Coloring of the infinite grid

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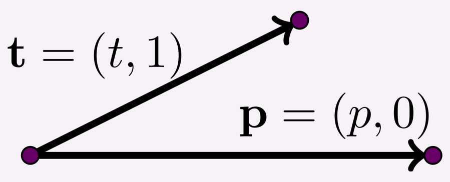
Coloring problem

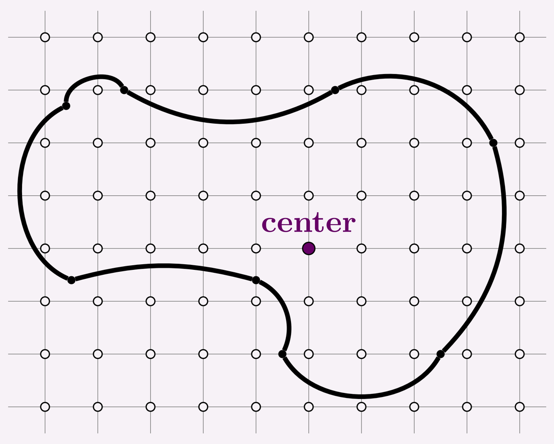
Is it possible to color the infinite grid $\mathbb{Z}^2$ with $\{\bullet, \circ\}$ such that

the coloring satisfies the **translations**



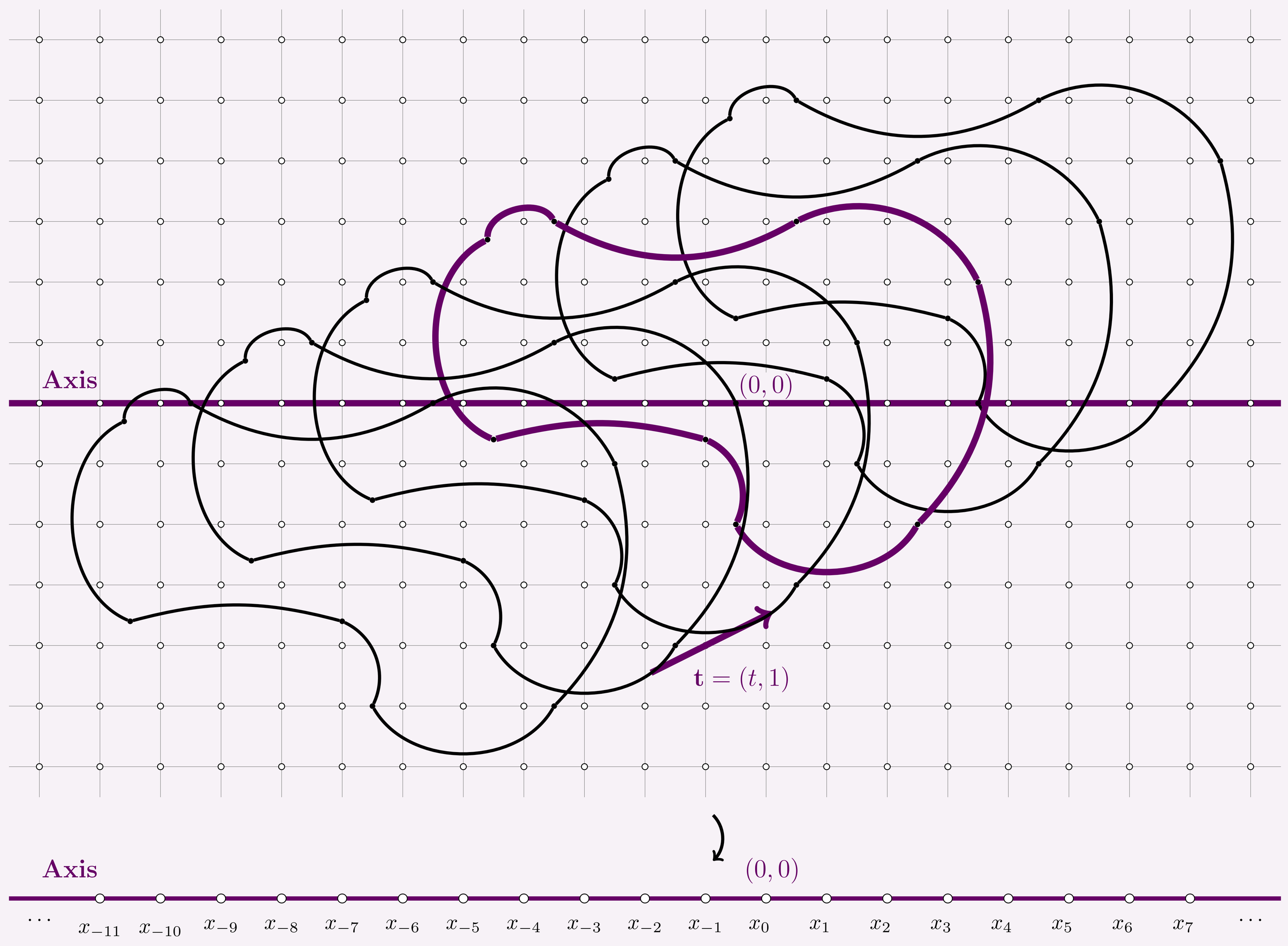
and, $\forall \mathbf{x} \in \mathbb{Z}^2$, a **frame** centered on $\mathbf{x}$ contains

- α black vertices if its center is black,
- β black vertices if its center is white ?



1. Projection

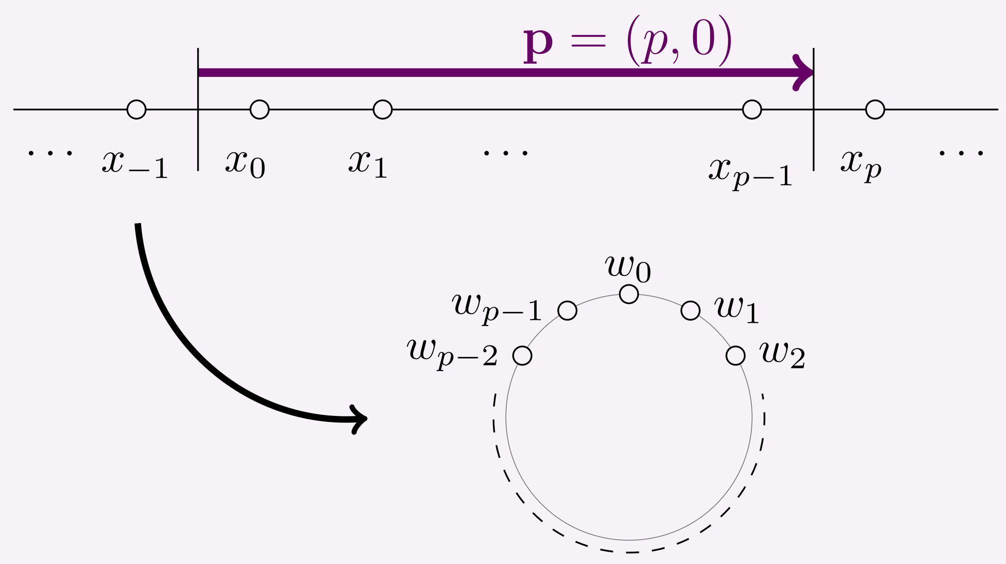
Using the translation $\mathbf{t} = (t, 1)$, we can project a frame on the **axis** $\{(i, 0) \mid i \in \mathbb{Z}\}$. Let $Trans$ be the set containing a frame and all its translated by multiples of $\mathbf{t}$. We have



$$\text{with } x_i = \#\{\mathcal{T} \in Trans \mid (i, 0) \in \mathcal{T}\} < \infty \qquad \forall i \in \mathbb{Z}.$$

2. Folding

Using the translation $\mathbf{p} = (p, 0)$, we can fold an axis of projection on a **cycle $\mathcal{C}_p$ with p weighted vertices**. We have



$$\text{with } w_i = \sum_{k \in \mathbb{Z}} x_{i+kp} < \infty \qquad \forall i \in \{0, \dots, p-1\}.$$

3. Equivalent problem

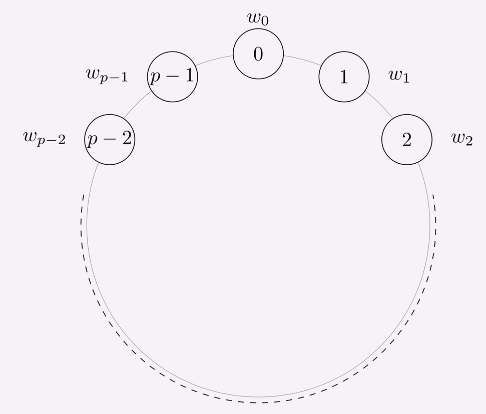
Let $\mathcal{C}_p$ be the cycle corresponding to the frame and translations given.

Is it possible to find a coloring φ of $\mathcal{C}_p$,

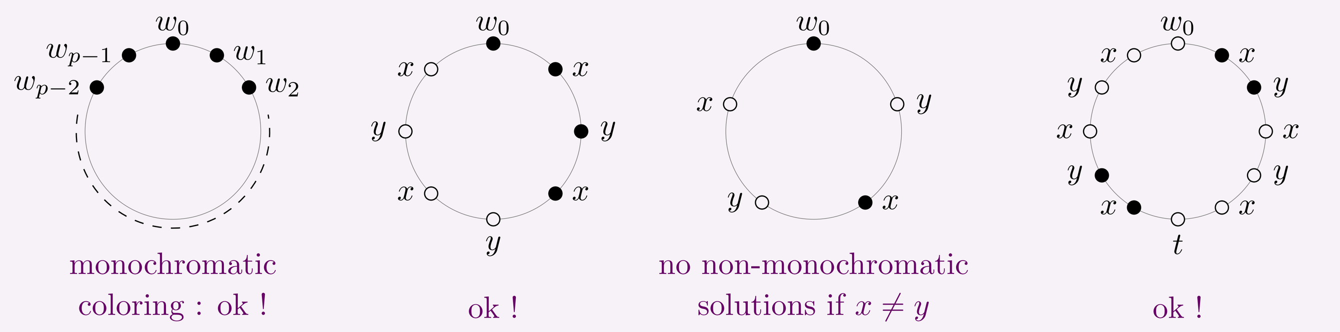
$$\varphi : \{0, \dots, p-1\} \to \{\bullet, \circ\}$$

such that, for all rotations ψ of φ, we have

- $\alpha = \sum_{i \in \psi^{-1}(\bullet)} w_i$ if $\psi(0) = \bullet$,
- $\beta = \sum_{i \in \psi^{-1}(\bullet)} w_i$ if $\psi(0) = \circ$?



Examples :

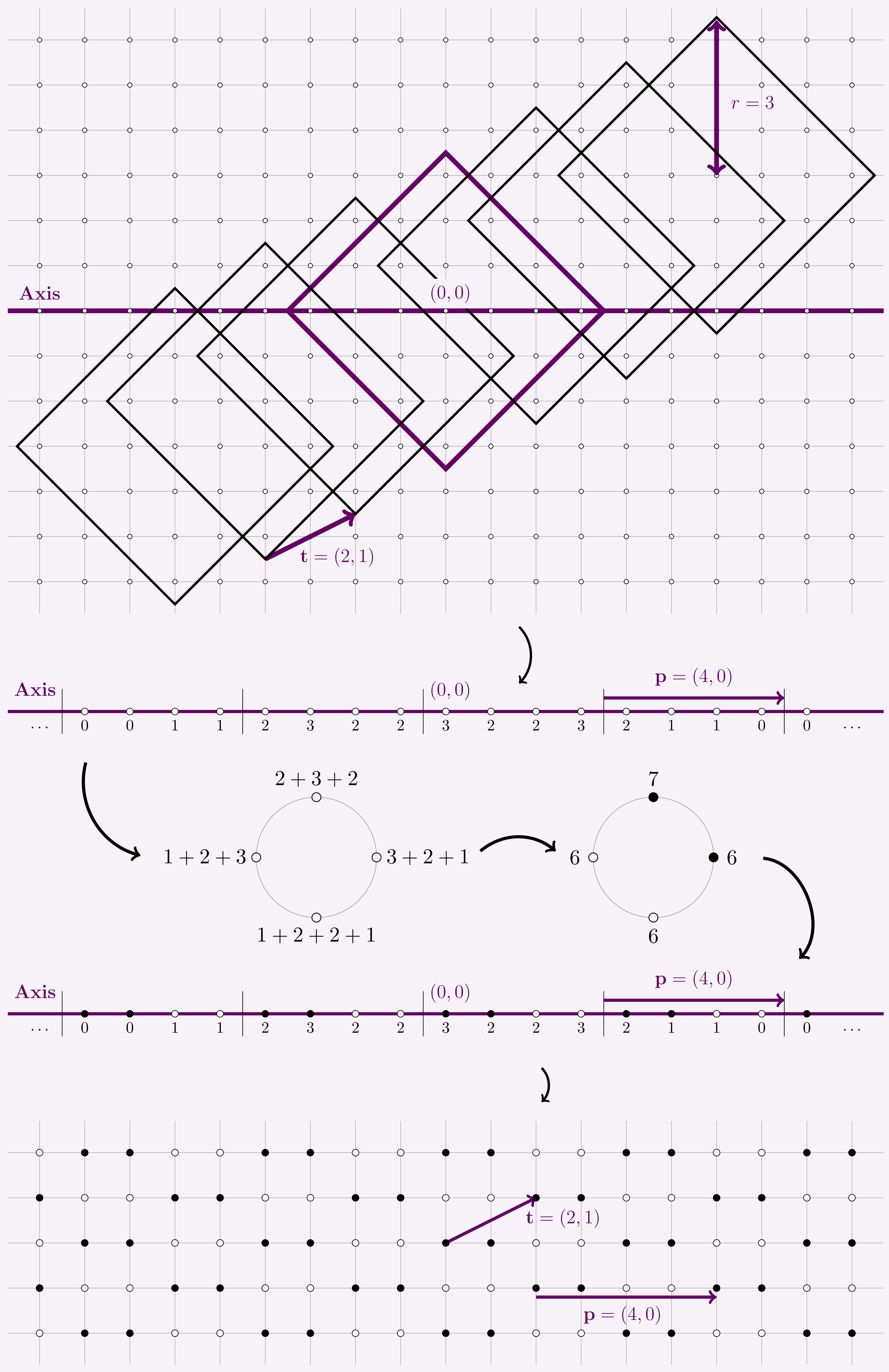


4. Application to (r, a, b)-codes

An (r, a, b)-code of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that

$$\begin{aligned} \forall \mathbf{x} \in S, \quad & a = \#\{B_r(\mathbf{y}) \mid \mathbf{x} \in B_r(\mathbf{y}),\ \mathbf{y} \in S\}, \\ \forall \mathbf{x} \in V \setminus S, \quad & b = \#\{B_r(\mathbf{y}) \mid \mathbf{x} \in B_r(\mathbf{y}),\ \mathbf{y} \in S\}. \end{aligned}$$

Consider the coloring problem of $\mathbb{Z}^2$ with $B_r(\mathbf{x})$, $\mathbf{x} \in \mathbb{Z}^2$, as frames.



$$\Rightarrow \exists \text{ a } (3, 13, 12)\text{-code of } \mathbb{Z}^2$$

References :

[1] M. A. Axenovich, On multiple coverings of the infinite rectangular grid with balls of constant radius, *Discrete Mathematics*, 268 (2003), 31–48.

[2] S. A. Puzynina, Perfect colorings of radius $r > 1$ of the infinite rectangular grid, *Siberian Electronic Mathematical Reports*, 5 (2008), 283–292.

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