Split Rank of Triangle and Quadrilateral Inequalities

Quentin Louveaux

Université de Liège - Montefiore Institute

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Joint work with Santanu Dey (CORE)
Outline

- Cuts from two rows of the simplex tableau
- The different cases to consider
- Split cuts and split ranks
- Finiteness proofs for the triangles
- The ideas for the quadrilaterals
- Conclusion
Cuts from two rows of the simplex tableau

Consider a mixed-integer program

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \mathbb{Z}_{+}^{n_1} \times \mathbb{R}_{+}^{n_2}.
\end{align*}
\]

We consider the problem of finding valid inequalities cutting off the linear relaxation optimum.

We consider the simplex tableau

\[
\begin{align*}
  x_1 - \bar{a}_{11}s_1 - \cdots - \bar{a}_{1n}s_n &= \bar{b}_1 \\
  \vdots & \quad \vdots \\
  x_m - \bar{a}_{m1}s_1 - \cdots - \bar{a}_{mn}s_n &= \bar{b}_m.
\end{align*}
\]

- Select two rows
- Relax the integrality requirements of the non-basic variables
- Relax the nonnegativity requirements of the basic variables but keeping integrality
Cuts from two rows of the simplex tableau

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\vdots & \,
\end{array}
\quad \begin{array}{c}
\vdots \\
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The 2 row-model

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} r_{1j} \\ r_{2j} \end{pmatrix} s_j, \quad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+
\]

Model studied in [Andersen, Louveaux, Weismantel, Wolsey, IPCO2007] (for the finite case) and [Cornuéjols, Margot, 2009] (for the infinite case).
The 2 row-model

The model

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\]

The geometry

\[
\begin{pmatrix}
  x_1 \\
  x_2 
\end{pmatrix} = \begin{pmatrix}
  1/4 \\
  1/2 
\end{pmatrix} + \begin{pmatrix}
  2 \\
  1 
\end{pmatrix} s_1 + \begin{pmatrix}
  1 \\
  1 
\end{pmatrix} s_2 + \begin{pmatrix}
  -3 \\
  2 
\end{pmatrix} s_3 + \begin{pmatrix}
  0 \\
  -1 
\end{pmatrix} s_4 + \begin{pmatrix}
  1 \\
  -2 
\end{pmatrix} s_5
\]
The geometry

The projection picture

\[ 2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1 \]

- We project the \( n + 2 \)-dim space onto the \( x \)-space
- The facet is represented by a polygon \( L_\alpha \)
- There is no integer point in the interior of \( L_\alpha \)
- The coefficients are a ratio of distances on the figure \( \alpha_1, \alpha_3 \)
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  \( \alpha_1, \alpha_3 \)
Classification of all possible facet-defining inequalities

**Theorem**: All facets are projected to triangles and quadrilaterals [Andersen et al 2007].

Split Cut

Triangle Cut

Quadrilateral Cut

Cook–Kannan–Schrijver

Disection Triangle

Disection Quadrilateral
The split rank question

- **Split cut**: applying a *disjunction* $\pi^T x \leq \pi_0 \lor \pi^T x \geq \pi_0 + 1$ to a polyhedron $P$

  \[
  x = f + RS \\
  s_1 \geq 0 \\
  \vdots \\
  s_n \geq 0 \\
  \pi^T x \leq \pi_0
  \]

- The first split closure $P^1$ of $P$ is what you obtain after having applied all possible split disjunctions $\pi$.

- The **split rank** of a valid inequality is the minimum $i$ such that the inequality is valid for $P^i$.

- Most inequalities used in commercial softwares are **split cuts**.

- Question: what is the split rank of the 2 row-inequalities?

- In how many rounds of **split cuts only** can we generate the inequalities?

- The Cook-Kannan-Schrijver has infinite rank and we prove that the other triangles have finite rank.
The split rank question

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Useful properties of the split rank

- The split rank is invariant up to integer translation and unimodular transformation.

- (Lifting) Consider a triangle (or quadrilateral) inequality for a 3-variable problem. If we keep the same shape of the polygon and consider an $n$-variable problem, the split rank does not increase.

It allows us to work with 3 variables only when trying to find the split rank of triangles.

- (Projection) Let $\sum_{i=1}^{n} \alpha_i s_i \geq 1$ be an inequality with split rank $\eta$. Then the projected inequality $\sum_{i=1}^{n-1} \alpha_i s_i \geq 1$ has a split rank of at most $\eta$ for the projected problem.
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The triangle case

Several cases to consider, after suitable unimodular transformation

An illustration of the proof in this talk
The triangle case

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Idea of the proof of upper bounds

- We prove an **upper bound** on the split rank.
- **Procedure**: We apply a sequence of two split disjunctions. Successively: \( x_1 \leq 0 \lor x_1 \geq 1 \) and \( x_2 \leq 0 \lor x_2 \geq 1 \).
- At step \( i \), we **keep one inequality** of rank at most \( i \) and proceed to the next disjunction.
- We prove that this procedure **converges in finite time** to the desired inequality.
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One proof for a non-degenerate non-maximal triangle

Rank 0
One proof for a non-degenerate non-maximal triangle

Rank 0
One proof for a non-degenerate non-maximal triangle

Rank 1

Split Rank of Triangles and Quadrilaterals

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One proof for a non-degenerate non-maximal triangle

Rank 1
One proof for a non-degenerate non-maximal triangle

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One proof for a non-degenerate non-maximal triangle
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Rank 2
One proof for a non-degenerate non-maximal triangle

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Rank 3
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Rank 4

Rank 4
One proof for a non-degenerate non-maximal triangle

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Rank 4
One proof for a non-degenerate non-maximal triangle
One proof for a non-degenerate non-maximal triangle

Rank 5

\[ \text{Rank 5} \]
One proof for a non-degenerate non-maximal triangle

Rank 5
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Rank 5
One proof for a non-degenerate non-maximal triangle
One proof for a non-degenerate non-maximal triangle

The goal inequality has a rank of at most 6
The geometry behind the convergence
The geometry behind the convergence
The geometry behind the convergence
The geometry behind the convergence
The geometry behind the convergence
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Assumptions for the following

- We have “proven” that a non-maximal triangle where the upward ray points to the left has a finite rank.
- We can prove that the constructed bound is logarithmic in the number of bits of the input.
- The proof for the upward ray pointing to the right works similarly (but not identically).
- In the following, we assume that any non-maximal triangle has a finite rank.
The maximal triangles
The maximal triangles

This inequality is a non-maximal triangle $\Rightarrow$ finite rank!
The maximal triangles
The maximal triangles

The goal inequality is valid for the disjunction.
The maximal triangles

The goal inequality has a finite rank
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point
The dissection triangle

Dissection ≡ each side is tight at exactly one integer point

This inequality is a non-maximal triangle ⇒ finite rank!
The dissection triangle

Dissection ≡ each side is tight at exactly one integer point

Similarly this inequality has a finite rank!
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point
The dissection triangle

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Brown line: set of points with a representation that satisfy both inequalities with equality
The dissection triangle

**Dissection** ≡ each side is tight at exactly one integer point
The dissection triangle

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**Dissection** ≡ each side is tight at exactly one integer point

The dissection cut has a finite rank
Conclusion

- All triangles except the Cook-Kannan-Schrijver have a finite rank.
- We provide a constructive split proof of that fact.
- Split cuts can essentially achieve all triangles in relatively few rounds.
- In constrast with the results of Basu et al. on the triangle closure compared to the split closure.
- All quadrilaterals have a finite rank.
- Open (and difficult) question: lower bounds on the split rank.
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