Cutting planes from lattice-point-free polyhedra

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Joint work with K. Andersen (Copenhagen), R. Weismantel (Magdeburg)

Outline

- Split cuts
- Lattice-point-free polyhedra
- High-dimensional split cuts and the d-dimensional split closure
- Some steps to prove that the d-dimensional split closure is a polyhedron
- Previous work on the 2-dimensional split cuts
- Conclusion

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Split cuts

The algebra

Based on a disjunction

$$\pi^T x \leq \pi_0 \quad \text{or} \quad \pi^T x \geq \pi_0 + 1$$

is valid for $x \in \mathbb{Z}^n$ when π, π_0 are integer.

The geometry

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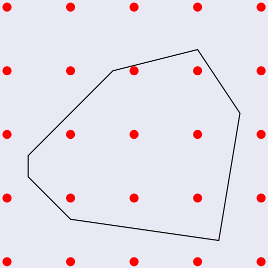

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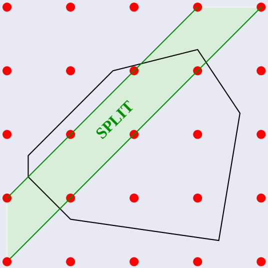

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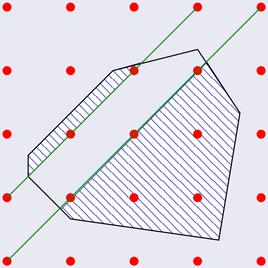

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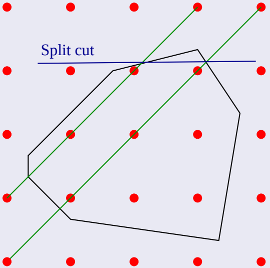

The split closure

Consider a polyhedron $P \subseteq \mathbb{R}^n$, the intersection of all split cuts of P is called the (first) split closure of P, denoted by $\mathrm{SC}(P)$.

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- Cook, Kannan, Schrijver [1990] The split closure is a polyhedron
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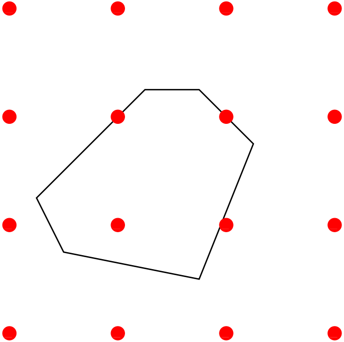

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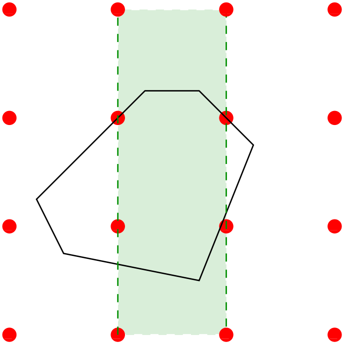

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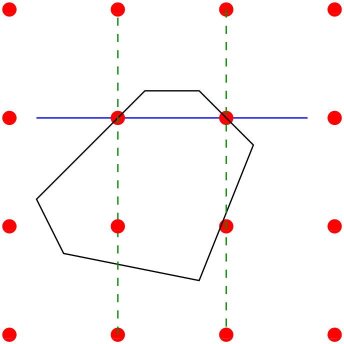

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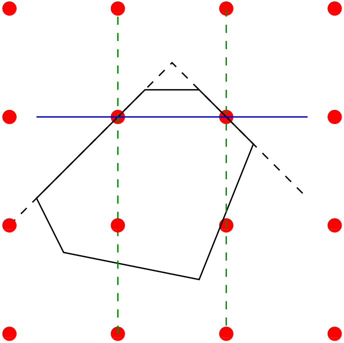

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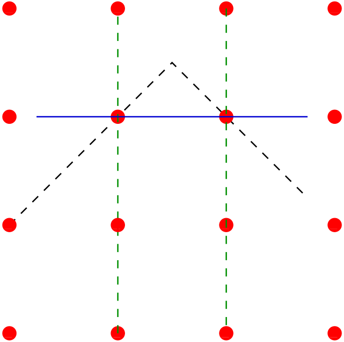

Lattice-point-free polyhedra

A polyhedron P is lattice-point-free when there is no integer point in its interior.

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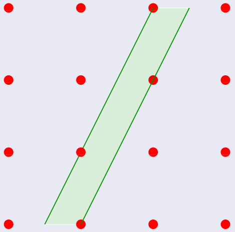

A basic split set in $\mathbb{R}^2$ is a lattice-point-free polyhedron

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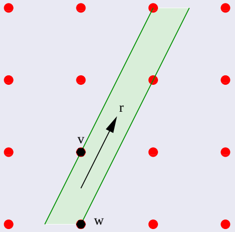

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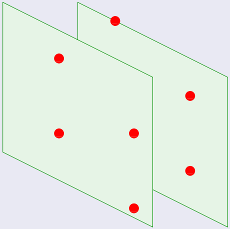

A basic split set in $\mathbb{R}^3$ is a lattice-point-free polyhedron

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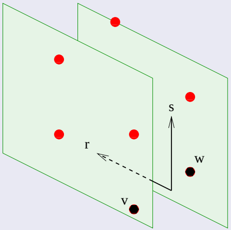

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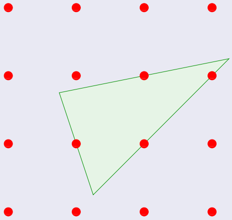

A triangle in $\mathbb{R}^2$ can be lattice-point-free

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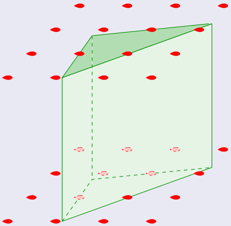

A triangle in $\mathbb{R}^2$ can be lattice-point-free
It can be lifted to a lattice-point-free polyhedron in $\mathbb{R}^3$

Towards high-dimensional split cuts

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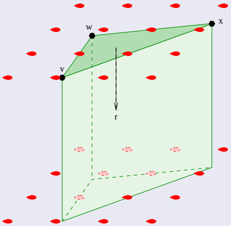

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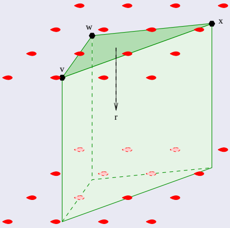

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Definition of the split dimension

A lattice-point-free polyhedron $P \subseteq \mathbb{R}^n$ can be written as

$$P = \text{conv}\{v^1, \ldots, v^p\} + \text{cone}\{w^1, \ldots, w^q\} + \text{span}\{r^1, \ldots, r^{n-d}\}.$$

The split-dimension of P is d.

Using the lattice-point-free polyhedra to generate cuts

The algebra

Let $P \subseteq \mathbb{R}^{n+m}$ be a polyhedron and $L \subseteq \mathbb{R}^n$ be a lattice-point-free polyhedron. We define a set of cuts, valid for $\{(x, y) \in \mathbb{R}^{n+m} | x \in P \cap \mathbb{Z}^n\}$ as

$$\text{cuts}_P(L) = \text{conv}\{(x, y) \in \mathbb{R}^{n+m} | (x, y) \in P \text{ and } x \notin \text{int}(L)\}.$$

The geometry

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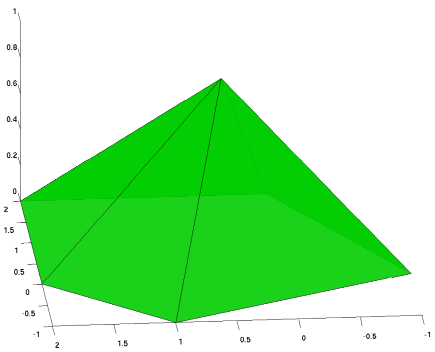

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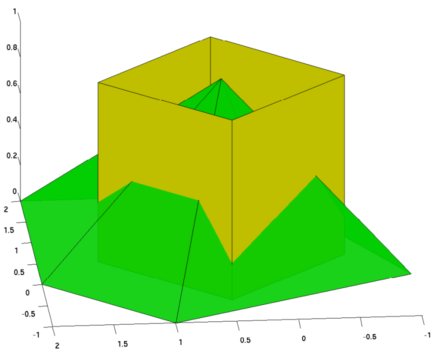

Back

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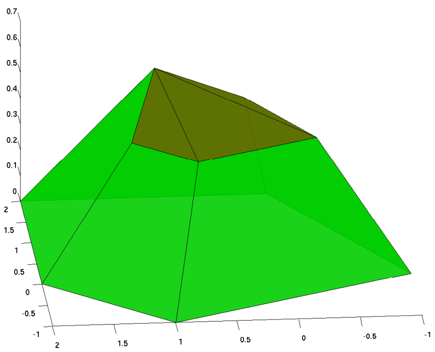

Definition

The *d-dimensional split closure* of P is the set of points in the intersection of all high-dimensional split cuts obtained from P with a *split-dimension less or equal to d*.

Steps to prove that the d-dimensional split closure is a polyhedron

- Every d-dimensional split cut can be obtained from a simpler set (but not a basis)
- Some d-dimensional splits generate cuts that are dominating compared to other d-dimensional splits
- Mapping of integer points and edges to dominating splits
- Finiteness on the number of coefficients of a given variable for fixed integer points and edges

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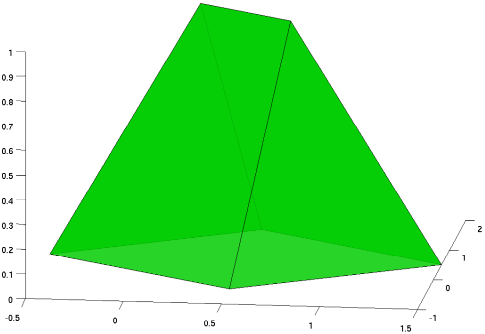
Some d-dimensional split cuts do not come from a basis



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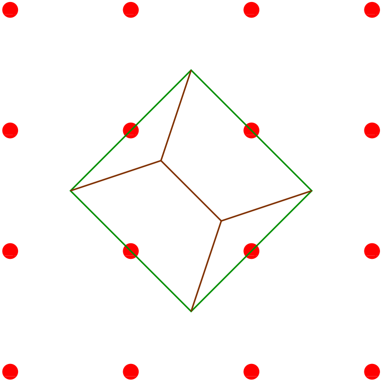

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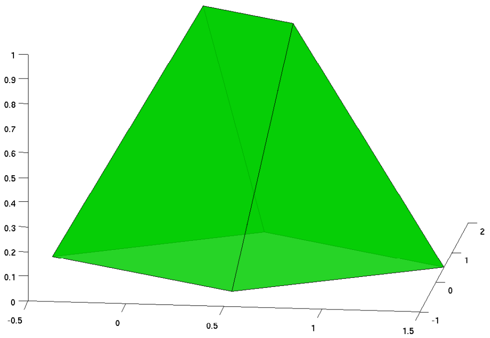

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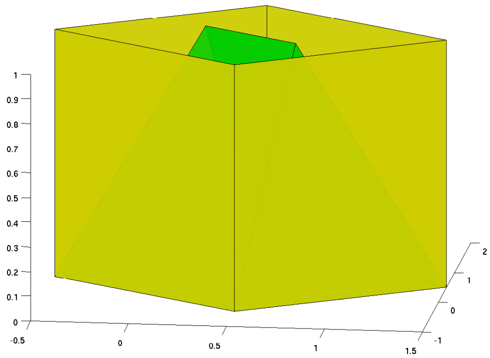
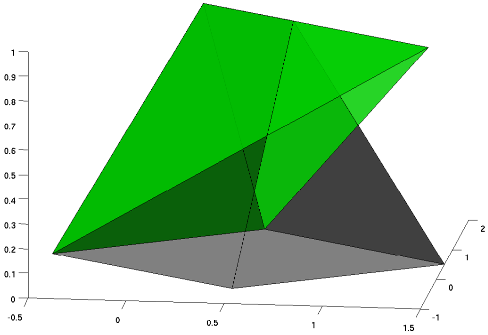


Natural 2-dimensional split cut : $x_3 \leq 0$



The first basis is a feasible basis (the other feasible basis is symmetric).

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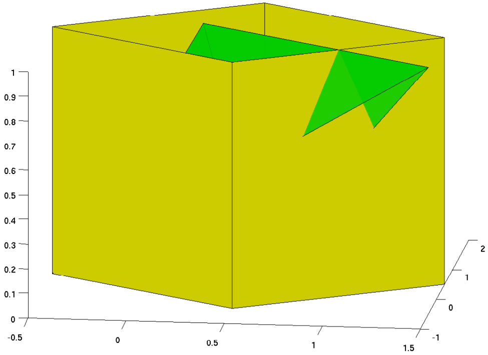

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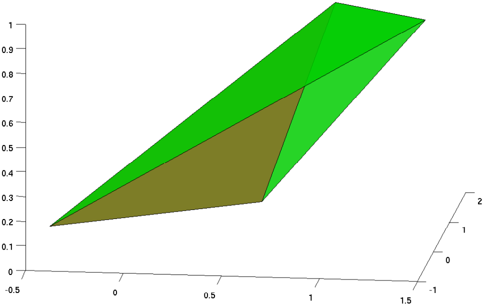

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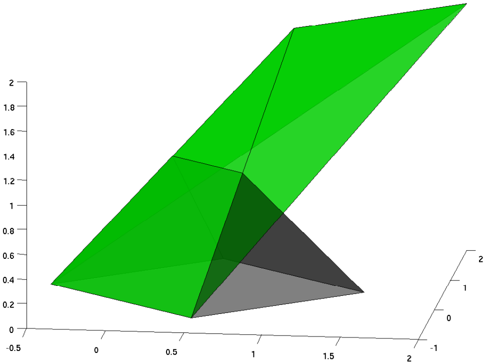
Cut generated : $-x_1 + x_2 + 2x_3 \leq 1$



The second basis is infeasible (and there is a symmetric one).

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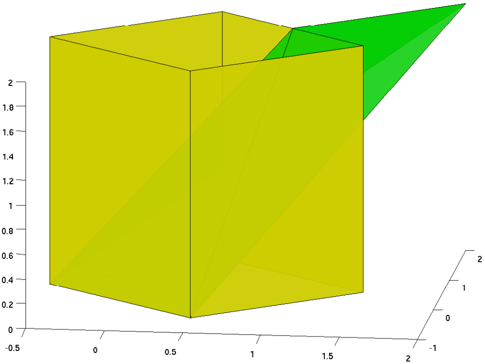

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No cut generated !