On the stability of the symmetric interior penalty method applied to RANS computations on hybrid boundary layer meshes

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The *symmetric interior penalty (SIP)* method is a discontinuous Galerkin Finite Element Method for elliptic problems. Applied to a scalar problem (solving for u) with diffusivity ν the SIP method defines the residual associated to the shape function $\phi_i \in \Phi^p$ as [1]:

$$\mathbf{r}_i = -\sum_E \int_E \nabla\phi_i \cdot \nu \nabla u \, dV + \sum_F \oint_F \left([\phi_i]^k \left\{ \nu \frac{\partial u}{\partial x^l} \right\} [\nu u]^k \left\{ \frac{\partial \phi_i}{\partial x^l} \right\} \right) + \sum_F \oint_F \sigma_F \, [u] \, [\phi_i] \, dS$$

Here E and F denote elements and faces respectively, while $[]$ and $\{\}$ indicate the face jump and average operators. The stability of the SIP method hinges on the value of the penalty parameter σ, which should be high enough to guarantee coercivity, but as small as possible, since it can impact severely on the conditioning of the system of equations. Shahbazi [2] has developed optimal values for σ for scalar, constant coefficient elliptic problems on simplex meshes [2]; these values were based on a coercivity analysis, based on sharp trace inequalities elaborated by Warburton [3]

$$\int_F u^2 dS \leq C(p) \cdot \frac{\mathcal{A}(F)}{\mathcal{V}(E)} \int_E u^2 dV \ , \ \forall u \in \Phi^p \tag{1}$$

In this inequality $\mathcal{A}(F)$ and $\mathcal{V}(E)$ denote face "area" and element "volume" respectively.

Reynolds-Averaged Navier-Stokes equations usually require high-aspect ratio meshes in the boundary layer, in order to resolve the very sharp velocity gradient. In order to stably apply the interior penalty method to this type of computations, we need to extend the stable values for the penalty parameter:

- sharp inequalities of the form 1 for all element types other than simplices are elaborated, and the definition of σ_F of Shahbazi is consequently modified;

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- an alternative definition of σ is proposed that genuinely takes the anisotropy of the mesh into account;
- the impact of a highly variable diffusion parameter, representative for the sharp variations of the eddy viscosity in RANS computations, on σ is investigated.

The stability of the definitions is compared on analytical problems, on meshes composed of a variety of element types. The impact of mesh anisotropy and variation of diffusion coefficient is equally investigated. Finally the variants are assessed on sample RANS computations.

References

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