Second order pseudo-maximum likelihood estimation and conditional variance misspecification

Bernard Lejeune
University of Liège, ERUDITE and CORE
Boulevard du Rectorat, 7, B33
4000 Liège Belgium
e-mail: B.Lejeune@ulg.ac.be

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1. Introduction

- Ingredients :

  – Second order semi-parametric models :

$$\begin{cases} E(Y_t|X_t) = m_t(X_t, \theta) \\ V(Y_t|X_t) = \Omega_t(X_t, \theta) \end{cases}, \quad t = 1, 2, ...$$

$\Leftrightarrow$

$$\begin{cases} Y_t = m_t(X_t, \theta) + u_t \\ E(u_t|X_t) = 0, \quad V(u_t|X_t) = \Omega_t(X_t, \theta) \end{cases}, \quad t = 1, 2, ...$$

  – Second order PML estimators : a class of estimators which jointly estimates, through the maximization of a pseudo log-likelihood function, the mean and variance parameters of the second order semi-parametric model (ex : gaussian PML)

  – Disruptive element : conditional variance misspecification

- Purpose of the paper : to study the behavior of second order PML estimators under conditional variance misspecification

2. General set-up

- The observed data are a realization of a (unknown) true DGP P_o: $W \equiv \{W_t = (Y_t', Z_t')', \ t = 1, 2, ...\}$
- X_t denotes some subset of the information set (Z_t, Ψ_{t-1}), where $\Psi_{t-1} \equiv (Y_{t-1}, Z_{t-1}, ..., Y_1, Z_1)$

  $\Rightarrow$ Interest lies in explaining Y_t in terms of X_t ($Y_t \subset \mathbb{R}^G$, $X_t \subset \mathbb{R}^{K_t}$)

- Semi-parametric model $\mathcal{S}$ for $E(Y_t|X_t)$ and $V(Y_t|X_t)$

$$\mathcal{S} \equiv \begin{cases} \{m_t(X_t, \theta): \ \theta \in \Theta_\theta\} \\ \{\Omega_t(X_t, \theta): \ \theta \in \Theta_\theta\} \end{cases}, \quad t = 1, 2, ...$$

- Concepts of correct specification: $\mathcal{S}$ is said

  (a) *first order correctly specified* if for some $\theta^o \in \Theta_\theta$

$$m_t(X_t, \theta^o) = E(Y_t|X_t), \qquad t = 1, 2, ...$$

  (b) *second order correctly specified* if for some $\theta^o \in \Theta_\theta$

$$\begin{cases} m_t(X_t, \theta^o) = E(Y_t|X_t) \\ \Omega_t(X_t, \theta^o) = V(Y_t|X_t) \end{cases}, \quad t = 1, 2, ...$$

  (c) *first order dynamically complete* if

$$E(Y_t|X_t) = E(Y_t|X_t, \Phi_{t-1}), \qquad t = 1, 2, ...$$

  (d) *second order dynamically complete* if (c) holds and

$$V(Y_t|X_t) = V(Y_t|X_t, \Phi_{t-1}), \qquad t = 1, 2, ...$$

  where $\Phi_{t-1} \equiv (Y_{t-1}, X_{t-1}, ..., Y_1, X_1)$

3. Second order pseudo-maximum likelihood estimators

- A second order pseudo-maximum likelihood estimator $\hat{\theta}_n$ of $\mathcal{S}$ is defined as a solution of

$$\text{Max}_{\theta \in \Theta_\theta} \ L_n(Y^n, X^n, \theta) \equiv \frac{1}{n} \sum_{t=1}^{n} \ln f_t \left(Y_t, m_t(X_t, \theta), \Omega_t(X_t, \theta) \right)$$

where the p.d.f. $f_t \left(Y, m, \Sigma \right)$ are indexed by their mean m and by their covariance matrix Σ and are "compatible" with $\mathcal{S}$

- $\lambda_t \left(., X_t, \theta \right) = f_t \left(., m_t(X_t, \theta), \Omega_t(X_t, \theta) \right)$ is a conditional density for Y_t given X_t whose the two first conditional moments are by definition $m_t(X_t, \theta)$ and $\Omega_t(X_t, \theta)$. The higher conditional moments depend on f_t.

  $\Rightarrow$ $\hat{\theta}_n$ is just a standard ML estimator of a possibly misspecified parametric model $\mathcal{P}$ implicitly defined by $\mathcal{S}$ and the sequence $\{f_t\}$

$$\mathcal{P} \equiv \left\{ \lambda_t \left(., X_t, \theta \right) = f_t \left(., m_t(X_t, \theta), \Omega_t(X_t, \theta) \right) : \ \theta \in \Theta_\theta \right\}, \ \ t = 1, 2, ...$$

- Further concepts of correct specification: $\mathcal{P}$ is said

  (a) *third order correctly specified* if $\mathcal{S}$ is second order correct and

$$Cov_{\lambda_t^o} \left[\left(\text{vec} \left(Y_t Y_t' \right), Y_t \right) \middle| X_t \right] = Cov \left[\left(\text{vec} \left(Y_t Y_t' \right), Y_t \right) \middle| X_t \right], \quad t = 1, 2, ...$$

  (b) *fourth order correctly specified* if $\mathcal{P}$ is third order correct and

$$V_{\lambda_t^o} \left[\text{vec} \left(Y_t Y_t' \right) \middle| X_t \right] = V \left[\text{vec} \left(Y_t Y_t' \right) \middle| X_t \right], , \quad t = 1, 2, ...$$

  where $Cov_{\lambda_t^o}[.|X_t]$ and $V_{\lambda_t^o}[.|X_t]$ are taken with respect to $\lambda_t \left(Y_t, X_t, \theta^o \right)$

  (c) *correctly specified for the conditional density* if for some $\theta^o \in \Theta_\theta$

$$\lambda_t \left(., X_t, \theta^o \right) = p_t(.|X_t), \qquad t = 1, 2, ...$$

4. Outline of the addressed questions

We have :

- A unknown true DGP P_o
- A second order semi-parametric model $\mathcal{S}$ for $E(Y_t|X_t)$ and $V(Y_t|X_t)$:

$$\mathcal{S} \equiv \begin{cases} \{m_t(X_t, \theta) : \ \theta \in \Theta_\theta\} \\ \{\Omega_t(X_t, \theta) : \ \theta \in \Theta_\theta\} \end{cases} , \ t = 1, 2, ...$$

- A class of second order PML estimators for $\mathcal{S}$:

$$\text{Max}_{\theta \in \Theta_\theta} \ L_n(Y^n, X^n, \theta) \equiv \frac{1}{n} \sum_{t=1}^{n} \ln f_t \left(Y_t, m_t(X_t, \theta), \Omega_t(X_t, \theta)\right)$$

$\Rightarrow$ Questions :

- Under which conditions does $\hat{\theta}_n$ provide a consistent estimator of both mean and variance parameters when $\mathcal{S}$ is second order correctly specified ?
- Under which conditions does $\hat{\theta}_n$ provide a consistent estimator of the mean parameters when $\mathcal{S}$ is first order correctly specified but second order misspecified ?
- Under which conditions does $\hat{\theta}_n$ provide a consistent estimator of both mean and variance parameters when $\mathcal{S}$ is second order correctly specified **and** continue to provide a consistent estimator of the mean parameters when $\mathcal{S}$ first order correctly specified but second order misspecified ?
- What are the limiting distribution properties of such a robust to conditional variance misspecification estimator ?

5. Pseudo-maximum likelihood of order 2 (PML2)

5.1. Quadratic exponential families

- A family of probability measures on $\mathbb{R}^G$ indexed by m and Σ is called *quadratic exponential* if every element of the family has a p.d.f. which may be written as

  $$l(Y, m, \Sigma) = \exp\left(A(m,\Sigma) + B(Y) + C(m,\Sigma)'Y + Y'D(m,\Sigma)Y\right)$$

  where $A(m,\Sigma)$ and $B(Y)$ are scalar, $C(m,\Sigma)$ is a $G \times 1$ vector and $D(m,\Sigma)$ is a $G \times G$ matrix

- Prominent member : the normal density

  $$A(m,\Sigma) = -\tfrac{G}{2}\ln 2\pi - \tfrac{1}{2}\ln|\Sigma| - \tfrac{1}{2}m'\Sigma^{-1}m,$$
  $$B(Y) = 0,\ C(m,\Sigma) = \Sigma^{-1}m \text{ and } D(m,\Sigma) = -\tfrac{1}{2}\Sigma^{-1}$$

- Key properties :

  (a) $\forall\ m,\ m_o, \forall\ \Sigma,\ \Sigma_o$, we have

  $$A(m_o,\Sigma_o) + C(m_o,\Sigma_o)'m_o + \operatorname{tr}\left(D(m_o,\Sigma_o)(\Sigma_o + m_o m_o')\right)$$
  $$\geq A(m,\Sigma) + C(m,\Sigma)'m_o + \operatorname{tr}\left(D(m,\Sigma)(\Sigma_o + m_o m_o')\right)$$

  where the equality holds if and only if $m = m_o$ and $\Sigma = \Sigma_o$

  (b) $\forall\ m_o,\ \forall\ \Sigma,\ \Sigma_o$ such that $\Sigma \neq \Sigma_o$, it may exist m such that $m \neq m_o$ and that we have

  $$A(m_o,\Sigma) + C(m_o,\Sigma)'m_o + \operatorname{tr}\left(D(m_o,\Sigma)(\Sigma_o + m_o m_o')\right)$$
  $$< A(m,\Sigma) + C(m,\Sigma)'m_o + \operatorname{tr}\left(D(m,\Sigma)(\Sigma_o + m_o m_o')\right)$$

5.2. Consistency of PML2 under second order correct specification

Proposition 1 Under usual regularity conditions,

if • $\mathcal{S}$ is second order correctly specified
• $\forall\, t,\; f_t$ belongs to the quadratic exponential family

then $\hat{\theta}_n \rightarrow \theta^o$ as $n \rightarrow \infty$

Proposition 2 $(G = 1)$ Under usual regularity conditions,

if for any P_o,
when $\mathcal{S}$ is second order correctly specified,
we have, $\forall\, n,\; \theta_n^* = \text{Argmax}_{\theta \in \Theta_\theta} E\left(L_n(Y^n, X^n, \theta)\right) = \theta^o$
(thus, $\hat{\theta}_n \rightarrow \theta^o$)

then $\forall\, t,\; f_t$ belongs to the quadratic exponential family

5.3. Inconsistency of PML2 under first order correct specification but second order misspecification

Suppose that $\mathcal{S}$ is such that mean and variance parameters vary independently, i.e., $\theta = \left(\theta_1', \theta_2'\right)' \in \Theta_\theta = \Theta_{\theta_1} \times \Theta_{\theta_2}$ and

$$\mathcal{S} \equiv \left\{ \begin{array}{l} \{m_t(X_t, \theta) = m_t(X_t, \theta_1) : \;\theta_1 \in \Theta_{\theta_1}\} \\ \{\Omega_t(X_t, \theta) = \Omega_t(X_t, \theta_2) : \;\theta_2 \in \Theta_{\theta_2}\} \end{array} \right. , \; t = 1, 2, ...$$

Proposition 3 Under usual regularity conditions,

if • mean and variance parameters vary independently
• $\mathcal{S}$ is first order correctly specified
but second order misspecified
• $\forall\, t,\; f_t$ belongs to the quadratic exponential family

then we may have $\forall\, n,\; \theta_n^* = \text{Argmax}_{\theta \in \Theta_\theta} E\left(L_n(Y^n, X^n, \theta)\right) \neq \left(\theta_1^{o\prime}, \theta_{2_n}^{*\prime}\right)'$
(thus, we may have $\hat{\theta}_{1_n} \nrightarrow \theta_1^o$)

6. Robust pseudo-maximum likelihood of order 2 (R1PML2 and RPML2)

6.1. Restricted generalized linear exponential families

- A family of probability measures on $\mathbb{R}^G$ indexed by m and Σ is called *restricted generalized linear exponential* if every element of the family has a p.d.f. which may be written as

$$l(Y, m, \Sigma) = \exp\left(A(m, \Sigma) + B(\Sigma, Y) + C(m, \Sigma)'Y\right)$$

  where $A(m, \Sigma)$ and $B(\Sigma, Y)$ are scalar, $C(m, \Sigma)$ is a $G \times 1$ vector
- The normal density belongs to the family
- This family does not contain the quadratic exponential family

$$l(Y, m, \Sigma) = \exp\left(A(m, \Sigma) + B(Y) + C(m, \Sigma)'Y + Y'D(m, \Sigma)Y\right)$$

  but is a special case of the generalized linear exponential family

$$l(Y, m, \eta) = \exp(A(m, \eta) + B(\eta, Y) + C(m, \eta)'Y)$$

- Key property :

  $\forall\ m,\ m_o, \forall\ \Sigma$, we have

$$\begin{aligned} &A(m_o, \Sigma) + C(m_o, \Sigma)'m_o \\ &\geq A(m, \Sigma) + C(m, \Sigma)'m_o \end{aligned}$$

  where the equality holds, $\forall\ \Sigma$, if and only if $m = m_o$

6.2. Consistency of R1PML2 under first order correct specification but possible second order misspecification

Suppose that $\mathcal{S}$ is such that $\theta = \left(\theta_1', \theta_2'\right)' \in \Theta_\theta = \Theta_{\theta_1} \times \Theta_{\theta_2}$ and

$$\mathcal{S} \equiv \left\{ \begin{array}{l} \{m_t(X_t, \theta) = m_t(X_t, \theta_1) : \ \theta_1 \in \Theta_{\theta_1}\} \\ \{\Omega_t(X_t, \theta) = \Omega_t(X_t, \theta_2) : \ \theta_2 \in \Theta_{\theta_2}\} \end{array} \right. , \ t = 1, 2, ...$$

Proposition 5 Under usual regularity conditions,

if • mean and variance parameters vary independently
• $\mathcal{S}$ is first order correctly specified
• $\forall\, t,\ f_t$ belongs to the restricted generalized linear exponential family

then $\hat{\theta}_{1_n} \rightarrow \theta_1^o$ and $\hat{\theta}_{2_n} - \theta_{2_n}^* \rightarrow 0$ as $n \rightarrow \infty$

where $\theta_{2_n}^* = \text{Argmax}_{\theta_2 \in \Theta_{\theta_2}} \frac{1}{n} \sum_{t=1}^{n} E\left[\ln f_t\left(Y_t, m_t(X_t, \theta_1^o), \Omega_t(X_t, \theta_2)\right)\right]$

Suppose that $\mathcal{S}$ is such that $\theta = \left(\theta_1', \theta_2'\right)' \in \Theta_\theta = \Theta_{\theta_1} \times \Theta_{\theta_2}$ and

$$\mathcal{S} \equiv \left\{ \begin{array}{l} \{m_t(X_t, \theta) = m_t(X_t, \theta_1) : \ \theta_1 \in \Theta_{\theta_1}\} \\ \{\Omega_t(X_t, \theta) = \Omega_t(X_t, \theta_1, \theta_2) : \ \theta_1 \in \Theta_{\theta_1}, \ \theta_2 \in \Theta_{\theta_2}\} \end{array} \right. , \ t = 1, 2, ...$$

Proposition 6 Under usual regularity conditions and $\mathcal{S}$ as given above,

if for any P_o,
when $\mathcal{S}$ is first order correctly specified
we have, $\forall\, n,\ \theta_n^* = \text{Argmax}_{\theta \in \Theta_\theta} E\left(L_n(Y^n, X^n, \theta_1, \theta_2)\right) = (\theta_1^{o\prime}, \theta_{2_n}^{*\prime})'$
(thus, $\hat{\theta}_{1_n} \rightarrow \theta_1^o$ and $\hat{\theta}_{2_n} - \theta_{2_n}^* \rightarrow 0$)

then • $\forall\, t,\ f_t$ belongs to the restricted generalized linear exponential family
• $\forall\, t,\ \Omega_t(X_t, \theta_1, \theta_2)$ does not depend on θ_1.
(thus, mean and variance parameters vary independently)

6.3. Restricted quadratic exponential families

- A family of probability measures on $\mathbb{R}^G$ indexed by m and Σ is called *restricted quadratic exponential* if every element of the family has a p.d.f. which may be written as

$$l(Y, m, \Sigma) = \exp\left(A(m, \Sigma) + B(Y) + C(m, \Sigma)'Y + Y'D(\Sigma)Y\right)$$

where $A(m, \Sigma)$ and $B(Y)$ are scalar, $C(m, \Sigma)$ is a $G \times 1$ vector and $D(\Sigma)$ is a $G \times G$ matrix

- The normal density belongs to the family.
- This family is a special case of the quadratic exponential family

$$l(Y, m, \Sigma) = \exp\left(A(m, \Sigma) + B(Y) + C(m, \Sigma)'Y + Y'D(m, \Sigma)Y\right)$$

and of the restricted generalized linear exponential family

$$l(Y, m, \Sigma) = \exp\left(A(m, \Sigma) + B(\Sigma, Y) + C(m, \Sigma)'Y\right)$$

- Key properties:

(a) $\forall\ m,\ m_o, \forall\ \Sigma,\ \Sigma_o$, we have

$$\begin{aligned} &A(m_o, \Sigma_o) + C(m_o, \Sigma_o)'m_o + \operatorname{tr}\left(D(\Sigma_o)(\Sigma_o + m_o m_o')\right) \\ &\geq A(m, \Sigma) + C(m, \Sigma)'m_o + \operatorname{tr}\left(D(\Sigma)(\Sigma_o + m_o m_o')\right) \end{aligned}$$

where the equality holds if and only if $m = m_o$ and $\Sigma = \Sigma_o$

(b) $\forall\ m,\ m_o, \forall\ \Sigma$, we have

$$\begin{aligned} &A(m_o, \Sigma) + C(m_o, \Sigma)'m_o \\ &\geq A(m, \Sigma) + C(m, \Sigma)'m_o \end{aligned}$$

where the equality holds, $\forall\ \Sigma$, if and only if $m = m_o$

6.4. Consistency of RPML2 under first order correct specification but possible second order misspecification

Suppose that $\mathcal{S}$ is such that $\theta = \left(\theta_1', \theta_2'\right)' \in \Theta_\theta = \Theta_{\theta_1} \times \Theta_{\theta_2}$ and

$$\mathcal{S} \equiv \begin{cases} \{m_t(X_t, \theta) = m_t(X_t, \theta_1) : \ \theta_1 \in \Theta_{\theta_1}\} \\ \{\Omega_t(X_t, \theta) = \Omega_t(X_t, \theta_2) : \ \theta_2 \in \Theta_{\theta_2}\} \end{cases}, \ t = 1, 2, ...$$

Proposition 7 Under usual regularity conditions,

if
- mean and variance parameters vary independently
- $\mathcal{S}$ is first order correctly specified
- $\forall\, t,\ f_t$ belongs to the restricted quadratic exponential family

then $\hat{\theta}_{1_n} \to \theta_1^o$ and $\hat{\theta}_{2_n} - \theta_{2_n}^* \to 0$ as $n \to \infty$

if, in addition, S is second order correctly specified

then $\hat{\theta}_{1_n} \to \theta_1^o$ and $\hat{\theta}_{2_n} \to \theta_2^o$ as $n \to \infty$

Suppose that $\mathcal{S}$ is such that $\theta = \left(\theta_1', \theta_2'\right)' \in \Theta_\theta = \Theta_{\theta_1} \times \Theta_{\theta_2}$ and

$$\mathcal{S} \equiv \begin{cases} \{m_t(X_t, \theta) = m_t(X_t, \theta_1) : \ \theta_1 \in \Theta_{\theta_1}\} \\ \{\Omega_t(X_t, \theta) = \Omega_t(X_t, \theta_1, \theta_2) : \ \theta_1 \in \Theta_{\theta_1}, \ \theta_2 \in \Theta_{\theta_2}\} \end{cases}, \ t = 1, 2, ...$$

Proposition 8 $(G = 1)$ Under usual regularity conditions and $\mathcal{S}$ as given above,

if for any P_o,
when $\mathcal{S}$ is first order correctly specified
we have, $\forall\, n,\ \theta_n^* = \text{Argmax}_{\theta\in\Theta_\theta} E\left(L_n(Y^n, X^n, \theta_1, \theta_2)\right) = (\theta_1^{o\prime}, \theta_{2_n}^{*\prime})'$
(thus, $\hat{\theta}_{1_n} \to \theta_1^o$ and $\hat{\theta}_{2_n} - \theta_{2_n}^* \to 0$)
and, when $\mathcal{S}$ is in addition second order correctly specified
we have, $\forall\, n,\ \theta_n^* = \text{Argmax}_{\theta\in\Theta_\theta} E\left(L_n(Y^n, X^n, \theta_1, \theta_2)\right) = (\theta_1^{o\prime}, \theta_2^{o\prime})'$
(thus, $\hat{\theta}_{1_n} \to \theta_1^o$ and $\hat{\theta}_{2_n} \to \theta_2^o$)

then
- $\forall\, t,\ f_t$ belongs to the restricted quadratic exponential family
- $\forall\, t,\ \Omega_t(X_t, \theta_1, \theta_2)$ does not depend on θ_1.
  (thus, mean and variance parameters vary independently)

7. Limiting distribution of RPML2

Proposition 9 Under usual regularity conditions,

$$\hat{\theta}_n - \theta_n^* \to 0 \text{ as } n \to \infty$$

and

$$\sqrt{n}\left(\hat{\theta}_n - \theta_n^*\right) \approx N(0, C_n^*), \quad C_n^* = A_n^{*-1} B_n^* A_n^{*-1}$$

where $A_n^* = E\left(\frac{\partial^2}{\partial\theta\partial\theta'} L_n(Y^n, X^n, \theta_n^*)\right)$ and $B_n^* = V\left(n^{1/2}\frac{\partial}{\partial\theta} L_n(Y^n, X^n, \theta_n^*)\right)$

Prominent results (Prop. 10-12) Under usual regularity conditions,

if
- $\forall\, t, f_t$ belongs to the restricted quadratic exponential family
- mean and variance parameters vary independently
- $\mathcal{S}$ is first order correctly specified

then $\forall\, n, \theta_n^* = \left(\theta_1^{o\prime}, \theta_{2_n}^{*\prime}\right)'$, $A_{n_{12}}^* = A_{n_{21}}^{*\prime} = E\left[\frac{\partial^2}{\partial\theta_1\partial\theta_2'} L_n(Y^n, X^n, \theta_n^*)\right] = 0$

such that $C_n^* = \begin{bmatrix} C_{n_{11}}^* & C_{n_{12}}^* \\ C_{n_{12}}^{*\prime} & C_{n_{22}}^* \end{bmatrix} = \begin{bmatrix} A_{n_{11}}^{*^{-1}} B_{n_{11}}^* A_{n_{11}}^{*^{-1}} & A_{n_{11}}^{*^{-1}} B_{n_{12}}^* A_{n_{22}}^{*^{-1}} \\ A_{n_{22}}^{*^{-1}} B_{n_{12}}^{*\prime} A_{n_{11}}^{*^{-1}} & A_{n_{22}}^{*^{-1}} B_{n_{22}}^* A_{n_{22}}^{*^{-1}} \end{bmatrix}$

where $B_{n_{ij}}^* = \frac{1}{n}\sum_{t=1}^{n} E\left[s_t^{i*} s_t^{j*\prime}\right] + \frac{1}{n}\sum_{\tau=1}^{n-1}\sum_{t=\tau+1}^{n}\left(E\left[s_t^{i*} s_{t-\tau}^{j*\prime}\right] + E\left[s_{t-\tau}^{i*} s_t^{j*\prime}\right]\right)$

if, in addition, S is first order dynamically complete

then $C_{n_{11}}^* = \overline{C}_{n_{11}}^* = A_{n_{11}}^{*^{-1}} \overline{B}_{n_{11}}^* A_{n_{11}}^{*^{-1}}$, where $\overline{B}_{n_{11}}^* = \frac{1}{n}\sum_{t=1}^{n} E\left[s_t^{1*} s_t^{1*\prime}\right]$

if, in addition, S is second order correctly specified

then $\forall\, n, \theta_n^* = \left(\theta_1^{o\prime}, \theta_2^{o\prime}\right)'$, $C_{n_{11}}^* = \overline{\overline{C}}_{n_{11}}^o = -A_{n_{11}}^{o^{-1}}$

and we have $\overline{C}_{n_{11}}^* - \overline{\overline{C}}_{n_{11}}^o \gg 0$

i.e, $\overline{\overline{C}}_{n_{11}}^o$ is the minimum asymptotic covariance matrix of a RPML2 mean parameters estimator of a semi-parametric model $\mathcal{S}$ first order correctly specified and first order dynamically complete

if, in addition, S is second order dynamically complete

then $C^*_{n_{12}} = \overline{C}^o_{n_{12}} = A^{o^{-1}}_{n_{11}} \overline{B}^o_{n_{12}} A^{o^{-1}}_{n_{22}}$, where $\overline{B}^o_{n_{12}} = \frac{1}{n} \sum_{t=1}^{n} E\left[s^{1o}_t s^{2o\prime}_t\right]$

$C^*_{n_{22}} = \overline{C}^o_{n_{22}} = A^{o^{-1}}_{n_{22}} \overline{B}^o_{n_{22}} A^{o^{-1}}_{n_{22}}$, where $\overline{B}^o_{n_{22}} = \frac{1}{n} \sum_{t=1}^{n} E\left[s^{2o}_t s^{2o\prime}_t\right]$

if, in addition, $\mathcal{P}$ is third order correctly specified

then $C^*_{n_{12}} = \overline{\overline{C}}^o_{n_{12}} = 0$

if, in addition, $\mathcal{P}$ is fourth order correctly specified

then $C^*_{n_{22}} = \overline{\overline{C}}^o_{n_{22}} = -A^{o^{-1}}_{n_{22}}$

and we have $\overline{C}^o_n - \overline{\overline{C}}^o_n \gg 0$

i.e., $\overline{\overline{C}}^o_n$ is the minimum asymptotic covariance matrix of a RPML2 estimator of a semi-parametric model $\mathcal{S}$ second order correctly specified and second order dynamically complete

Proposition 13 Under usual regularity conditions,

if
- $\forall\, t, f_t$ belongs to the restricted quadratic exponential family
- mean and variance parameters vary independently
- $\mathcal{S}$ is first order correctly specified
- the observations are independent across t

then $\forall\, n, \theta^*_n = \left(\theta^{o\prime}_1, \theta^{*\prime}_{2_n}\right)'$

$C^*_{n_{11}} = \overline{C}^*_{n_{11}} = A^{*^{-1}}_{n_{11}} \overline{B}^*_{n_{11}} A^{*^{-1}}_{n_{11}}$, where $\overline{B}^*_{n_{11}} = \frac{1}{n} \sum_{t=1}^{n} E\left[s^{1*}_t s^{1*\prime}_t\right]$

$C^*_{n_{12}} = \overline{C}^*_{n_{12}} = A^{*^{-1}}_{n_{11}} \overline{B}^*_{n_{12}} A^{*^{-1}}_{n_{22}}$, where $\overline{B}^*_{n_{12}} = \frac{1}{n} \sum_{t=1}^{n} E\left[s^{1*}_t s^{2*\prime}_t\right]$

$C^*_{n_{22}} \ll \overline{Q}^*_{n_{22}} = A^{*^{-1}}_{n_{22}} \overline{B}^*_{n_{22}} A^{*^{-1}}_{n_{22}}$, where $\overline{B}^*_{n_{22}} = \frac{1}{n} \sum_{t=1}^{n} E\left[s^{2*}_t s^{2*\prime}_t\right]$