Split Rank of Triangle and Quadrilateral Inequalities

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Joint work with Santanu Dey (CORE)
Outline

- Cuts from two rows of the simplex tableau
- The different cases to consider
- Split cuts and split ranks
- Finiteness proofs for the triangles
- The ideas for the quadrilaterals
- Conclusion
Cuts from two rows of the simplex tableau

Consider a mixed-integer program

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+.
\end{align*}
\]

We consider the problem of finding valid inequalities cutting off the linear relaxation optimum.

We consider the simplex tableau

\[
\begin{align*}
& \quad x_1 - \bar{a}_{11}s_1 - \cdots - \bar{a}_{1n}s_n = \bar{b}_1 \\
& \quad \vdots \\
& \quad x_m - \bar{a}_{m1}s_1 - \cdots - \bar{a}_{mn}s_n = \bar{b}_m.
\end{align*}
\]

- Select two rows
- Relax the integrality requirements of the non-basic variables
- Relax the nonnegativity requirements of the basic variables but keeping integrality
Cuts from two rows of the simplex tableau

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The 2 row-model

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix}
+ \sum_{j=1}^{n} \begin{pmatrix}
  r_1^j \\
  r_2^j
\end{pmatrix} s_j,
\quad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+
\]

Model studied in [Andersen, Louveaux, Weismantel, Wolsey, IPCO2007] (for the finite case) and [Cornuéjols, Margot, 2009] (for the infinite case).
The 2 row-model

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The geometry

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} = \begin{pmatrix}
  1/4 \\
  1/2
\end{pmatrix} s_1 + \begin{pmatrix}
  2 \\
  1
\end{pmatrix} s_2 + \begin{pmatrix}
  1 \\
  1
\end{pmatrix} s_3 + \begin{pmatrix}
  -3 \\
  2
\end{pmatrix} s_4 + \begin{pmatrix}
  0 \\
  -1
\end{pmatrix} s_5
\]
The geometry

The projection picture

\[ 2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1 \]

- We project the \( n + 2 \)-dim space onto the \( x \)-space
- The facet is represented by a polygon \( L_\alpha \)
- There is no integer point in the interior of \( L_\alpha \)
- The coefficients are a ratio of distances on the figure

\( \alpha_1, \alpha_3 \)
The geometry

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\( \alpha_1, \alpha_3 \)
Classification of all possible facet-defining inequalities

**Theorem**: All facets are projected to triangles and quadrilaterals [Andersen et al 2007].

\[\text{Split Cut} \hspace{2cm} \text{Triangle Cut} \hspace{2cm} \text{Quadrilateral Cut}\]

\[\text{Cook–Kannan–Schrijver} \hspace{2cm} \text{Dissection Triangle} \hspace{2cm} \text{Dissection Quadrilateral}\]
The split rank question

- Split cut: applying a disjunction $\pi^T x \leq \pi_0 \lor \pi^T x \geq \pi_0 + 1$ to a polyhedron $P$

\[
\begin{align*}
x &= f + RS \\
s_1 &\geq 0 \\
\vdots \\
s_n &\geq 0 \\
\pi^T x &\leq \pi_0
\end{align*}
\]

- The first split closure $P^1$ of $P$ is what you obtain after having applied all possible split disjunctions $\pi$.
- The split rank of a valid inequality is the minimum $i$ such that the inequality is valid for $P^i$.
- Most inequalities used in commercial softwares are split cuts.
- Question: what is the split rank of the 2 row-inequalities?
  In how many rounds of split cuts only can we generate the inequalities?
- The Cook-Kannan-Schrijver has infinite rank and we prove that the other triangles have finite rank.
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Split Rank of Triangles and Quadrilaterals

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Useful properties of the split rank

- The split rank is invariant up to **integer translation and unimodular transformation**
- **(Lifting)** Consider a triangle (or quadrilateral) inequality for a 3-variable problem. If we keep the same shape of the polygon and consider an \( n \)-variable problem, the split rank does not increase.

It allows us to work with 3 variables only when trying to find the split rank of triangles.

- **(Projection)** Let \( \sum_{i=1}^{n} \alpha_i s_i \geq 1 \) be an inequality with split rank \( \eta \). Then the projected inequality \( \sum_{i=1}^{n-1} \alpha_i s_i \geq 1 \) has a split rank of at most \( \eta \) for the projected problem.
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The triangle case

Several cases to consider, after suitable unimodular transformation
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An illustration of the proof in this talk
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Idea of the proof of upper bounds

- We prove an **upper bound** on the split rank.
- Procedure: We apply a sequence of two split disjunctions. Successively:
  \[ x_1 \leq 0 \lor x_1 \geq 1 \text{ and } x_2 \leq 0 \lor x_2 \geq 1 \]
- At step \( i \), we **keep one inequality** of rank at most \( i \) and proceed to the next disjunction.
- We prove that this procedure **converges in finite time** to the desired inequality.
Idea of the proof of upper bounds

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One proof for a non-degenerate non-maximal triangle
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Rank 0
One proof for a non-degenerate non-maximal triangle

Rank 1
One proof for a non-degenerate non-maximal triangle

Rank 1

Diagram of a non-degenerate non-maximal triangle with points marked.
One proof for a non-degenerate non-maximal triangle

Rank 1
One proof for a non-degenerate non-maximal triangle
One proof for a non-degenerate non-maximal triangle

Rank 2
One proof for a non-degenerate non-maximal triangle

Rank 2

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure}
\caption{Rank 2}
\end{figure}
One proof for a non-degenerate non-maximal triangle

Rank 2
One proof for a non-degenerate non-maximal triangle

Rank 3

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One proof for a non-degenerate non-maximal triangle

Rank 3
One proof for a non-degenerate non-maximal triangle

Rank 3

Rig 3
One proof for a non-degenerate non-maximal triangle
One proof for a non-degenerate non-maximal triangle

Rank 4
One proof for a non-degenerate non-maximal triangle

Rank 4

Figure: A diagram illustrating a non-degenerate non-maximal triangle with rank 4.
One proof for a non-degenerate non-maximal triangle

Rank 4

\[\text{Rank 4}\]
One proof for a non-degenerate non-maximal triangle
One proof for a non-degenerate non-maximal triangle

Rank 5
One proof for a non-degenerate non-maximal triangle

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Split Rank of Triangles and Quadrilaterals

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One proof for a non-degenerate non-maximal triangle

The goal inequality has a rank of at most 6
The geometry behind the convergence
The geometry behind the convergence
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Assumptions for the following

- We have “proven” that a non-maximal triangle where the upward ray points to the left has a finite rank.
- We can prove that the constructed bound is logarithmic in the number of bits of the input.
- The proof for the upward ray pointing to the right works similarly (but not identically).
- In the following, we assume that any non-maximal triangle has a finite rank.
The maximal triangles
The maximal triangles

This inequality is a non-maximal triangle $\Rightarrow$ finite rank!
The maximal triangles
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The goal inequality is valid for the disjunction.
The maximal triangles

The goal inequality has a finite rank
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point

This inequality is a non-maximal triangle $\Rightarrow$ finite rank!
The dissection triangle

**Dissection** \(\equiv\) each side is tight at exactly one integer point

Similarly this inequality **has a finite rank**!
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point
The dissection triangle

Dissection ≡ each side is tight at exactly one integer point

Brown line: set of points with a representation that satisfy both inequalities with equality
The dissection triangle

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Dissection $\equiv$ each side is tight at exactly one integer point
The disection triangle

Disection ≡ each side is tight at exactly one integer point

The disection cut has a finite rank
The quadrilateral cuts

- Two cases: non-maximal quadrilateral and dissection quadrilateral.
- By the projection Lemma, we can deal with most non-maximal quadrilaterals.
- One exception: if the lifted triangle has infinite rank.
The quadrilateral cuts

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![Diagram of quadrilateral cuts](image-url)
Conclusion

- All triangles except the Cook-Kannan-Schrijver have a finite rank.
- We provide a constructive split proof of that fact.
- Split cuts can essentially achieve all triangles in relatively few rounds.
- In contrast with the results of Basu et al. on the triangle closure compared to the split closure.
- Ongoing work: (almost?) all quadrilaterals have a finite rank.
- Open (and difficult) question: lower bounds on the split rank.
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