Split Rank of Triangle and Quadrilateral Inequalities

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January 2009

Joint work with Santanu Dey (CORE)

Outline

- Cuts from two rows of the simplex tableau
- The different cases to consider
- Split cuts and split ranks
- Finiteness proofs for the triangles
- The ideas for the quadrilaterals
- Conclusion

Cuts from two rows of the simplex tableau

Consider a mixed-integer program

$$
\begin{aligned}
\min\ & c^T x \\
\text{s.t. } & Ax = b \\
& x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2}.
\end{aligned}
$$

We consider the problem of finding valid inequalities cutting off the linear relaxation optimum.

We consider the simplex tableau

$$
\begin{array}{ccc}
x_1 & & -\bar{a}_{11}s_1 - \cdots - \bar{a}_{1n}s_n = \bar{b}_1 \\
& \ddots & \vdots \\
& & x_m - \bar{a}_{m1}s_1 - \cdots - \bar{a}_{mn}s_n = \bar{b}_m.
\end{array}
$$

- Select two rows
- Relax the integrality requirements of the non-basic variables
- Relax the nonnegativity requirements of the basic variables but keeping integrality

Cuts from two rows of the simplex tableau

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The 2 row-model

The model

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j, \qquad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+$$

Model studied in [Andersen, Louveaux, Weismantel, Wolsey, IPCO2007] (for the finite case) and [Cornuéjols, Margot, 2009] (for the infinite case).

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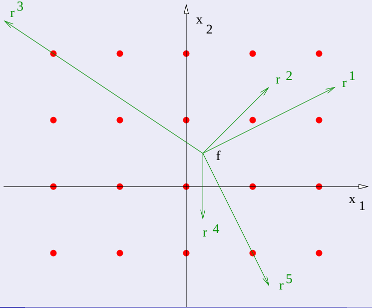
The geometry

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5$$



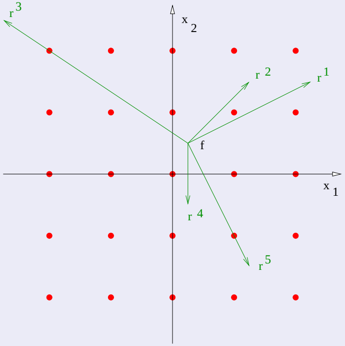
The geometry

The projection picture

$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1$$

- We project the $n+2$-dim space onto the x-space
- The facet is represented by a polygon L_α
- There is no integer point in the interior of L_α
- The coefficients are a ratio of distances on the figure

α_1 α_3



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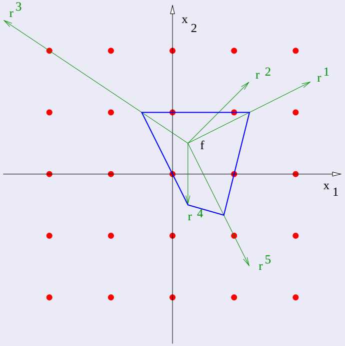

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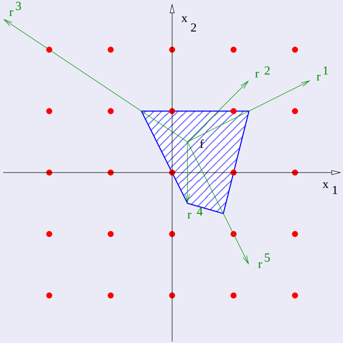

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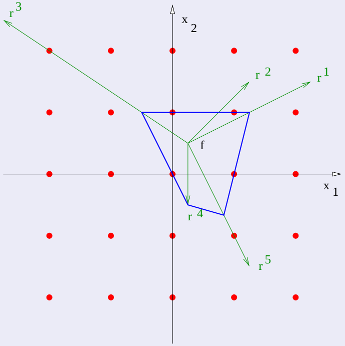

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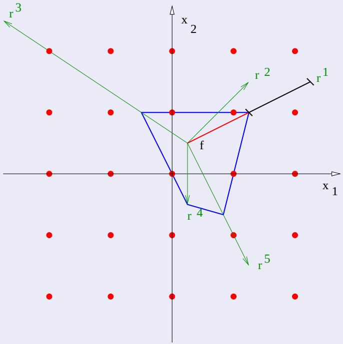

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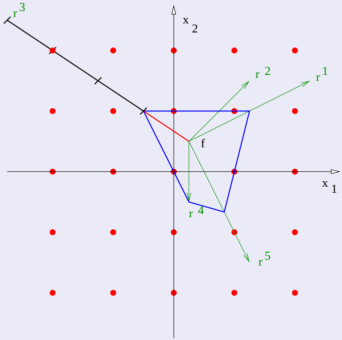
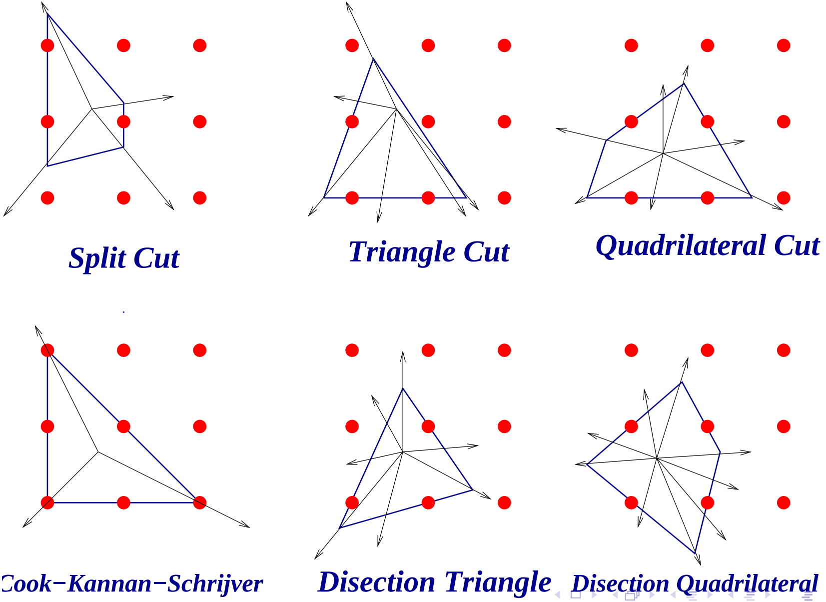


Classification of all possible facet-defining inequalities

Theorem : All facets are projected to triangles and quadrilaterals [Andersen et al 2007].



The split rank question

- Split cut : applying a disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ to a polyhedron P

$$
\begin{aligned}
x &= f + RS \\
s_1 &\geq 0 \\
&\vdots \\
s_n &\geq 0 \\
\pi^T x &\leq \pi_0
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- The first split closure P^1 of P is what you obtain after having applied all possible split disjunctions π.
- The split rank of a valid inequality is the minimum i such that the inequality is valid for P^i
- Most inequalities used in commercial softwares are split cuts
- Question : what is the split rank of the 2 row-inequalities ? In how many rounds of split cuts only can we generate the inequalities ?
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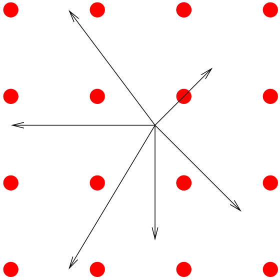

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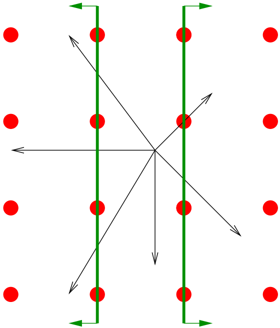

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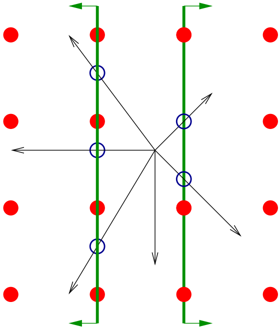

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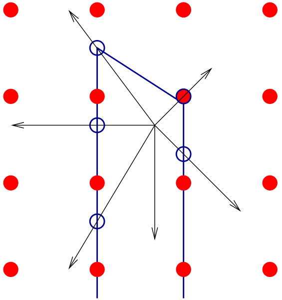

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Useful properties of the split rank

- The split rank is invariant up to integer translation and unimodular transformation
- (Lifting) Consider a triangle (or quadrilateral) inequality for a 3-variable problem. If we keep the same shape of the polygon and consider an n-variable problem, the split rank does not increase.

  It allows us to work with 3 variables only when trying to find the split rank of triangles.
- (Projection) Let $\sum_{i=1}^{n} \alpha_i s_i \geq 1$ be an inequality with split rank η. Then the projected inequality $\sum_{i=1}^{n-1} \alpha_i s_i \geq 1$ has a split rank of at most η for the projected problem.

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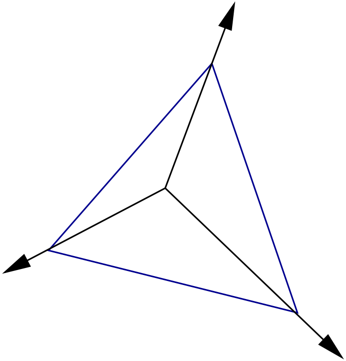

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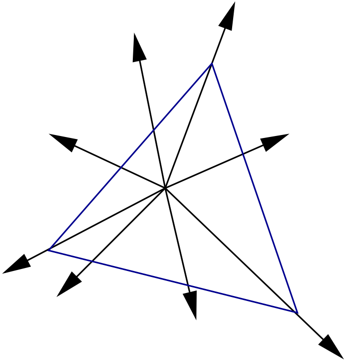

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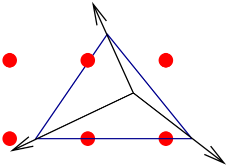
The triangle case

Several cases to consider, after suitable unimodular transformation



An illustration of the proof in this talk

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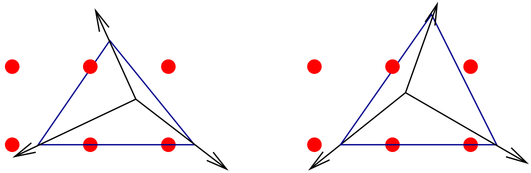

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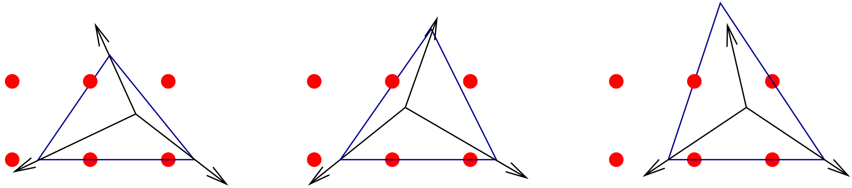

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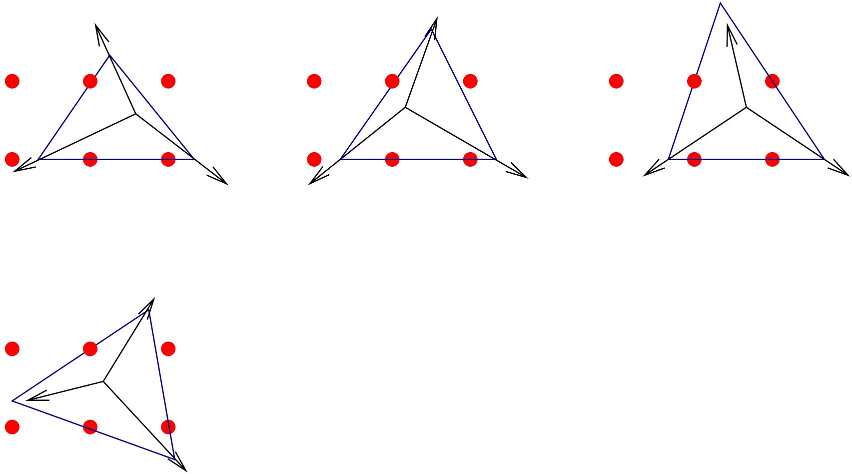

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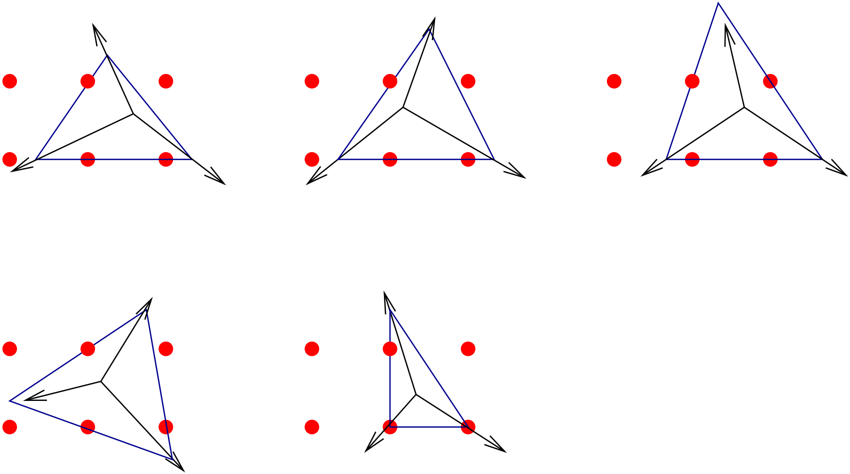

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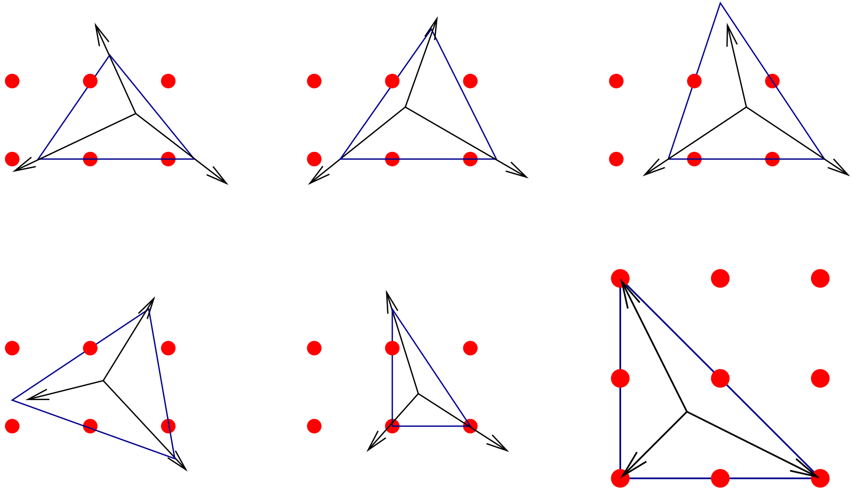

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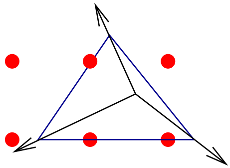

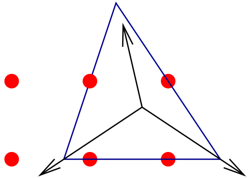

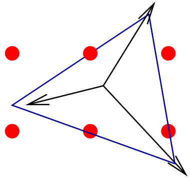

An illustration of the proof in this talk

Idea of the proof of upper bounds

- We prove an upper bound on the split rank.
- Procedure : We apply a sequence of two split disjunctions.
  Successively : $x_1 \leq 0 \vee x_1 \geq 1$ and $x_2 \leq 0 \vee x_2 \geq 1$
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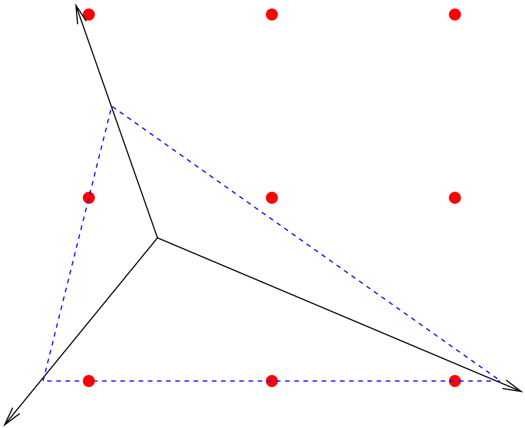
One proof for a non-degenerate non-maximal triangle



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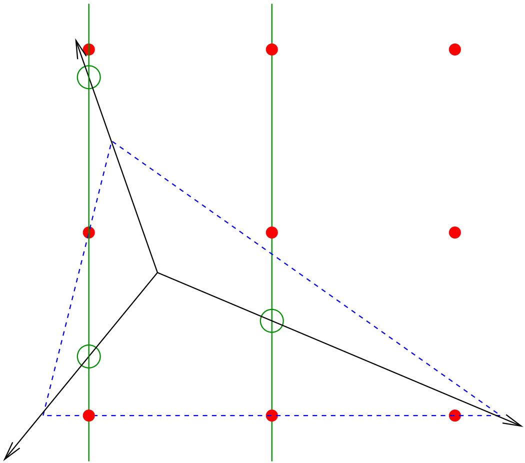
One proof for a non-degenerate non-maximal triangle



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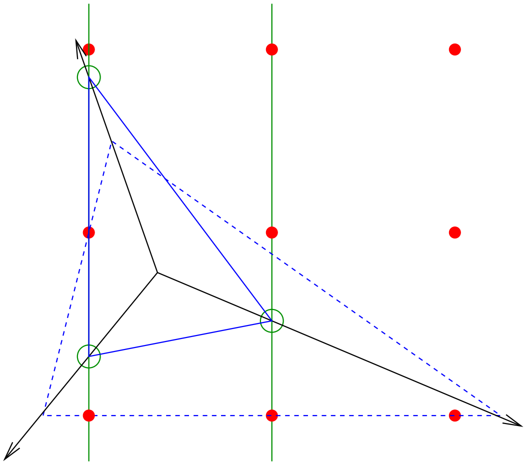

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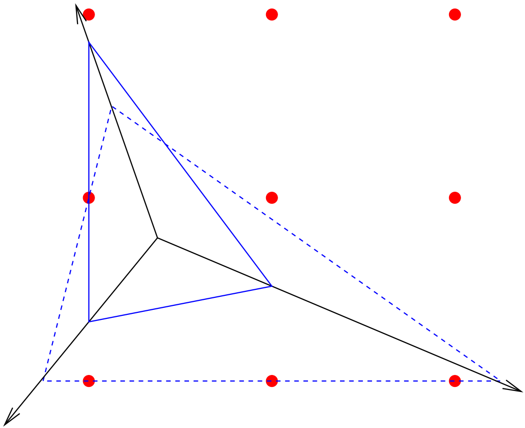

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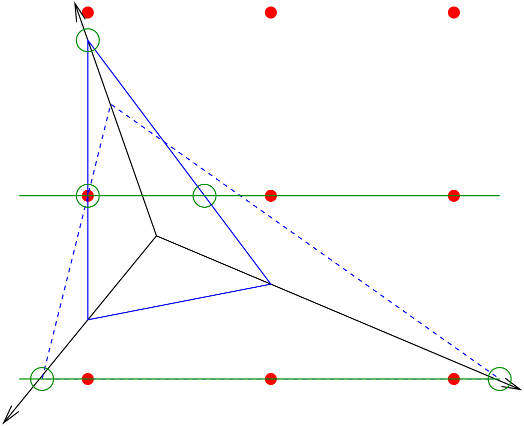

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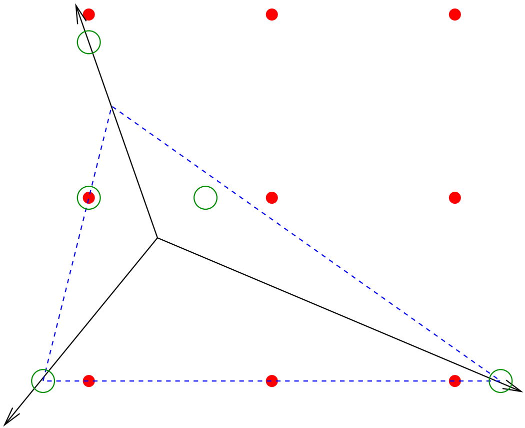

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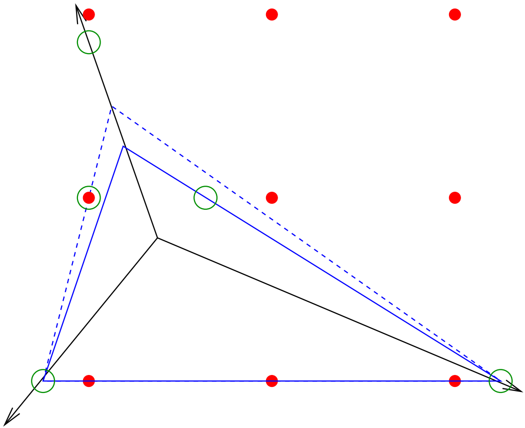

Rank 2

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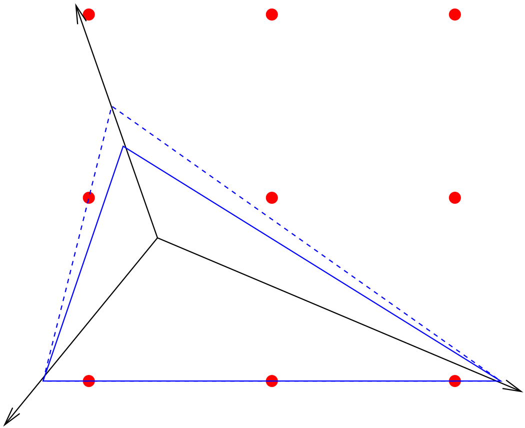

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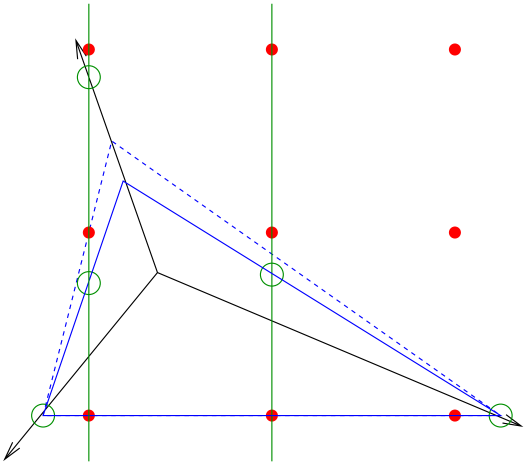

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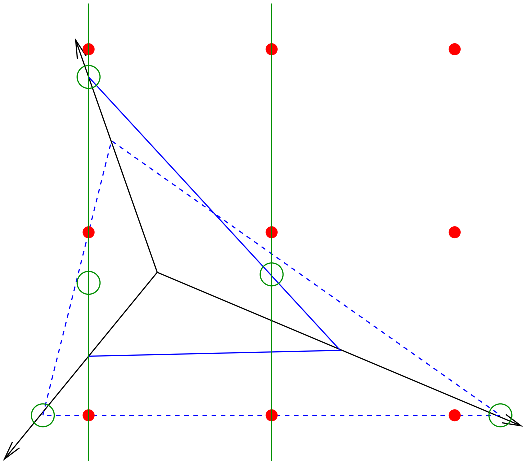
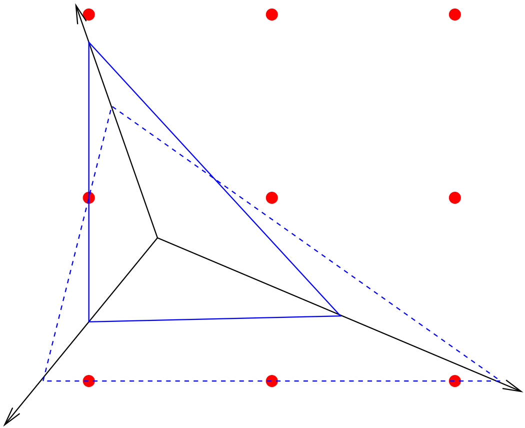


Rank 3

One proof for a non-degenerate non-maximal triangle



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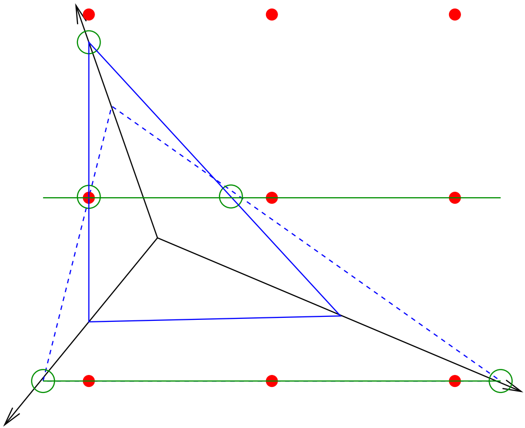

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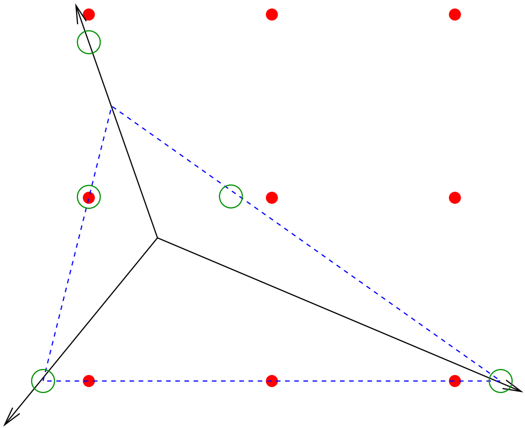

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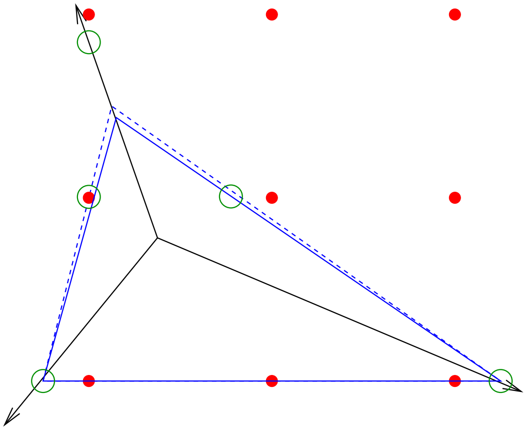
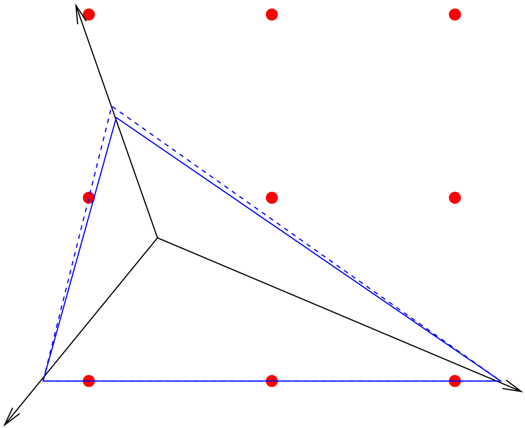


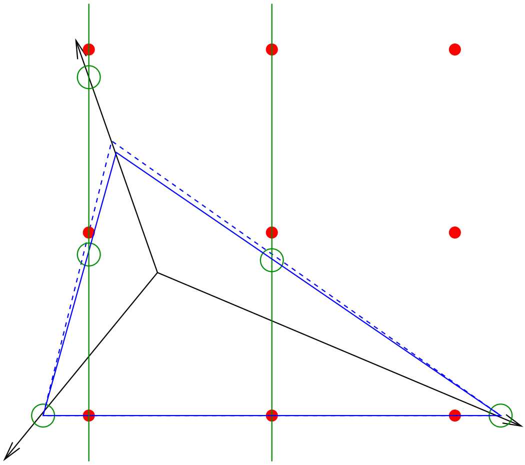
Rank 4

One proof for a non-degenerate non-maximal triangle



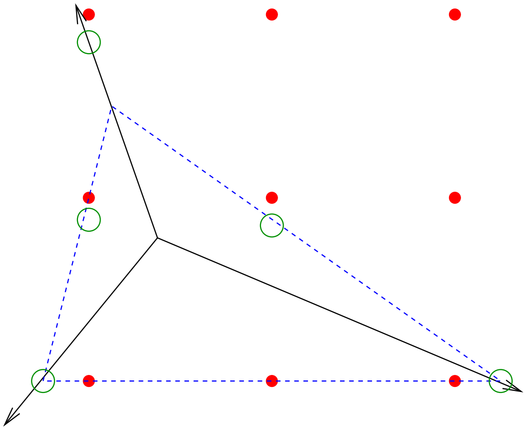
Rank 4

One proof for a non-degenerate non-maximal triangle

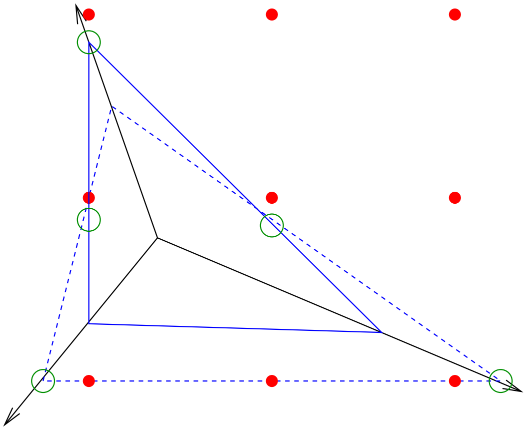


Rank 4

One proof for a non-degenerate non-maximal triangle

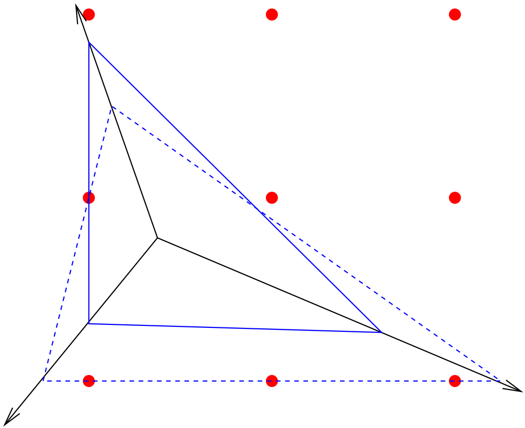


One proof for a non-degenerate non-maximal triangle



Rank 5

One proof for a non-degenerate non-maximal triangle



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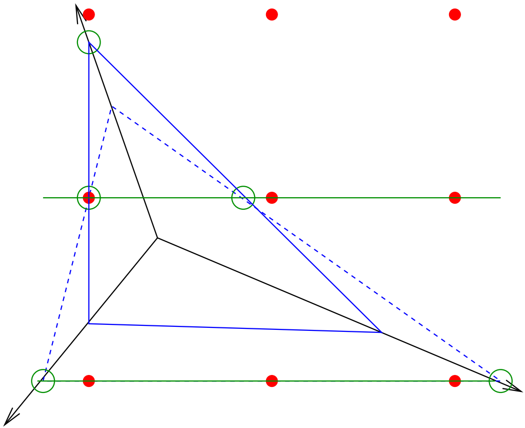

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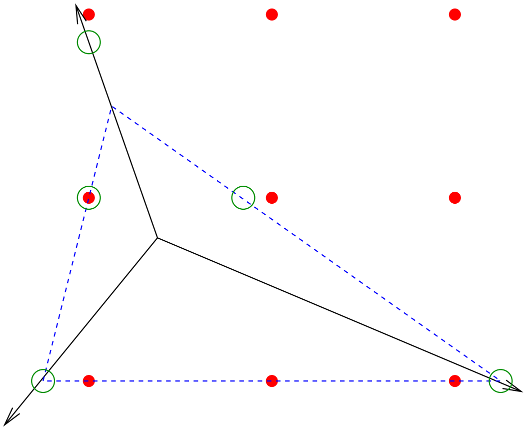

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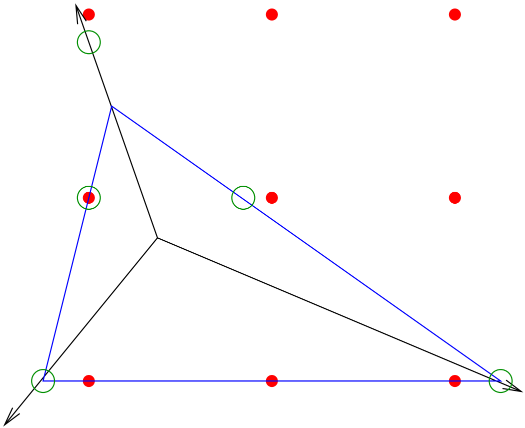
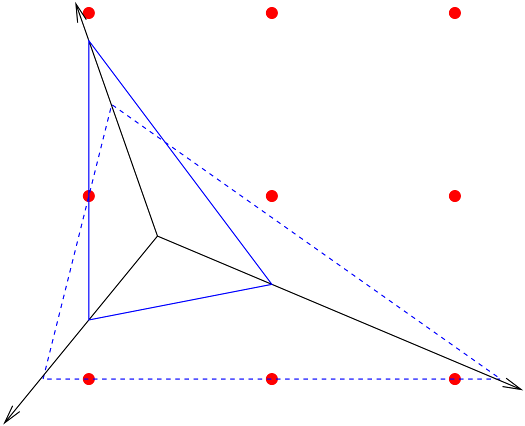


The goal inequality has a rank of at most 6

The geometry behind the convergence



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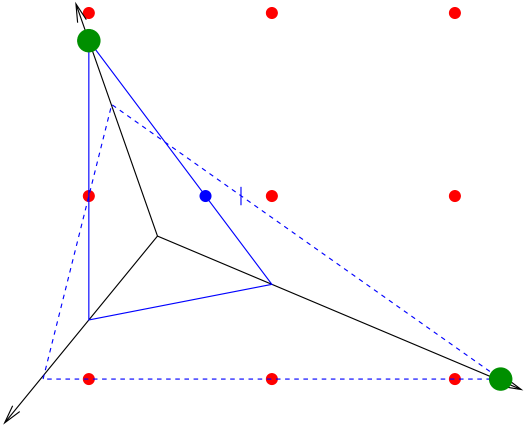

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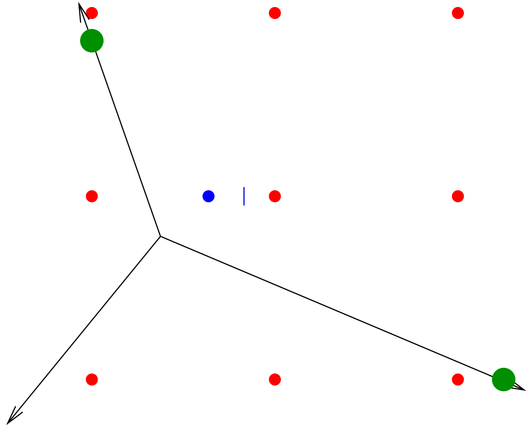

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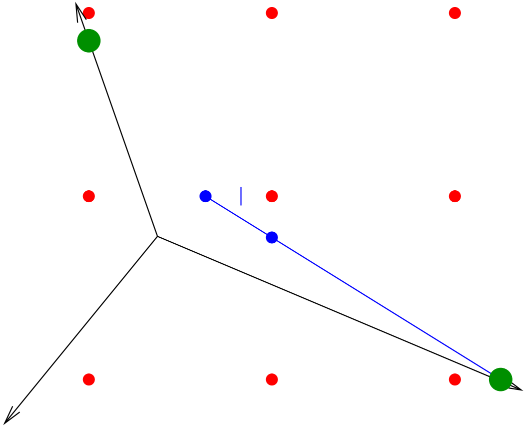

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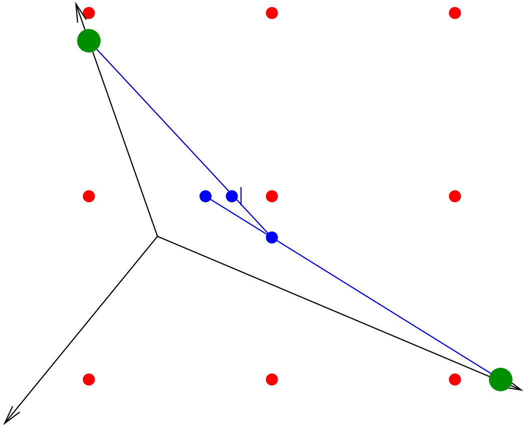

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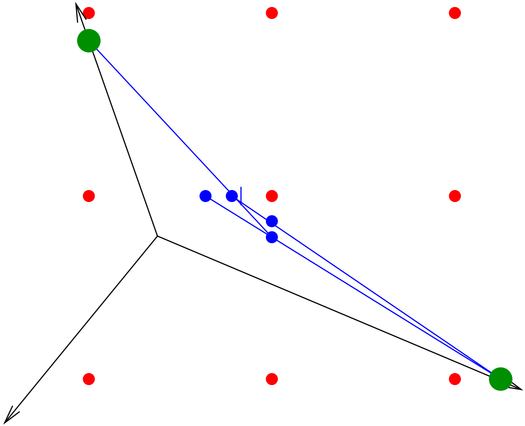

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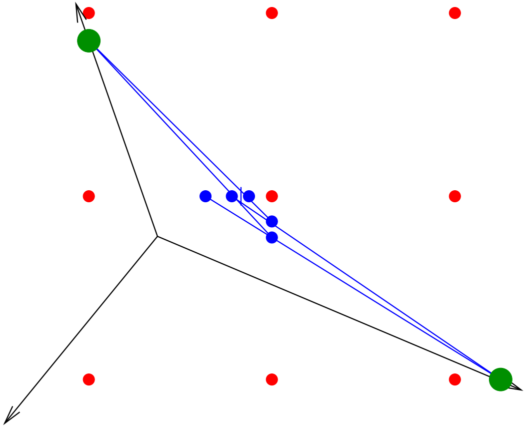

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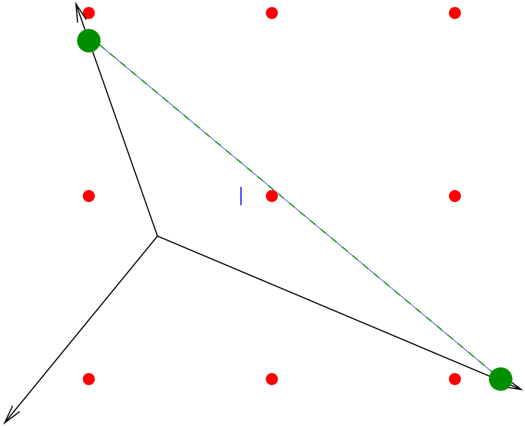
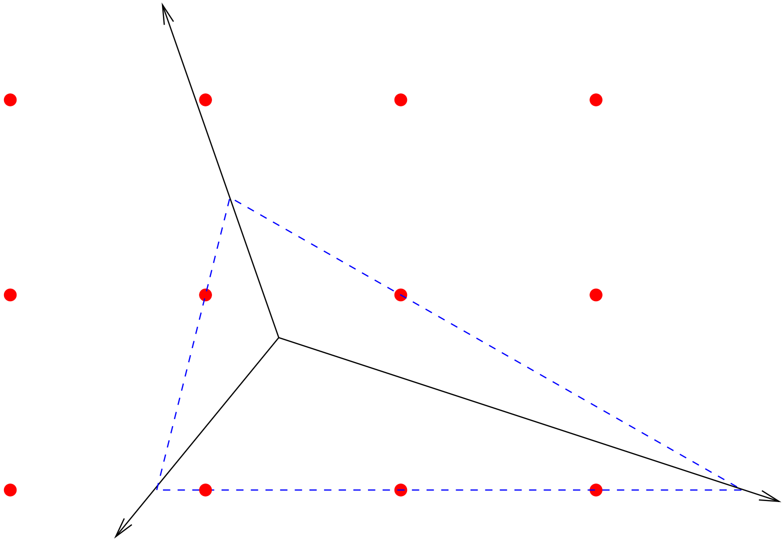


Assumptions for the following

- We have "proven" that a non-maximal triangle where the upward ray points to the left has a finite rank.
- We can prove that the constructed bound is logarithmic in the number of bits of the input.
- The proof for the upward ray pointing to the right works similarly (but not identically).
- In the following, we assume that any non-maximal triangle has a finite rank.

The maximal triangles



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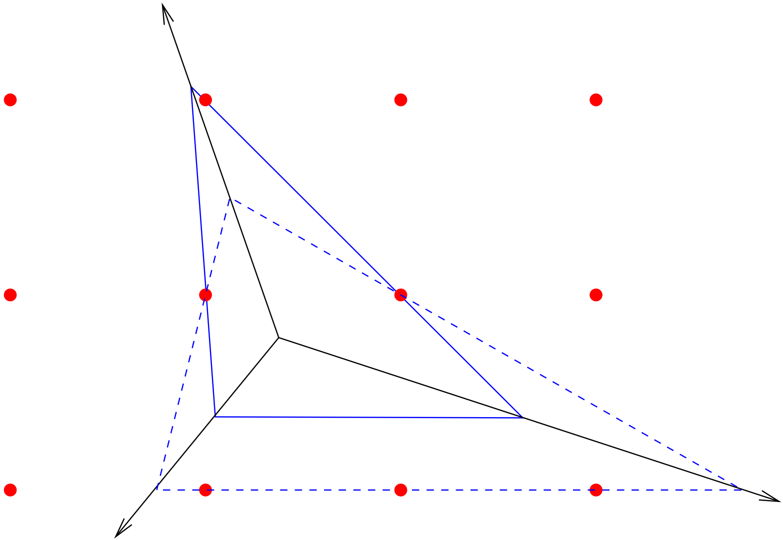
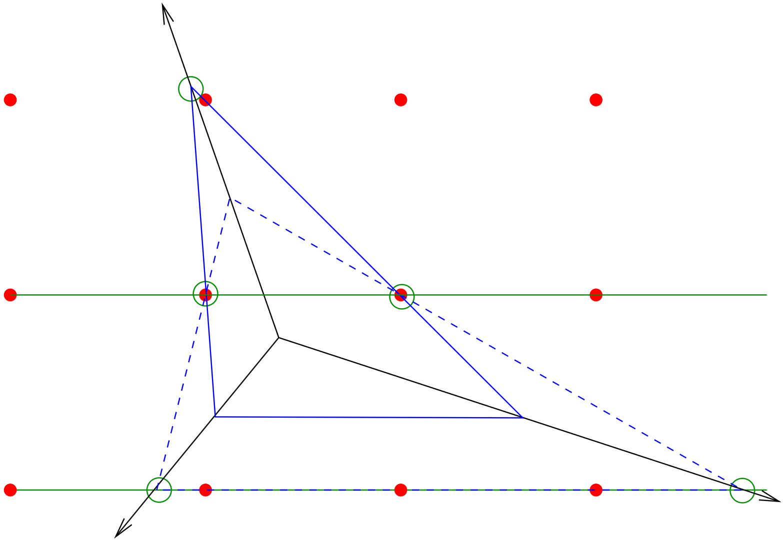


This inequality is a non-maximal triangle ⇒ finite rank !

The maximal triangles



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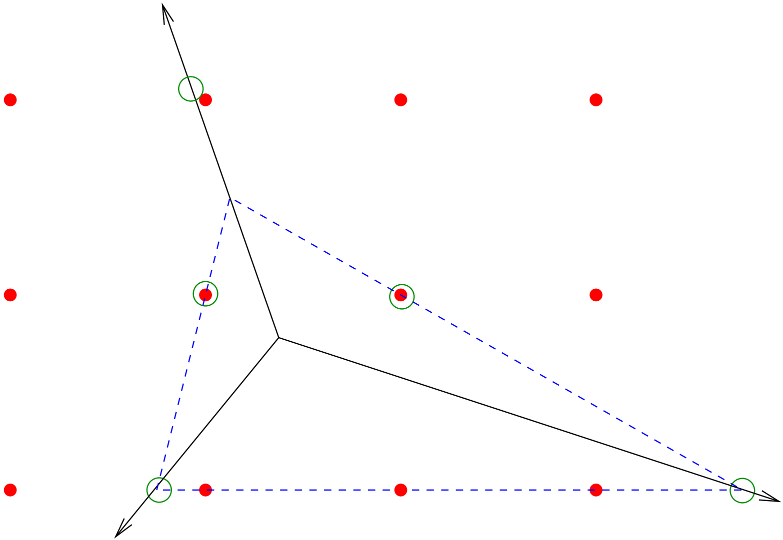
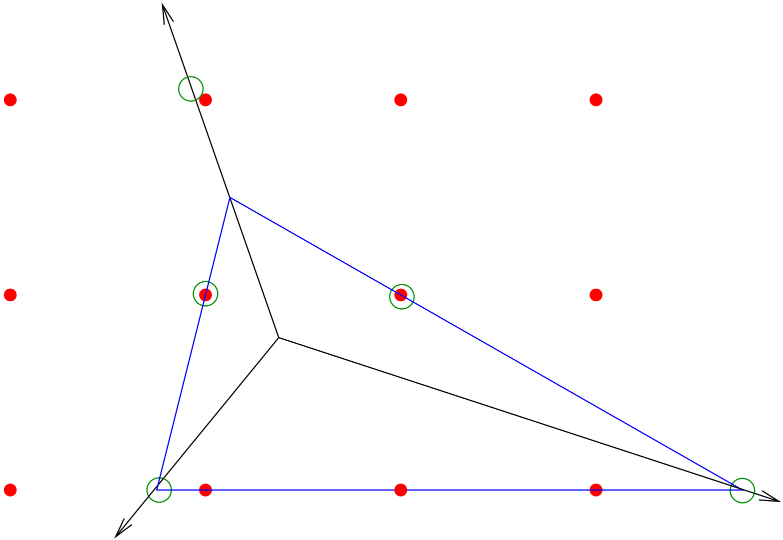


The goal inequality is valid for the disjunction.

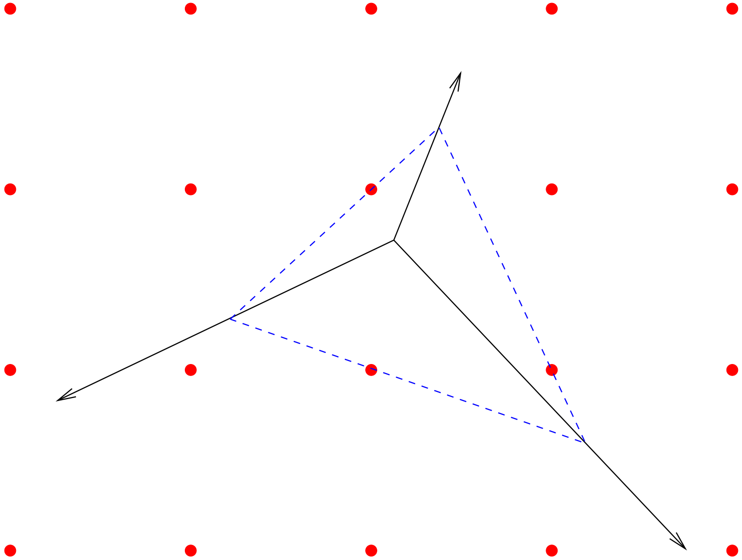
The maximal triangles



The goal inequality has a finite rank

The disection triangle

Disection $\equiv$ each side is tight at exactly one integer point



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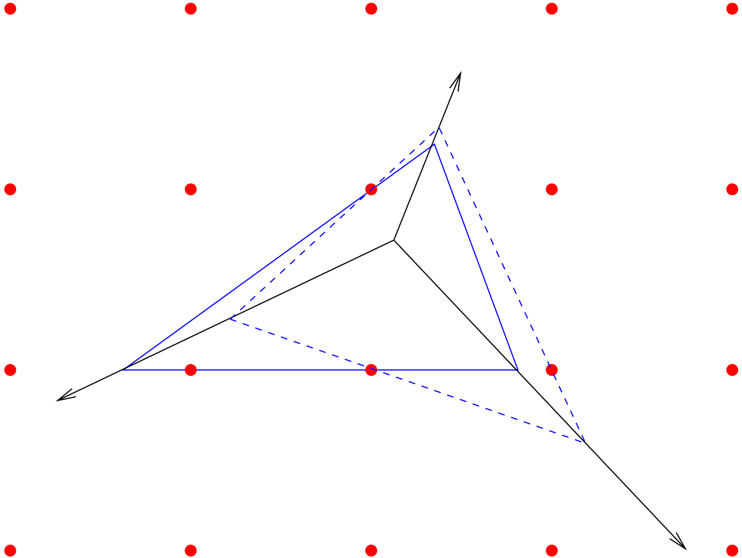

This inequality is a non-maximal triangle $\Rightarrow$ finite rank !

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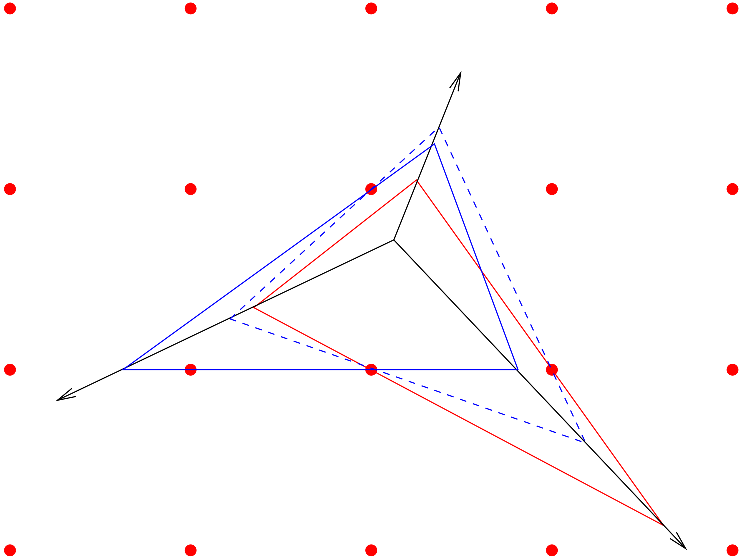

Similarly this inequality has a finite rank !

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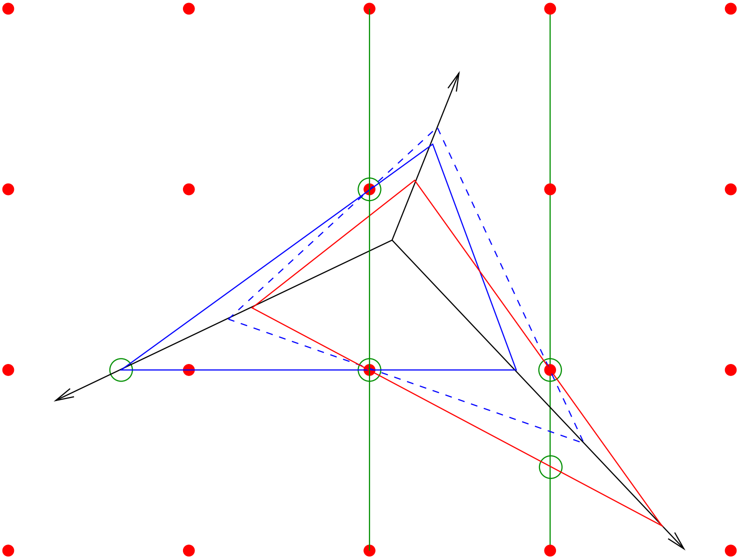

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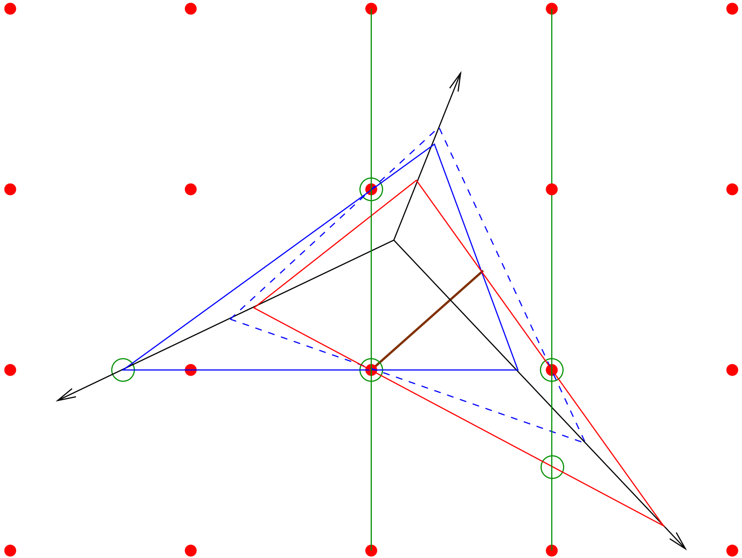

Brown line : set of points with a representation that satisfy both inequalities with equality

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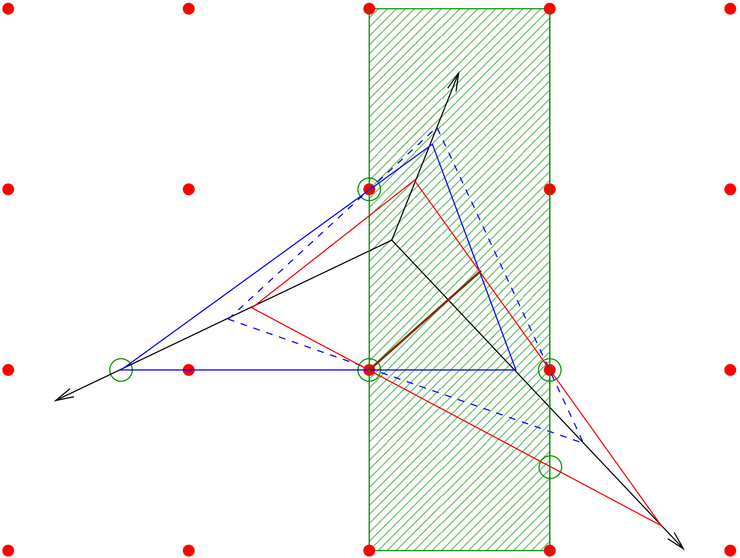

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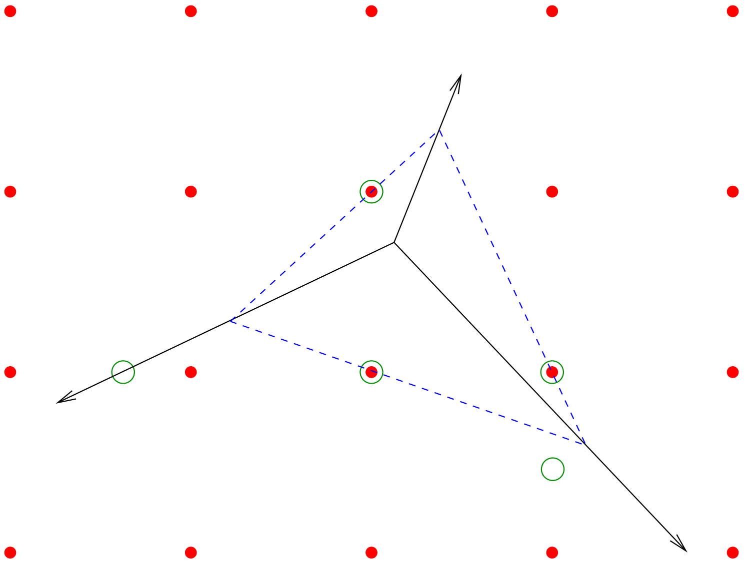

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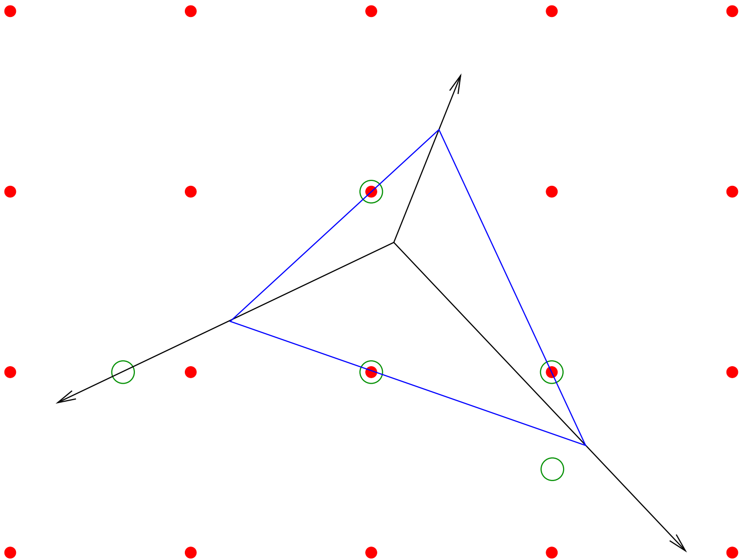

The disection cut has a finite rank

The quadrilateral cuts

- Two cases : non-maximal quadrilateral and disection quadrilateral.
- By the projection Lemma, we can deal with most non-maximal quadrilaterals
- One exception : if the lifted triangle has infinite rank.

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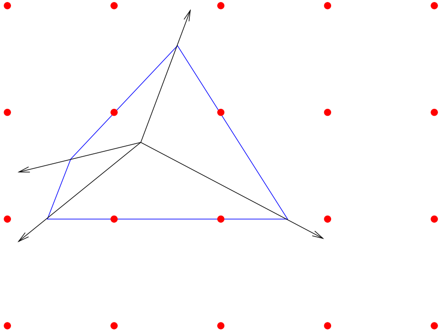

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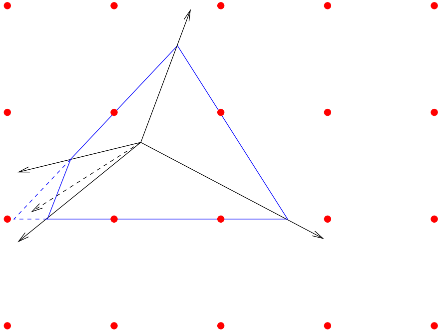

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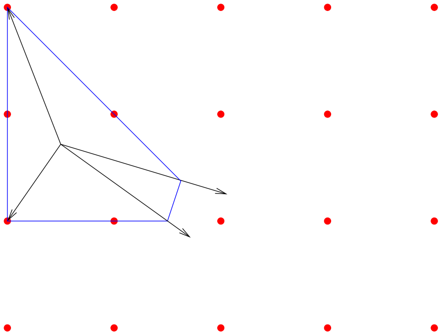

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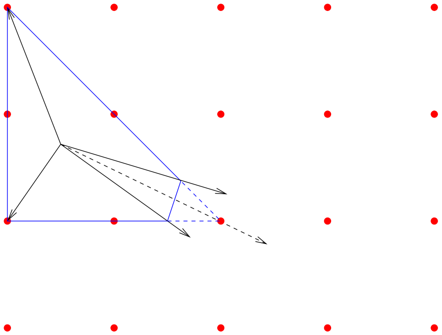

Conclusion

- All triangles except the Cook-Kannan-Schrijver have a finite rank.
- We provide a constructive split proof of that fact.
- Split cuts can essentially achieve all triangles in relatively few rounds.
- In constrast with the results of Basu et al. on the triangle closure compared to the split closure.
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