About the Regularity of Cantor's Bijection

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Functional Analysis:
Applications to Complex Analysis and Partial Differential Equations
Będlewo (Poland) – May 6-12, 2012

Introduction

In 1878, Cantor constructed a bijection between $[0,1]$ and $[0,1]^2$, bijection defined via continued fractions.

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1878.

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Notations

$$E = [0,1], \qquad D = E \cap \mathbb{Q} \qquad \text{and} \qquad I = E \setminus D.$$

Continued Fractions

A. Ya. Khintchine, *Continued fractions*, P. Noordhoff, 1963.

Let $\boldsymbol{a} = (a_j)_{j \in \{1,\dots,n\}}$ a finite sequence of positive real numbers ($n \in \mathbb{N}$); the expression $[a_1, \dots, a_n]$ is recursively defined as follows:

$$[a_1] = \frac{1}{a_1} \quad \text{and} \quad [a_1, \dots, a_m] = \frac{1}{a_1 + [a_2, \dots, a_m]},$$

for any $m \in \{2, \dots, n\}$. If $\boldsymbol{a} \in \mathbb{N}^n$, we say that

$$[a_1, \dots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

is a (simple) **finite continued fraction**.

Proposition

For any $\boldsymbol{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$), $[a_1, \dots, a_n]$ belongs to D. Conversely, for any $x \in D$, there exists a natural number n and a sequence $\boldsymbol{a} \in \mathbb{N}^n$ such that $x = [a_1, \dots, a_n]$.

Continued Fractions and Convergents

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. For all $j \in \mathbb{N}$, the quantities $p_j(\boldsymbol{a})$ and $q_j(\boldsymbol{a})$ are recursively defined as follows:

$$p_{-1}(\boldsymbol{a}) = 1,\ q_{-1}(\boldsymbol{a}) = 0,\ p_0(\boldsymbol{a}) = 0,\ q_0(\boldsymbol{a}) = 1$$

and, for $k \in \{1, \dots, j\}$,

$$\begin{cases} p_k(\boldsymbol{a}) = a_k p_{k-1}(\boldsymbol{a}) + p_{k-2}(\boldsymbol{a}) \\ q_k(\boldsymbol{a}) = a_k q_{k-1}(\boldsymbol{a}) + q_{k-2}(\boldsymbol{a}) \end{cases}.$$

The quotient $\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})}$ is called the **convergent of order** j of $\boldsymbol{a}$.

Proposition

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. For all $j \in \mathbb{N}$, we have

$$[a_1, \dots, a_j] = \frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})}.$$

With the properties of convergents, we can show that the sequence $x_j = [a_1, \dots, a_j]$ converges. The limit is called an **infinite continued fraction** and is denoted $[a_1, \dots]$.

Continued Fractions

If the real number $x \in E$ is equal to $[a_1, \ldots]$, we say that $[\boldsymbol{a}] = [a_1, \ldots]$ is a continued fraction corresponding to x.

Theorem – Representation of the real numbers (of E)

Any element of D can be expressed as a finite continued fraction. We have $x \in I$ if and only if there exists an infinite continued fraction corresponding to x; moreover, this infinite continued fraction is unique.

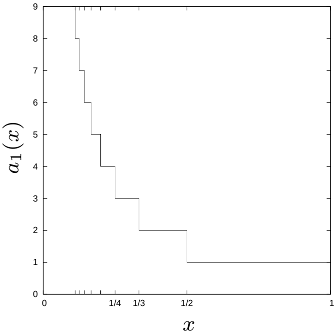
Remark

An element x of I is a quadratic number if and only if the corresponding continued fraction $[\boldsymbol{a}]$ is ultimately periodic, i.e. there exist $k, J \in \mathbb{N}$ such that $a_{j+k} = a_j$ for any $j \geq J$.

Metric Theory of Continued Fractions

For any $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, $[\boldsymbol{a}]$ corresponds to an irrational number $x \in I$. Let us consider, for each $j \in \mathbb{N}$, the term a_j as a function of x : $a_j = a_j(x)$.

- Function a_1



We can write

$$\frac{1}{x} = a_1 + [a_2, \ldots].$$

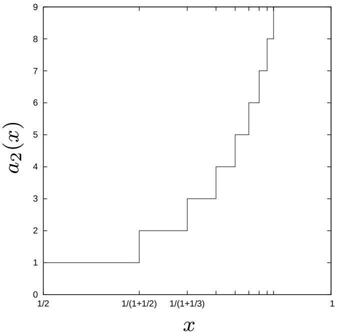
For any $k \in \mathbb{N}$, we have

$$a_1 = k \quad \Leftrightarrow \quad \frac{1}{k+1} < x \leq \frac{1}{k}.$$

Then, a_1 is a piecewise constant and non-increasing function.

Metric Theory of Continued Fractions

- Function a_2 (Representation with $a_1 = 1$)



We can write

$$x = [a_1, r_2] = \cfrac{1}{a_1 + \cfrac{1}{r_2}}$$

with $r_2 \in [1, \infty)$. For any $k \in \mathbb{N}$, we have

$$a_2 = k \quad \Leftrightarrow \quad k \leq r_2 < k + 1.$$

Then, a_2 is a piecewise constant and non-decreasing function.

- Function a_j ($j \in \mathbb{N}$)

If j is odd, a_j is non-increasing piecewise constant function.

If j is even, a_j is non-decreasing piecewise constant function.

Metric Theory of Continued Fractions

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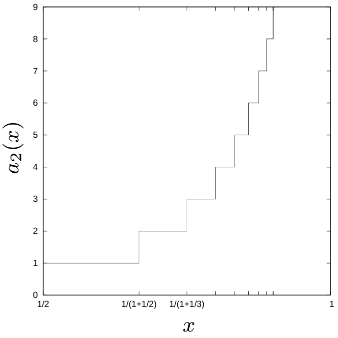

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Metric Theory of Continued Fractions

Let $x = [\boldsymbol{a}]$ be an irrational number; for $n \in \mathbb{N}$, we set

$$I_n(x) = \big\{ y = [\boldsymbol{b}] \in I : b_j = a_j \text{ if } j \in \{1, \ldots, n\} \big\}.$$

We will say that $I_n(x)$ is an interval of rank n. For any $n \in \mathbb{N}$, $I_n(x)$ is an irrational subinterval of I, $I_{n+1}(x) \subset I_n(x)$ and $\lim_n I_n(x) = \{x\}$. In fact, one gets

$$I_n(x) = \left(\frac{p_n(\boldsymbol{a})}{q_n(\boldsymbol{a})}, \frac{p_n(\boldsymbol{a}) + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a})} \right) \cap I,$$

if n is even (if n is odd, the endpoints of the interval are reversed). Every interval of rank n is partitioned into a countable infinite number of intervals of rank $n+1$. By denoting $|I_n(x)|$ the Lebesgue measure of $I_n(x)$, one has

$$|I_n(x)| = \frac{1}{q_n(\boldsymbol{a})(q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a}))}.$$

Cantor's Bijection

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1878.

If $x \in I$, let $[a_1, \ldots]$ the corresponding continued fraction and define the applications f_1 and f_2 as follows:

$$f_1(x) = [a_1, a_3, \ldots, a_{2j+1}, \ldots] \quad \text{and} \quad f_2(x) = [a_2, a_4, \ldots, a_{2j}, \ldots].$$

The application

$$f : I \to I^2 \; ; x \mapsto (f_1(x), f_2(x))$$

is the **Cantor's Bijection** on I.

Remark

- If Q denotes the quadratic numbers of I, f is a one-to-one mapping between Q to Q^2.
- Since the cardinals of E and I are equal, f can be extended to a one-to-one mapping from E to E^2.
- For any $n \in \mathbb{N}$ and any $x \in I$, f_1 maps the interval $I_n(x)$ to $I_m(f_1(x))$, where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. This shows that f_1 is a continuous function.

Cantor's Bijection

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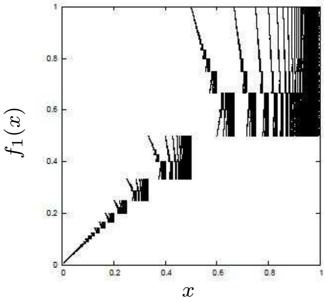
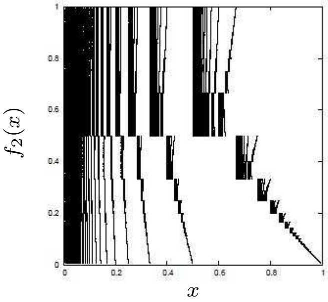
Cantor's Bijection

Representations of the functions f_1 (left panel) and f_2 (right panel)





Continuity of Cantor's Bijection

For all $x \in I$, we write $\varphi(x) = \boldsymbol{a}$ if $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies $x = [\boldsymbol{a}]$.

A usual distance on $\mathbb{N}^{\mathbb{N}}$ is given by

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}$$

if $\boldsymbol{a} = (a_j)_{j \in \mathbb{N}}$ and $\boldsymbol{b} = (b_j)_{j \in \mathbb{N}}$ are two elements of $\mathbb{N}^{\mathbb{N}}$. We implicitly consider that $\mathbb{N}^{\mathbb{N}}$ is equipped with this distance. The set I (like D and E) is endowed with the Euclidean distance.

Proposition

The application φ is an homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$. In particular, Cantor's bijection f is an homeomorphism between I and I^2.

Remark

Since $(\mathbb{N}^{\mathbb{N}}, d)$ is a separable complete metric space, the space I is a Polish space.

Continuity of Cantor's Bijection

Netto's theorem

Any bijective map $g : [0, 1] \to [0, 1]^2$ is necessarily discontinuous.

H. Sagan, *Space-filling curves*, Universitext, New-York : Springer-Verlag, 1994.

Then, Cantor's bijection f can not be extended to a continuous function from E to E^2.

Proposition

Any extension of Cantor's bijection to E is discontinuous at any rational number.

Hölder Regularity

S. Jaffard, *Wavelet Techniques in Multifractal Analysis*, In Proceedings of Symposia in Pure Mathematics, **Vol. 72**, 91-152, 2004.

Let $\alpha \in [0,1]$. A continuous and bounded real function g defined on $A \subset \mathbb{R}$ belongs to the Hölder space $\Lambda^\alpha(x)$ with $x \in A$ if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x-y|^\alpha,$$

for any $y \in A$. The Hölder exponent $h_g(x)$ of g at x is defined as follows:

$$h_g(x) = \sup\{\alpha \in [0,1] : g \in \Lambda^\alpha(x)\}.$$

Hölder Regularity of Cantor's Bijection

Theorem

For almost every $x \in I$, we have $h_{f_1}(x) = 1/2$ and $h_{f_2}(x) = 1/2$.

Remark

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ the sequence defined by

$$a_j = \begin{cases} 2^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases},$$

for any $j \in \mathbb{N}$ and set $x = [\boldsymbol{a}]$. For this particular point, we have $h_{f_1}(x) = 0$, so that f_1 is a multifractal function.