Multiplication by a Constant and Recognizability in an Abstract Numeration System

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Introduction

One of the main interesting concerns in work about numeration systems is studying the relationships that exist between arithmetic properties of numbers and syntactical properties of their representation. More precisely, we prefer to manipulate set of numbers whose representations are characterized by very simple syntactical rules, that is, forming a regular language. Such sets are called S-recognizable sets, where S is the numeration system we are working with. In particular, we like to obtain numeration systems in which the whole set N is recognizable, partly due to the fact that, if the set of representations of all integers is regular, there are very simple algorithms making it possible to decide if a word stands or does not stand for a number.

Some general questions in this area are the following.

- Characterization of the recognizable parts in a fixed numeration system.
- Determination of the numeration systems for which a given set of numbers is recognizable.
- To examine the stability of recognizability under arithmetic operations.

In this paper we present our research about that last problem. First, we will define the abstract numeration systems based on a regular language. Then we will study the relationship between multiplication by a constant and the S-recognizability for languages of the type \( \{a_1 \ldots a_k^n \} \).

Numeration Systems

Definition 1. A numeration system is simply a bijection from \( \mathbb{N} \) onto a language, called the numeration language. In other words, it is a way of representing numbers by words. The entire base-k systems and the Fibonacci system are particular cases of the positional numeration systems which are defined as a strictly increasing sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of integers such that \( \mathcal{U}_0 = 1 \) and \( \frac{\mathcal{U}_{n+1}}{\mathcal{U}_n} \) is bounded. The representation bijection is noted \( r_\mathcal{S} \). Such numeration systems are based on the greedy algorithm.

Numeration Systems on a Regular Language

As we prefer to work with regular numeration languages, the idea of P. Lecomte and M. Rigo was to define new numeration systems which would be based on a regular language. More precisely, instead of searching for a classical numeration system with a regular numeration language, the idea was to start immediately with a regular language and then construct the numeration system on this language.

Let \( \mathcal{L} \) be a regular language. The corresponding abstract numeration system is given by a triple \( (\mathcal{L}, \prec, n) \).

Proposition 3. Let \( \mathcal{S} \) be the numeration system associated with the numeration language based on a strictly increasing sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of integers such that \( \mathcal{U}_0 = 1 \) and consider the genealogical order on \( \mathcal{L} \). Then \( r_\mathcal{S} : \mathbb{N} \rightarrow \mathcal{S} \) is a strictly increasing function.

This proposition shows that positional numeration systems are entirely characterized by the numeration language and the order on the alphabet.

The following definition was introduced by P. Lecomte and M. Rigo in [1].

Definition 4. An abstract numeration system or numeration system on a regular language is a triple \( (\mathcal{S}, \prec, n) \) where \( \mathcal{S} \) is an infinite language over the totally ordered alphabet \( \mathcal{C} \). Enumerating the elements of \( \mathcal{S} \) genealogically with respect to \( \prec \) leads to a one-to-one map \( r_\mathcal{S} \) from \( \mathbb{N} \) onto \( \mathcal{L} \). To any natural number \( n \), we assign the \( n \)-th word of \( \mathcal{S} \) in \( \mathcal{L} \)-representation, while the inverse map \( \mathcal{S} \rightarrow \mathbb{N} \) sends any word belonging to \( \mathcal{S} \) onto its numerical value.

Definition 5. A subset \( \mathcal{S} \) is said to be \( S \)-recognizable if \( r_\mathcal{S}(X) \) is a regular subset of \( \mathcal{L} \).

Multiplication by a Constant and \( S \)-recognizability

Having defined those generalized numeration systems at our disposal, we can consider the effect of addition on \( S \)-recognizable sets. If addition preserves \( S \)-recognizability then multiplication by 2 also preserves the \( S \)-recognizability. So, a natural question about the stability of \( S \)-recognizability arises.

\( \mathcal{S} \) is said to be \( S \)-recognizable if \( r_\mathcal{S}(X) \) is a regular subset of \( \mathcal{L} \).

Theorem 7. Let \( \mathcal{S} = \{a_1 \ldots a_k^n \} \). We have

\[ v \mathcal{S}(a_1 \ldots a_k^n) = \sum_{i=1}^{k} (i-1)(a_i + 1) \]

Example 8. Consider the words of length 5 in the language \( \mathcal{L} = \{0, 1, 2\} \), \( 0 < a < b < c < d < e < f < g < h < i < j \). We have \( v \mathcal{S}(00000) = 5 \) and \( v \mathcal{S}(11111) = 15 \). If we apply the erasing morphism \( \varphi : \{a, b, c\} \rightarrow \{a, b, c\} \) defined by \( \varphi(a) = c, \varphi(b) = b \) and \( \varphi(c) = c \) on the words of length 3, we get

\[ x < b < c < b < c < b < c < b < c < b < c \]

So the ordered list of words of length 3 in the language \( \mathcal{L} = \{0, 1, 2\} \) and the ordered list of words of length 3 in the language \( \mathcal{L} = \{0, 1, 2\} \) are obtained using \( \mathcal{S}(a, b, c) \) and \( \mathcal{S}(a, b, c) \).

Corollary 9. Let \( f \in \mathbb{N} \setminus \{0\} \) for any integer \( n \) can be uniquely written as

\[ n = (n_1)_f + (n_2)_f + \ldots + (n_l)_f \]

and solving the system

\[ n_1 + n_2 + n_3 = 12 \]

\[ n_2 + n_3 = 11 \]

\[ n_3 = 1 \]

we have \( r_\mathcal{S}(278) = \beta_2 \).

Example 10. For \( f = 3 \), one gets for instance

\[ 12345678901234567890 \]

\[ \frac{\mathcal{S}((12345678901234567890) = 112223270} \]

and solving the system

\[ n_1 + n_2 + n_3 = 12 \]

\[ n_2 + n_3 = 11 \]

\[ n_3 = 1 \]

we have \( r_\mathcal{S}(278) = \beta_2 \).

The Case of \( S = (a^*b^*c^*) \)

In their paper [1], P. Lecomte and M. Rigo have studied that question \( Q \) on the particular abstract numeration system \( S = (a^*b^*c^*) \).

Theorem 11. Let \( \phi \) be the system \( (a^*b^*c^*) \).

We have that \( \alpha \) is not \( S \)-recognizable.

Multiplication for Polynomial Languages

Another interesting result of [11] concerns the relationship between the complexity of the numeration language, multiplication by a constant and \( S \)-recognizability.

Theorem 12. Let \( \mathcal{L} \) be a regular language such that its complexity function \( u_\mathcal{L}(n) = n^k + 1 \) for some integer \( k \). Then, there exists a subset \( X \) of \( \mathbb{N}^* \) such that \( r_\mathcal{L}(X) \) is regular and that \( \mathcal{S}(X) \) is not. In other words, multiplication by a constant \( \lambda \) preserves \( \mathcal{L} \)-recognizability only if \( \lambda \) is a form of \( \lambda^n + 1 \) for some \( n \in \mathbb{N}^* \).

The Case of \( S = (a^*b^*c^*) \)

Thanks to the theorem 12, we already know that only the multiplication by constants of the form \( \beta \) with \( \beta \) an integer, can preserve the \( S \)-recognizability. The following result shows that the condition for \( \beta \) is not so restrictive.

Theorem 13. For the abstract numeration system \( S = (a^*b^*c^*) \), if \( \beta \in \mathbb{N} \setminus \{0\} \) is such that \( \beta \neq \pm \lambda (mod 6) \) then the multiplication by \( \beta \) does not preserve the \( S \)-recognizability.

Conjecture for the General Case

Thanks to computer experiments, we conjecture the following result.

Theorem 14. Multiplication by \( \beta \) preserves \( S \)-recognizability for the abstract numeration system \( S = (a_1 \ldots a_k^n) \), if all \( p_i \) prime numbers strictly greater than \( \beta \).

References