Abstract Numeration Systems and Recognizability

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Outline of the talk

Abstract Numeration Systems

Some natural Questions

First Results about Recognizability

Bounded Languages

B_ℓ-Representation of an Integer

Multiplication by $\lambda = \beta^\ell$

Definition

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where L is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of L with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\mathrm{rep}_S : \mathbb{N} \to L \qquad \mathrm{val}_S = \mathrm{rep}_S^{-1} : L \to \mathbb{N}.$$

Example

$L = a^*, \ \Sigma = \{a\}$

n	0	1	2	3	4	$\cdots$
$\mathrm{rep}(n)$	ε	a	aa	aaa	$aaaa$	$\cdots$

Example

$L = \{a, b\}^*, \ \Sigma = \{a, b\}, \ a < b$

n	0	1	2	3	4	5	6	7	$\cdots$
$\mathrm{rep}(n)$	ε	a	b	aa	ab	ba	bb	aaa	$\cdots$

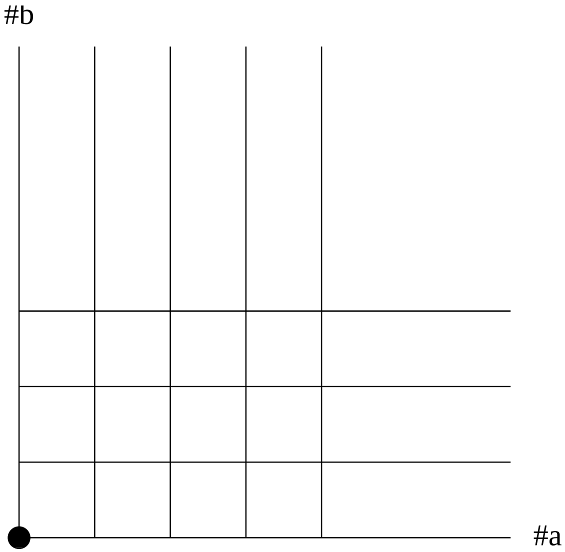
Example

$L = a^* b^*,\ \Sigma = \{a, b\},\ a < b$

n	0	1	2	3	4	5	6	$\cdots$
$\mathrm{rep}(n)$	ε	a	b	aa	ab	bb	aaa	$\cdots$

$$\mathrm{val}(a^p b^q) = \frac{1}{2}(p+q)(p+q+1) + q$$

Abstract Numeration Systems



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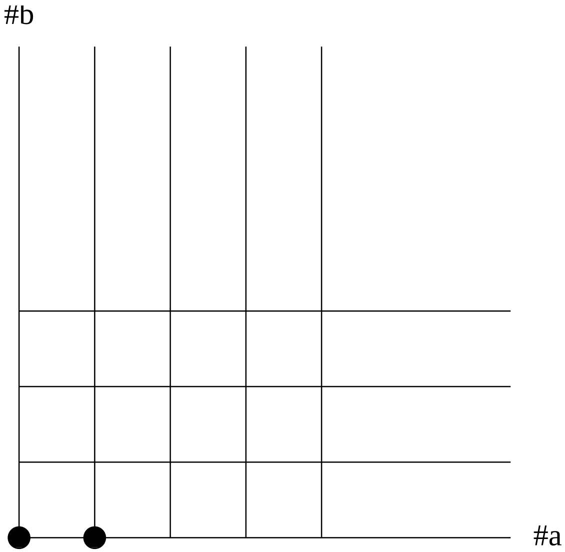

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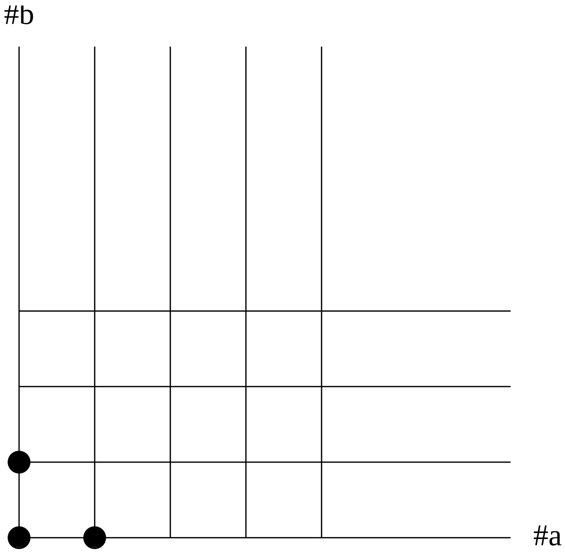

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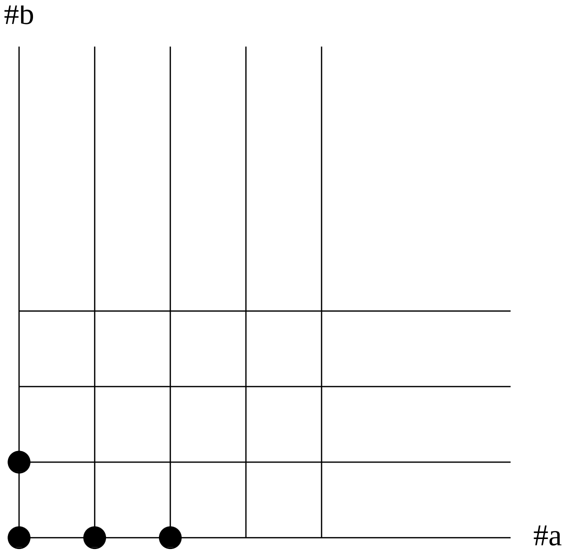

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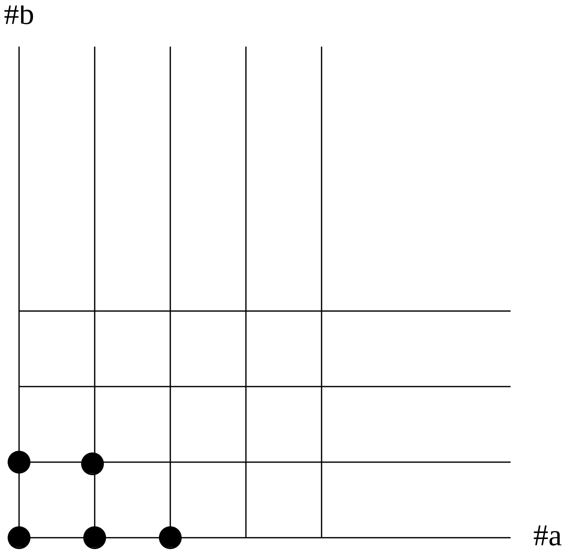

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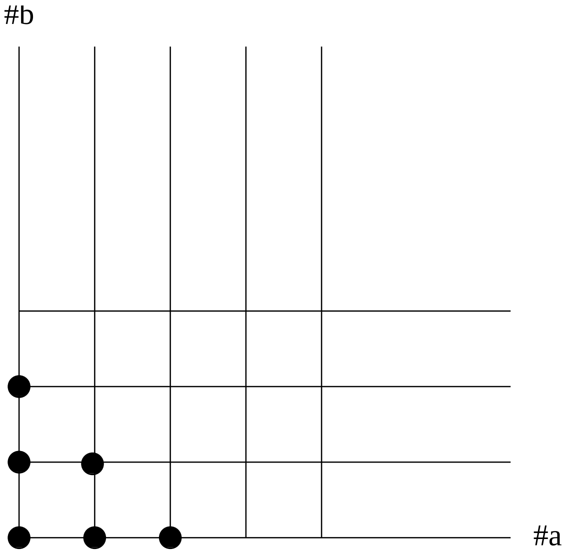

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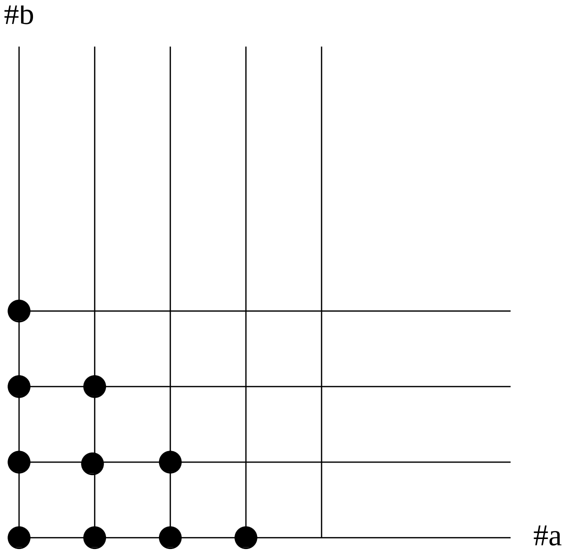

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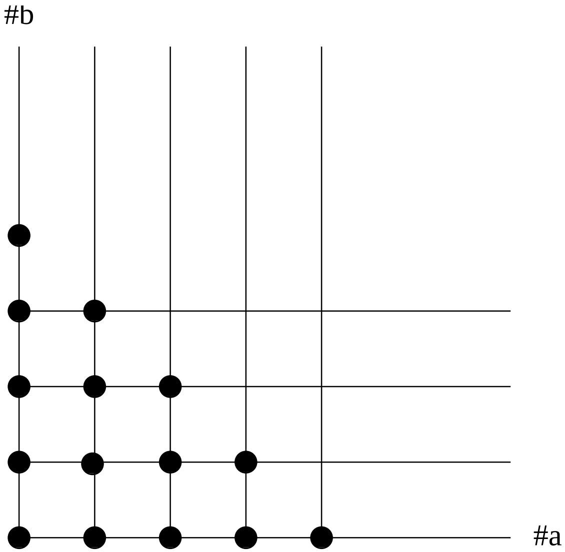

Remark

This generalizes "classical" Pisot systems like integer base systems or Fibonacci system.

$$L = \{\varepsilon\} \cup \{1, \ldots, k-1\}\{0, \ldots, k-1\}^* \text{ or } L = \{\varepsilon\} \cup 1\{0, 01\}^*$$

Definition

A set $X \subseteq \mathbb{N}$ is *S-recognizable if* $\mathrm{rep}_S(X) \subseteq \Sigma^*$ *is a regular language (accepted by a DFA).*

Some natural Questions

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- Dynamics, odometer, tilings, logic...

Theorem

Let $S = (L, \Sigma, <)$ be an abstract numeration system. Any arithmetic progression is S-recognizable.

Well-known Fact (see Eilenberg's book)

The set of squares is never recognizable in any integer base system.

Example

Let $L = a^*b^* \cup a^*c^*$, $\Sigma = \{a, b, c\}$, $a < b < c$.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
$rep(n)$	ε	a	b	c	aa	ab	ac	bb	cc	aaa	$\cdots$

Theorem (Translation)

Let $S = (L, \Sigma, <)$ be an abstract numeration system and $X \subseteq \mathbb{N}$. For each $t \in \mathbb{N}$, $X + t$ is S-recognizable if and only if X is S-recognizable.

Question : Multiplication by a Constant

If $S = (L, \Sigma, <)$ is an abstract numeration system, can we find some necessary and sufficient condition on $\lambda \in \mathbb{N}$ such that for any S-recognizable set X, the set λX is still S-recognizable ?

$$X \ S\text{-rec} \quad \stackrel{?}{\longrightarrow} \quad \lambda X \ S\text{-rec}$$

Definition

We denote by $\mathbf{u}_L(n)$ the number of words of length n belonging to L.

Theorem (Polynomial Case)

Let $L \subseteq \Sigma^*$ be a regular language such that $\mathbf{u}_L(n) \in \Theta(n^k)$, $k \in \mathbb{N}$ and $S = (L, \Sigma, <)$. Preservation of S-recognizability after multiplication by λ holds only if $\lambda = \beta^{k+1}$ for some $\beta \in \mathbb{N}$.

Definition

A language L is *slender* if $\mathbf{u}_L(n) \in O(1)$.

Theorem (Slender Case)

Let $L \subset \Sigma^*$ be a slender regular language and $S = (L, \Sigma, <)$. A set $X \subseteq \mathbb{N}$ is S-recognizable if and only if X is a finite union of arithmetic progressions.

Corollary

Let S be a numeration system built on a slender language. If $X \subseteq \mathbb{N}$ is S-recognizable then λX is S-recognizable for all $\lambda \in \mathbb{N}$.

Theorem

Let $\beta > 0$. For the abstract numeration system

$$S = (a^* b^*, \{a, b\}, a < b),$$

multiplication by β^2 preserves S-recognizability if and only if β is an odd integer.

Notation

We denote by $\mathcal{B}_\ell = a_1^* \cdots a_\ell^*$ the bounded language over the totally ordered alphabet $\Sigma_\ell = \{a_1 < \ldots < a_\ell\}$ of size $\ell \geq 1$.

We consider abstract numeration systems of the form $(\mathcal{B}_\ell, \Sigma_\ell)$ and we denote by rep_ℓ and val_ℓ the corresponding bijections.

A set $X \subseteq \mathbb{N}$ is said to be $\mathcal{B}_\ell$-*recognizable* if $\mathrm{rep}_\ell(X)$ is a regular language over the alphabet Σ_ℓ.

In this context, multiplication by a constant λ can be viewed as a transformation

$$f_\lambda : \mathcal{B}_\ell \rightarrow \mathcal{B}_\ell.$$

The question becomes then :

Can we determine some necessary and sufficient condition under which this transformation preserves regular subsets of $\mathcal{B}_\ell$ *?*

Example

Let $\ell = 2$, $\Sigma_2 = \{a, b\}$ and $\lambda = 25$.

$$\begin{array}{rcl} 8 & \xrightarrow{\times 25} & 200 \\ \mathrm{rep}_2 \downarrow & & \downarrow \mathrm{rep}_2 \\ a\,b^2 & \xrightarrow{f_{25}} & a^9\,b^{10} \end{array} \qquad \begin{array}{rcl} \mathbb{N} & \xrightarrow{\times \lambda} & \mathbb{N} \\ \mathrm{rep}_\ell \downarrow & & \downarrow \mathrm{rep}_\ell \\ \mathcal{B}_\ell & \xrightarrow{f_\lambda} & \mathcal{B}_\ell \end{array}$$

Thus multiplication by $\lambda = 25$ induces a mapping f_λ onto $\mathcal{B}_2$ such that for $w, w' \in \mathcal{B}_2$, $f_\lambda(w) = w'$ if and only if $\mathrm{val}_2(w') = 25\,\mathrm{val}_2(w)$.

We set

$$\mathbf{u}_\ell(n) := \mathbf{u}_{\mathcal{B}_\ell}(n) = \#(\mathcal{B}_\ell \cap \Sigma_\ell^n)$$

and

$$\mathbf{v}_\ell(n) := \#(\mathcal{B}_\ell \cap \Sigma_\ell^{\leq n}) = \sum_{i=0}^{n} \mathbf{u}_\ell(i).$$

Lemma

For all $\ell \geq 1$ and $n \geq 0$, we have

$$\mathbf{u}_{\ell+1}(n) = \mathbf{v}_\ell(n) \tag{1}$$

and

$$\mathbf{u}_\ell(n) = \binom{n+\ell-1}{\ell-1}. \tag{2}$$

Lemma

Let $S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$. *We have*

$$\mathrm{val}_\ell(a_1^{n_1} \cdots a_\ell^{n_\ell}) = \sum_{i=1}^{\ell} \binom{n_i + \cdots + n_\ell + \ell - i}{\ell - i + 1}.$$

Corollary (Katona, 1966)

Let $\ell \in \mathbb{N} \setminus \{0\}$. *Any integer* n *can be uniquely written as*

$$n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell - 1} + \cdots + \binom{z_1}{1} \qquad (3)$$

with $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0$.

Example

Consider the words of length 3 in the language $a^*b^*c^*$,

$$aaa < aab < aac < abb < abc < acc < bbb < bbc < bcc < ccc.$$

We have $\mathrm{val}_3(aaa) = \binom{5}{3} = 10$ and $\mathrm{val}_3(acc) = 15$. If we apply the erasing morphism $\varphi : \{a, b, c\} \to \{a, b, c\}^*$ defined by

$$\varphi(a) = \varepsilon, \varphi(b) = b, \varphi(c) = c$$

on the words of length 3, we get

$$\varepsilon < b < c < bb < bc < cc < bbb < bbc < bcc < ccc.$$

So we have $\mathrm{val}_3(acc) = \mathrm{val}_3(aaa) + \mathrm{val}_2(cc)$ where val_2 is considered as a map defined on the language b^*c^*.

Algorithm computing $\mathrm{rep}_\ell(n)$.

Let `n` be an integer and `l` be a positive integer.

```
For i=l,l-1,...,1 do
if n>0,
find t such that (t choose i) ≤ n < (t+1 choose i)
z(i)←t
n←n-(t choose i)
otherwise, z(i)←i-1
```

Consider now the triangular system having $\alpha_1, \ldots, \alpha_\ell$ as unknowns

$$\alpha_i + \cdots + \alpha_\ell = \mathtt{z}(\ell - i + 1) - \ell + i, \quad i = 1, \ldots, \ell.$$

One has $\mathrm{rep}_\ell(\mathtt{n}) = a_1^{\alpha_1} \cdots a_\ell^{\alpha_\ell}$.

Example

For $\ell = 3$, one gets for instance

$$12345678901234567890 = \binom{4199737}{3} + \binom{3803913}{2} + \binom{1580642}{1}$$

and solving the system

$$\left\{ \begin{array}{rcl} n_1 + n_2 + n_3 & = & 4199737 - 2 \\ n_2 + n_3 & = & 3803913 - 1 \\ n_3 & = & 1580642 \end{array} \right.$$

$$\Leftrightarrow (n_1, n_2, n_3) = (395823, 2223270, 1580642),$$

we have

$$\mathrm{rep}_3(12345678901234567890) = a^{395823} b^{2223270} c^{1580642}.$$

Remark

We have $\mathbf{u}_{\mathcal{B}_\ell}(n) \in \Theta(n^{\ell-1})$.

So we have to focus only on multiplicators of the kind

$$\lambda = \beta^\ell.$$

Multiplication by $\lambda = \beta^\ell$

Lemma

For $n \in \mathbb{N}$ large enough, we have

$$|\mathrm{rep}_\ell(\beta^\ell n)| = \beta\,|\mathrm{rep}_\ell(n)| + \frac{(\beta-1)(\ell-1)}{2} + i$$

with $i \in \{-1, 0, \ldots, \beta-1\}$.

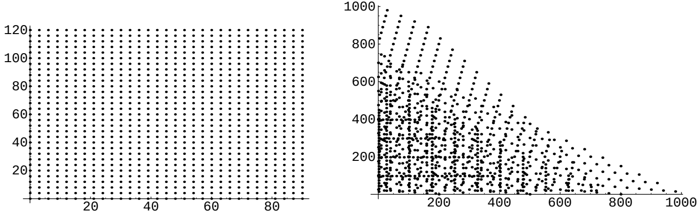
Definition

For all $i \in \{-1, 0, \ldots, \beta-1\}$ and $k \in \mathbb{N}$ large enough, we define

$$\mathcal{R}_{i,k} := \Big\{ n \in \mathbb{N} : |\mathrm{rep}_\ell(n)| = k \text{ and } |\mathrm{rep}_\ell(\beta^\ell n)| = \beta\,k + \frac{(\beta-1)(\ell-1)}{2} + i \Big\}.$$

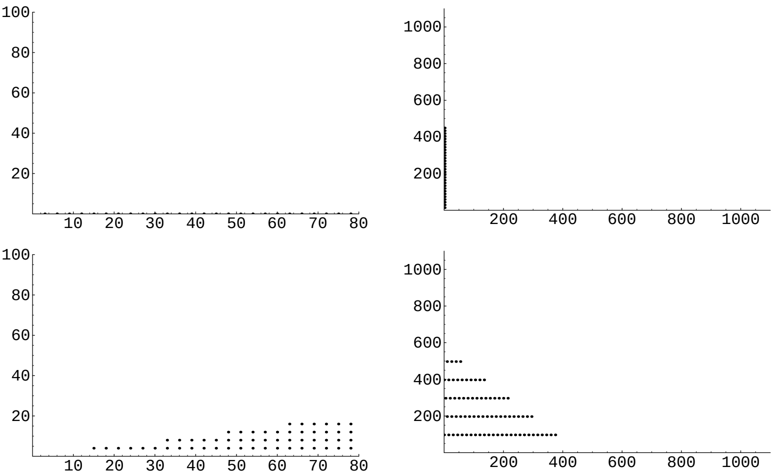
Multiplication by $\lambda = \beta^\ell$

Example (Multiplication by 25 in $\mathcal{B}_2$)



Multiplication by $\lambda = \beta^\ell$

Example (The R_i before and after Multiplication by 25.)



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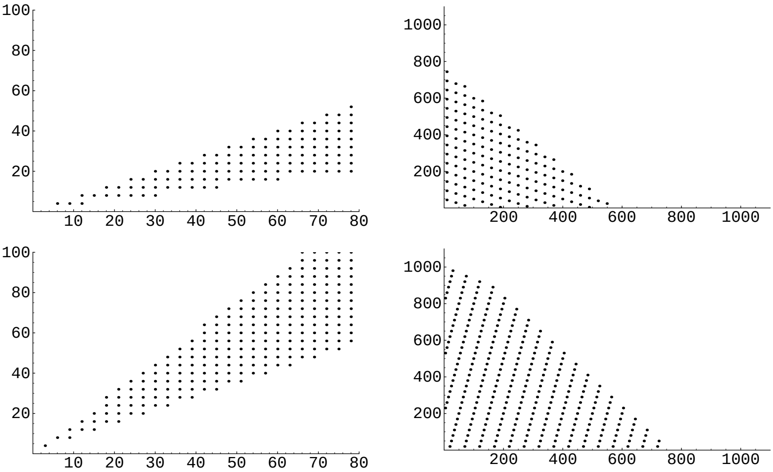

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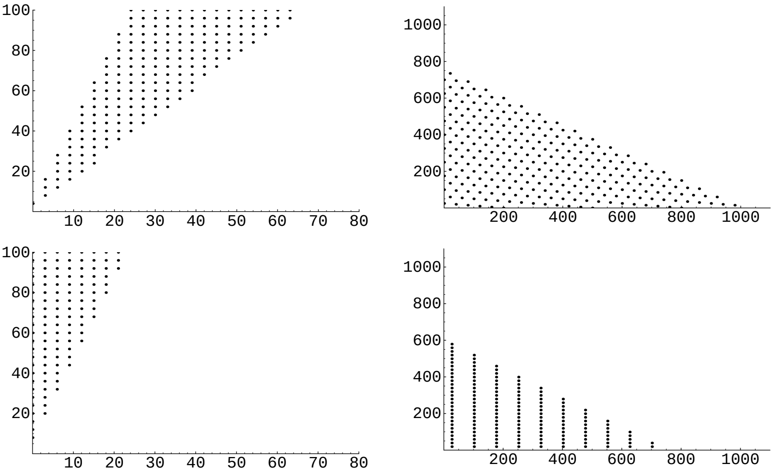

Multiplication by $\lambda = \beta^\ell$

Theorem

Let $S = (a^*b^*c^*, \{a < b < c\})$*. For any constant* $\beta \in \mathbb{N}$*, multiplication by* β^3 *does not preserve* S*-recognizability.*

Corollary

Let $S = (a^*b^*c^*, \{a < b < c\})$*. For any constant* $\lambda \in \mathbb{N}$*, multiplication by* λ *does not preserve* S*-recognizability.*

Past Conjecture

Multiplication by β^ℓ preserves S-recognizability for the abstract numeration system

$$S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$$

built on the bounded language $\mathcal{B}_\ell$ over ℓ letters if and only if

$$\beta = \prod_{i=1}^{k} p_i^{\theta_i}$$

where $p_1, \ldots, p_k$ are prime numbers strictly greater than ℓ.
In other words, multiplication by β^ℓ does not preserve S-recognizability if and only if

$$\exists M \in \{2, \ldots, \ell\} : \beta \equiv 0 \pmod{M}.$$