subseteq \mathbb{N}$ is $S$-recognizable if and only if $X$ is a finite union of arithmetic progressions.

Corollary. Let $S$ be a numeration system built on a slender language. If $X \subseteq \mathbb{N}$ is $S$-recognizable then $\lambda X$ is $S$-recognizable for all $\lambda \in \mathbb{N}$. 
Theorem 4. Let $\beta > 0$. For the abstract numeration system

$$S = (a^*b^*, \{a < b\}),$$

the multiplication by $\beta^2$ preserves $S$-recognizability if and only if $\beta$ is an odd integer.
4) Bounded languages, notation

We denote by $B_\ell = a_1^* \cdots a_\ell^*$ the bounded language over the totally ordered alphabet $\Sigma_\ell = \{ a_1 < \ldots < a_\ell \}$ of size $\ell \geq 1$.

We consider abstract numeration systems of the form $(B_\ell, \Sigma_\ell)$ and we denote by $\text{rep}_\ell$ and $\text{val}_\ell$ the corresponding bijections.

A set $X \subseteq \mathbb{N}$ is said to be $B_\ell$-recognizable if $\text{rep}_\ell(X)$ is a regular language over the alphabet $\Sigma_\ell$. 
In this context, multiplication by a constant \( \lambda \) can be viewed as a transformation

\[ f_\lambda : \mathcal{B}_\ell \rightarrow \mathcal{B}_\ell. \]

The question becomes then:

*Can we determine some necessary and sufficient condition under which this transformation preserves regular subsets of \( \mathcal{B}_\ell \)?*
Example

Let $\ell = 2$, $\Sigma_2 = \{a, b\}$ and $\lambda = 25$.

$\begin{align*}
8 & \xrightarrow{\times 25} 200 \\
\text{rep}_2 \downarrow & \quad \downarrow \text{rep}_2 & \quad \text{rep}_\ell \downarrow & \quad \downarrow \text{rep}_\ell \\
ab ab^2 & \xrightarrow{\times 25} a^9 b^{10} & \quad & \quad \\
\end{align*}$

Thus multiplication by $\lambda = 25$ induces a mapping $f_\lambda$ onto $\mathcal{B}_2$ such that for $w, w' \in \mathcal{B}_2$, $f_\lambda(w) = w'$ if and only if $\text{val}_2(w') = 25 \cdot \text{val}_2(w)$. 
5) $B_\ell$-representation of an integer

We set

$$u_\ell(n) := u_{B_\ell}(n) = \#(B_\ell \cap \Sigma_\ell^n)$$

and

$$v_\ell(n) := \#(B_\ell \cap \Sigma_\ell^{\leq n}) = \sum_{i=0}^n u_\ell(i).$$

**Lemma 1.** For all $\ell \geq 1$ and $n \geq 0$, we have

$$u_{\ell+1}(n) = v_\ell(n) \quad (1)$$

and

$$u_\ell(n) = \binom{n + \ell - 1}{\ell - 1}. \quad (2)$$
Lemma 2. Let $S = (a_1^* \cdots a_{\ell}^*, \{a_1 < \cdots < a_{\ell}\})$. We have

$$\text{val}_{\ell}(a_1^{n_1} \cdots a_{\ell}^{n_{\ell}}) = \sum_{i=1}^{\ell} \left( n_i + \cdots + n_{\ell} + \ell - i \right).$$

Consequently, for any $n \in \mathbb{N}$,

$$|\text{rep}_{\ell}(n)| = k \iff \begin{array}{c}
\begin{aligned}
\left\{ \text{val}_{\ell}(a_1^k) \right\} \leq n \leq \sum_{i=1}^{\ell} \binom{k+i-1}{i} \\
\left\{ \text{val}_{\ell}(a_{\ell}^k) \right\}
\end{aligned}
\end{array}.$$
Example

Consider the words of length 3 in the language $a^*b^*c^*$,

$$aaa < aab < aac < abb < abc < acc < bbb < bbc < bcc < ccc.$$  

We have $\text{val}_3(aaa) = \binom{5}{3} = 10$ and $\text{val}_3(acc) = 15$.

If we apply the erasing morphism $\varphi : \{a, b, c\} \rightarrow \{a, b, c\}^*$ defined by

$$\varphi(a) = \varepsilon, \varphi(b) = b, \varphi(c) = c$$

on the words of length 3, we get

$$\varepsilon < b < c < bb < bc < cc < bbb < bbc < bcc < ccc.$$  

So we have $\text{val}_3(\text{acc}) = \text{val}_3(\text{aaa}) + \text{val}_2(\text{cc})$ where $\text{val}_2$ is considered as a map defined on the language $b^*c^*$. 
**Remark.** In particular, we have $u_{B_\ell}(n) \in \Theta(n^{\ell-1})$.

So we have to focus only on multiplicators of the kind

$$\lambda = \beta^\ell.$$
**Corollary.** Let $\ell \in \mathbb{N} \setminus \{0\}$. Any integer $n$ can be uniquely written as

$$n = \binom{\bar{z}_\ell}{\ell} + \binom{\bar{z}_{\ell-1}}{\ell - 1} + \cdots + \binom{\bar{z}_1}{1}$$

with $\bar{z}_\ell > \bar{z}_{\ell-1} > \cdots > \bar{z}_1 \geq 0$. 
Algorithm computing $\text{rep}_{\ell}(n)$.

Let $n$ be an integer and $1$ be a positive integer.

For $i=1,1-1,\ldots,1$ do
if $n>0$,
find $t$ such that $\binom{t}{i} \leq n < \binom{t+1}{i}$
z(i)$\leftarrow t$
n$\leftarrow n-\binom{t}{i}$
otherwise, $z(i)\leftarrow i-1$

Consider now the triangular system having $\alpha_1,\ldots,\alpha_{\ell}$ as unknowns

$$\alpha_i + \cdots + \alpha_{\ell} = z(\ell - i + 1) - \ell + i, \quad i = 1,\ldots,\ell.$$ 

One has $\text{rep}_{\ell}(n) = a_1^{\alpha_1} \cdots a_{\ell}^{\alpha_{\ell}}$. 
Example

For \( \ell = 3 \), one gets for instance

\[
12345678901234567890 = \binom{4199737}{3} + \binom{3803913}{2} + \binom{1580642}{1}
\]

and solving the system

\[
\begin{align*}
    n_1 + n_2 + n_3 &= 4199737 - 2 \\
    n_2 + n_3 &= 3803913 - 1 \\
    n_3 &= 1580642
\end{align*}
\]

\[ \Leftrightarrow (n_1, n_2, n_3) = (395823, 2223270, 1580642), \]

we have

\[
\text{rep}_3(12345678901234567890) = a^{395823} b^{2223270} c^{1580642}.
\]
6) Multiplication by $\lambda = \beta^\ell$

**Theorem.** For the abstract numeration system

$$S = (a*b*c^*, \{a < b < c\}),$$

if $\beta \in \mathbb{N} \setminus \{0, 1\}$ is such that $\beta \not\equiv \pm 1 \pmod{6}$ then the multiplication by $\beta^3$ does not preserve the $S$-recognizability.
**Conjecture.** Multiplication by $\beta^\ell$ preserves $S$-recognizability for the abstract numeration system

$$S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$$

built on the bounded language $\mathcal{B}_\ell$ over $\ell$ letters if and only if

$$\beta = \prod_{i=1}^{k} p_i^{\theta_i}$$

where $p_1, \ldots, p_k$ are prime numbers strictly greater than $\ell$. 
Lemma. For $n \in \mathbb{N}$ large enough, we have

$$|\text{rep}_\ell(\beta^n)| = \beta |\text{rep}_\ell(n)| + \frac{(\beta - 1)(\ell - 1)}{2} + i$$

with $i \in \{-1, 0, \ldots, \beta - 1\}$. 