Extensions of superalgebras of Krichever-Novikov type

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Abstract

An explicit construction of central extensions of Lie superalgebras of Krichever-Novikov type is given. In the case of Jordan superalgebras related to the superalgebras of Krichever-Novikov type we calculate a 1-cocycle with coefficients in the dual space.

Key Words: Krichever-Novikov Lie superalgebras, Jordan superalgebras, Lie antialgebras, Gelfand-Fuchs cocycle.

1 Introduction

In 1987, I. M. Krichever and S. P. Novikov [11], [12] and [13] introduced and studied a family of Lie algebras generalizing the Virasoro algebra. Krichever-Novikov algebras are obtained as central extensions of the Lie algebras of meromorphic vector fields on a Riemann surface of arbitrary genus g with two marked points. M. Schlichenmaier studied the Krichever-Novikov Lie algebras for more than two marked points [23], [24] and [25]. He showed, in particular, the existence of local 2-cocycles and central extensions for multiple-point Krichever-Novikov algebras [27] extending the explicit formula of 2-cocycles due to Krichever and Novikov. Deformations on theses algebras were studied in [4] and [5].

The notion of Lie antialgebra was introduced by V. Ovsienko in [21], where the geometric origins were explained. It was then shown in [15] that these algebras are particular cases of Jordan superalgebras. The most important property of Lie antialgebras is their relationships with Lie superalgebras see [21], [19], [15] and [16]; different from the classical Kantor-Koecher-Tits construction for general Jordan superalgebras. One of the main example of [21] is the conformal Lie antialgebra $\mathcal{AK}(1)$ closely related to the Virasoro algebra and the Neveu-Schwarz Lie superalgebra $\mathcal{K}(1)$. In [19], S. Morier-Genoud studied an other important finite dimensional Lie antialgebra: $\mathcal{K}_3$, called the Kaplansky Jordan superalgebra which is related to $osp(1|2)$.

Lie superalgebras of Krichever-Novikov type, denoted $\mathcal{L}_{g,N}$ and the relation with Jordan superlagebras of Krichever-Novikov type, denoted $\mathcal{J}_{g,N}$, were studied by S. Leidwanger and S. Morier-Genoud in [16]. In this article, they found examples of Lie antialgebras generalizing $\mathcal{AK}(1)$, as the same way that $\mathcal{L}_{g,N}$ generalizes $\mathcal{K}(1)$. In this article, we are studying extensions on $\mathcal{L}_{g,N}$ and $\mathcal{J}_{g,N}$.

Our first theorem is an explicit formula for a non-trivial 2-cocycle on $\mathcal{L}_{g,N}$. This formula uses projective connections and is very similar to the formula of Krichever-Novikov and Schlichenmaier. We prove that the cohomology class of this 2-cocycle is independent of the choice of the projective connection. In the case of punctured Riemann sphere ($g = 0$), the constructed cocycle is unique provided it vanishes on the Lie subalgebra osp(1|2).

Our second theorem is an explicit formula for a 1-cocycle on $\mathcal{L}_{g,N}$ with coefficient in the dual space. Recently, P. Lecomte and V. Ovsienko introduced a cohomology theory of Lie antialgebras in [14]. In particular, they discovered two non-trivial cohomology classes of the conformal Lie antialgebra $\mathcal{AK}(1)$ analogous to the celebrated Gelfand-Fuchs class and to the Godbillon-Vey class. The cocycle on $\mathcal{L}_{g,N}$ studied in this article satisfies similar properties than those found in [14]. It is given by a very simple and geometrically natural formula that, perhaps, explains the geometric nature of Lie antialgebras associated to Riemann surfaces.

Interesting explicit examples of superalgebras arise in the case of the Riemann sphere with three marked points. These examples were thoroughly studied in [23] and [16]. The Lie superalgebra is denoted by $\mathcal{L}_{0,3}$ and the corresponding Jordan superalgebra by $\mathcal{J}_{0,3}$. These two algebras are closely related since $\mathcal{L}_{0,3}$ is the adjoint superalgebra of $\mathcal{J}_{0,3}$. Moreover, these algebras contain the conformal algebras:

$$\mathcal{L}_{0,3} \supset \mathcal{K}(1) \supset \mathrm{osp}(1|2) \qquad \text{and} \qquad \mathcal{J}_{0,3} \supset \mathcal{AK}(1) \supset \mathcal{K}_3.$$

We calculate explicitly the 2-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ that is unique provided it vanishes on the Lie subalgebra $\mathrm{osp}(1|2)$. This 2-cocycle induces a 1-cocycle on $\mathcal{L}_{0,3}$ with values in its dual space. Finally, we give an explicit formula for the 1-cocycle on $\mathcal{J}_{0,3}$ with values in its dual space.

The paper is organized as follows. In section two, we recall some definitions and main results on the Krichever-Novikov Lie algebras. In particular, we consider 2-cocycles on these Lie algebras and recall some tools that we will use in the computation of cocycles in the case of the Riemann sphere. In Section 3, we give the basic definitions of Lie superalgebras and of $\mathcal{L}_{g,N}$ with significant examples : $\mathcal{K}(1)$ and $\mathcal{L}_{0,3}$. We give a non trivial 2-cocycle on $\mathcal{L}_{g,N}$ (Theorem 1) and also go further with the structure $\mathcal{L}_{0,3}$. In Section 4, we recall the basic notions of Lie antialgebras with examples and relations to Lie superalgebras. In section 5, first We construct a 1-cocycle on $\mathcal{L}_{0,3}$ related to the 2-cocycle found in section 3. After, we give a 1-cocycle on $\mathcal{J}_{g,N}$ (Theorem 2) and construct the unique 1-cocycle on $\mathcal{J}_{0,3}$ that vanishes on $\mathcal{K}_3$.

2 Lie algebras of Krichever-Novikov type

In [11], [12] and [13], Krichever and Novikov introduced some generalizations of the well known Witt algebra and its central extension the Virasoro algebra [1]. In this section, we recall the definitions and main facts needed for the sequel. All the structures in this article will be considered over the field $\mathbb{C}$.

2.1 Definition and examples

Let M be a compact Riemann surface of genus g (i.e., a smooth projective curve over $\mathbb{C}$). Consider the union of two sets of ordered disjoint points called *punctures*

$$A = \underbrace{(P_1, \ldots, P_K)}_{:=I} \cup \underbrace{(Q_1, \ldots, Q_{N-K})}_{:=O}$$

where $N, K \in \mathbb{N}\backslash\{0\}$ with $N \geq 2$ and $1 \leq K < N$. We call I, the set of *in-points*, and O the set of *out-points*. Denote by $\mathfrak{a}_{g,N}$ the associative algebra of meromorphic functions on M which are holomorphic outside of A.

[1] A global overview (made in 2003) of this theory can be found in [29].

The Krichever-Novikov algebra $\mathfrak{g}_{g,N}$ is the Lie algebra of meromorphic vector fields on M which are holomorphic outside of A ($\mathfrak{g}_{g,N}$ is equipped with the usual Lie bracket of vector fields). We will use the same symbol for the vector field and its local representation so that the Lie bracket is

$$\left[e(z)\frac{d}{dz}, f(z)\frac{d}{dz}\right] = \big(e(z)f'(z) - f(z)e'(z)\big)\frac{d}{dz}.$$

If $g=0$, one considers the Riemann sphere $\mathbb{CP}^1$ with punctures. The moduli space $\mathcal{M}_{0,N}$ is of dimension $N-3$. This means that, for $N \leq 3$, the points can be chosen in an arbitrary way providing isomorphic algebraic structures. Note also that $\mathbb{CP}^1$ can be equipped with a "quasi-global" coordinate z.

In the case $g=0$ and $N=2$, one can take $I=\{0\}$ and $O=\{\infty\}$ and the Krichever-Novikov algebra $\mathfrak{g}_{0,2}$ is nothing but the Witt algebra. It admits a basis $\{e_n = z^{n+1}\frac{d}{dz} : n \in \mathbb{Z}\}$ satisfying the relations:

$$[e_n, e_m] = (m-n)e_{n+m}.$$

The (unique) non-trivial central extension of the Witt algebra is well-known, it is called the Virasoro algebra. This algebra has a basis $\{e_n = z^{n+1}\frac{d}{dz} : n \in \mathbb{Z}\}$ together with the central element c, such that

$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n,-m}c, \qquad [e_n, c] = 0.$$

The algebra of functions $\mathfrak{a}_{0,2}$ is the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$.

Another simple example considered in [26] and further in [4] is the case $g=0$ and $N=3$. The marked points are then chosen as follows: $I=\{\alpha, -\alpha\}$ and $O=\{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$. The Lie algebra $\mathfrak{g}_{0,3}$ is spanned by the following vector fields, for all $k \in \mathbb{Z}$:

$$V_{2k}(z) = z(z-\alpha)^k(z+\alpha)^k\frac{d}{dz}, \qquad V_{2k+1}(z) = (z-\alpha)^{k+1}(z+\alpha)^{k+1}\frac{d}{dz}. \tag{1}$$

2.2 Construction of a 2-cocycle on $\mathfrak{g}_{g,N}$

Let us recall the construction of a 2-cocycle on $\mathfrak{g}_{g,N}$ due to Krichever and Novikov [11] and [12] also studied by Schlichenmaier [27].

Given a Riemann surface and $(U_\alpha, z_\alpha)_{\alpha\in J}$ a covering by holomorphic coordinates with transition functions $z_\beta = g_{\beta\alpha}(z_\alpha)$, a *projective connection* is a system of functions $R = (R_\alpha(z_\alpha))_{\alpha\in J}$ transforming as

$$R_\beta(z_\beta).\left(g'_{\beta\alpha}\right)^2 = R_\alpha(z_\alpha) + S(g_{\beta\alpha}),$$

where

$$S(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$$

is the *Schwarzian derivative* (see [22]) and where $'$ denotes differentiation with respect to the coordinate z_α. It is a classical result that every Riemann surfaces admits a holomorphic projective connection, see [8] or [10] (p202).

Krichever and Novikov defined a 2-cocycle on $\mathfrak{g}_{g,N}$:

$$\gamma_{\mathcal{C},R}\left(e(z)\frac{d}{dz}, f(z)\frac{d}{dz}\right) = \frac{1}{2i\pi}\int_{\mathcal{C}}\left(\frac{1}{2}(e'''f - ef''') - R(e'f - ef')\right)dz, \tag{2}$$

where $\mathcal{C}$ can be taken as follows: $\mathcal{C} = \sum_{i=1}^{K} \mathcal{C}_i$ and where $\mathcal{C}_i$ are (small) circles around the points in I. Note that, in the case $g = 0$, one can consider $R \equiv 0$. The 2-cocycle (2) can be understood as a generalization of the famous Gelfand-Fuchs cocycle, see [6].

M. Schlichenmaier [27] proved that the cohomology class of $\gamma_{\mathcal{C},R}$ does not depend on the chosen connection R (one can prove it by simple calculation). He also showed that this cocycle is cohomologically non-trivial, local and that every cocycle of $\mathfrak{g}_{g,N}$ is either a coboundary or a scalar multiple of $\gamma_{\mathcal{C},R}$ with R a meromorphic projective connection which is holomorphic outside A.

This integral in (2) is written in the complex analytic setting, since one integrates over a circle around a point, the result is given by the theorem of residues. The Riemann sphere can be viewed as the structure of the extended complex plane $\widehat{\mathbb{C}}$, see [17]. In the next section, we calculate the residue at ∞ and considering the function $f_{1/z} : z \mapsto f(\frac{1}{z})$, one has:

$$Res_{\infty}(f) = -Res_0 \left(\frac{f_{1/z}}{z^2} \right),$$

and moreover, if $z_0 \in \mathbb{C}$ is a pole of f, of order $p \in \mathbb{N}\backslash\{0\}$, then

$$Res_{z_0} f = \frac{1}{(p-1)!} \lim_{z \to z_0} D^{p-1}[(z - z_0)^p f(z)].$$

3 Lie superalgebras of K-N type and their central extensions

In this section, we recall the notion of Lie superalgebra of Krichever-Novikov type, $\mathcal{L}_{g,N}$. We show the existence of a non-trivial 2-cocycle on $\mathcal{L}_{g,N}$ satisfying similar properties to those of the cocycle (2). We consider, in particular, the case $g = 0$ and $N = 3$, namely, the Lie superalgebra $\mathcal{L}_{0,3}$ and compute the 2-cocycle explicitly.

3.1 Definition and examples of Lie superalgebras

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space, $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, equipped with a bilinear product (Lie bracket), such that

(LS1) super skewsymmetry : $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$

(LS2) super Jacobi identity : $(-1)^{\bar{x}\bar{z}}\,[x, [y, z]] + (-1)^{\bar{y}\bar{x}}\,[y, [z, x]] + (-1)^{\bar{z}\bar{y}}\,[z, [x, y]] = 0$

for all homogeneous elements x, y, z in $\mathcal{L}$. The subspace $\mathcal{L}_0$ is the space of *even elements* and the subspace $\mathcal{L}_1$ is that of *odd elements*. The *degree* of a homogeneous element x is denoted by $\bar{x}$, i.e. $\bar{x} = i$ for $x \in \mathcal{L}_i$.

Example 3.1. The *conformal Lie superalgebra* $\mathcal{K}(1)$ is an infinite-dimensional Lie superalgebra with basis $\{e_n,\ n \in \mathbb{Z}\}$ of the even part and $\{b_i,\ i \in \mathbb{Z} + \frac{1}{2}\}$ of the odd part satisfying the relations:

$$\begin{cases} [e_n, e_m] = (m - n)e_{n+m} \\ [e_n, b_i] = \frac{1}{2}(i - \frac{n}{2})b_{i+n} \\ [b_i, b_j] = e_{i+j}. \end{cases}$$

The even part of $\mathcal{K}(1)$ coincides thus with the Witt algebra $\mathfrak{g}_{0,2}$. The following elements $\{b_{-\frac{1}{2}}, b_{\frac{1}{2}}, e_{-1}, e_0, e_1\}$ span the classical simple Lie superalgebra osp(1|2).

3.2 The Lie superalgebras $\mathcal{L}_{g,N}$

The Lie superalgebras of Krichever-Novikov type were studied in [16]. Let us briefly recall the main definition.

We denote by $\mathcal{F}_\lambda$, where $\lambda \in \mathbb{Z} \cup \frac{1}{2} + \mathbb{Z}$, the space of meromorphic tensor densities of weight λ on M. The space $\mathcal{F} = \oplus_\lambda \mathcal{F}_\lambda$ is a Poisson algebra with the following bilinear operations (given in local coordinates):

$$\begin{array}{rcllcl} \bullet & : & \mathcal{F}_\lambda \times \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\lambda+\mu} : & (e(z)dz^\lambda, f(z)dz^\mu) & \longmapsto & e(z)f(z)dz^{\lambda+\mu} \\ \{,\} & : & \mathcal{F}_\lambda \times \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\lambda+\mu+1} : & (e(z)dz^\lambda, f(z)dz^\mu) & \longmapsto & \big(\mu e'(z)f(z) - \lambda e(z)f'(z)\big)\, dz^{\lambda+\mu+1}. \end{array}$$

One checks that the above operations are independent of the choice of the coordinate and, therefore, are globally defined.

We have the Lie algebra isomorphism $\mathfrak{g}_{g,N} \cong \mathcal{F}_{-1}$, and the natural action of the Lie algebra $\mathfrak{g}_{g,N}$ on $\mathcal{F}_{-1/2}$ is given by the above Poisson bracket.

Definition 3.2. The *Lie superalgebra of Krichever-Novikov*, denoted by $\mathcal{L}_{g,N}$, is the vector space $(\mathcal{L}_{g,N})_0 \oplus (\mathcal{L}_{g,N})_1 = \mathfrak{g}_{g,N} \oplus \mathcal{F}_{-1/2}$ with the Lie bracket defined by

$$[e(z)(dz)^{-1}, f(z)(dz)^{-1}] = \{e(z)(dz)^{-1}, f(z)(dz)^{-1}\}$$

$$[e(z)(dz)^{-1}, \psi(z)(dz)^{-1/2}] = \{e(z)(dz)^{-1}, \psi(z)(dz)^{-1/2}\}$$

$$[\varphi(z)(dz)^{-1/2}, \psi(z)(dz)^{-1/2}] = \frac{1}{2}\varphi(z)(dz)^{-1/2} \bullet \psi(z)(dz)^{-1/2}.$$

The axioms of Lie superalgebras can be easily checked.

More precisely, we can write in coordinates:

$$\begin{cases} \left[e(dz)^{-1}, f(dz)^{-1}\right] = (-e'f + ef')\,(dz)^{-1} \\ \left[e(dz)^{-1}, \psi(dz)^{-1/2}\right] = \left(-\frac{1}{2}e'\psi + e\psi'\right)(dz)^{-1/2} \\ \left[\varphi(dz)^{-1/2}, \psi(dz)^{-1/2}\right] = \frac{1}{2}\,\varphi\psi(dz)^{-1}. \end{cases}$$

Example 3.3. a) In the case of two marked points $A = \{0\} \cup \{\infty\}$ on the Riemann sphere, we can identify $\mathcal{L}_{0,2}$ with $\mathcal{K}(1)$, see example 3.1. We have the following identification:

$$e_n = z^{n+1}(dz)^{-1}, \quad b_i = z^{i+1/2}(dz)^{-1/2}.$$

b) Consider the Lie superalgebra $\mathcal{L}_{0,3}$ associated with the Riemann sphere with three punctures $A = \{-\alpha, \alpha\} \cup \{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$. According to [26], see also [4], the even part of $\mathcal{L}_{0,3}$, namely $\mathfrak{g}_{0,3}$, has the basis (1). The odd part, $\mathcal{F}_{-1/2}$, according to [16], has the basis, for all $k \in \mathbb{Z}$:

$$\varphi_{2k+\frac{1}{2}}(z) = \sqrt{2}z(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2}, \qquad \varphi_{2k-\frac{1}{2}}(z) = \sqrt{2}(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2}. \quad (3)$$

The explicit Lie bracket of this algebra is expressed in details in [16] and, moreover, it is shown that the sub-superalgebra $\mathcal{L}_{0,3}^- = \left\langle V_n \,:\, n \le 0\,;\; \varphi_i \,:\, i \le \frac{1}{2}\right\rangle$ of $\mathcal{L}_{0,3}$ is isomorphic to $\mathcal{K}(1)$.

3.3 A non-trivial 2-cocycle on $\mathcal{L}_{g,N}$

In this section, we show that every Lie superalgebra $\mathcal{L}_{g,N}$ has a non-trivial central extension. To this end, we construct a non-trivial 2-cocycle quite similar to (2).

Recall that a 2-*cocycle* on a Lie superalgebra $\mathcal{L}$ is an even bilinear function $c : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ satisfying the following conditions:

(C1) super skewsymmetry: $c(u, v) = -(-1)^{\bar{u}\bar{v}} c(v, u)$

(C2) super Jacobi identity: $c(u, [v, w]) = c([u, v], w) + (-1)^{\bar{u}\bar{v}} c(v, [u, w])$

for every homogeneous elements $u, v, w \in \mathcal{L}$. As in the usual Lie case, a 2-cocycle defined a central extension of $\mathcal{L}$. A 2-cocycle is called *trivial*, or a *coboundary* if it is of the form $c(u, v) = f([u, v])$, where f is a linear function on $\mathcal{L}$. Otherwise, c is called *non-trivial*. The space of all 2-cocycles is denoted by $Z^2(\mathcal{L})$ and the space of 2-coboundaries by $B^2(\mathcal{L})$, the quotient-space $H^2(\mathcal{L}) = Z^2(\mathcal{L})/B^2(\mathcal{L})$ is called the second cohomology space of $\mathcal{L}$. This space classifies non-trivial central extensions of $\mathcal{L}$.

The first result of this paper is the following.

Theorem 1. *(i) The even bilinear map $c : \mathcal{L}_{g,N} \times \mathcal{L}_{g,N} \to \mathbb{C}$ given by*

$$\begin{aligned}
c\left(e(z)\tfrac{d}{dz}, f(z)\tfrac{d}{dz}\right) &= \frac{-1}{2i\pi}\int_{\mathcal{C}} \frac{1}{2}(e'''f - ef''') - R(e'f - ef')dz,\\
c\left(\varphi(z)dz^{-1/2}, \psi(z)dz^{-1/2}\right) &= \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{1}{2}\left(\varphi''\psi + \varphi\psi''\right) - \frac{1}{2}R\,\varphi\psi dz,\\
c\left(e(z)\tfrac{d}{dz}, \psi(z)dz^{-1/2}\right) &= 0
\end{aligned} \tag{4}$$

is a non-trivial 2-cocycle (where $\mathcal{C}$ is as described in (2)).

(ii) The cohomology class of c does not depend on the choice of the projective connection.

Proof. Part (i). To show that the above integral is well defined, one notices that, after coordinate changes $z_\beta = g_{\beta,\alpha}(z_\alpha)$, the expressions in the both parts of (4) are transformed as 1-forms. The cocycle condition is then straightforward. Note that the minus sign in the first equation comes from the super Jacoby identity in the definition of a 2-cocycle. To check that the cocycle c is non-trivial, it suffices to notice that the corresponding central extension of $\mathcal{L}_{g,N}$ always contains the Virasoro algebra. Or either, since c is cohomologically non-trivial on the even part (see [4], p933) it is also the case on $\mathcal{L}_{g,N}$.

Part (ii). Let R' be a different projective connection, then $R - R'$ is a well-defined quadratic differential. The 2-cocycle $c - c'$ depends only on the Lie bracket of the elements, on the odd part, we have :

$$c_R(\varphi, \psi) - c'_{R'}(\varphi, \psi) = \frac{1}{2i\pi}\int_{\mathcal{C}} -\frac{1}{2}(R - R')\varphi\psi dz = \frac{1}{2i\pi}\int_{\mathcal{C}} \left[(R' - R)(dz)^2 \bullet [\varphi, \psi](dz)^{-1}\right]$$

and therefore is a coboundary. On the even part, Schlichenmaier has already done it in [27], p64.

□

3.4 The case of genus zero

Let us now assume that $g = 0$ and consider the Lie superalgebra $\mathcal{L}_{0,N}$. Choose the projective connection $R \equiv 0$ (in the standard flat coordinate z) adapted to the standard projective structure on $\mathbb{CP}^1$.

An important property of $\mathcal{L}_{0,N}$ is that it contains a subalgebra isomorphic to osp(1|2) that consists in holomorphic vector fields and $-1/2$-densities. The Lie superalgebra $\mathcal{L}_{0,N}$ also contains many copies of the conformal Lie superalgebra $\mathcal{K}(1)$ consisting in densities holomorphic outside two points of the set A.

Proposition 3.4. *There exists a unique (up to a constant) 2-cocycle on $\mathcal{L}_{0,N}$ that vanishes on the Lie subalgebra* osp(1|2)*.*

Proof. The cocycle (4) with $R \equiv 0$ vanishes on osp(1|2). When $N = 2$, since the cocycle vanishes on osp(1|2), it is unique (see [20], section 3.2). We conclude by induction using the embedding $\mathcal{L}_{0,N-1} \subset \mathcal{L}_{0,N}$. □

Let us now compute the explicit formula for the 2-cocycle (4) with $R \equiv 0$ on $\mathcal{L}_{0,3}$.

Proposition 3.5. *Up to a constant, the 2-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ is given by*

$$\begin{array}{rcl} c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l+\frac{1}{2}}\right) & = & 0 \\ c\left(\varphi_{2k-\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) & = & 0 \\ c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) & = & 4k(2k+1)\delta_{k+l,0} + 8\alpha^2 k(k-1)\delta_{k+l,1} \\ c\left(V_{2k}, V_{2l}\right) & = & -2k(4k^2-1)\delta_{k+l,0} - 8\alpha^2 k(k-1)(2k-1)\delta_{k+l,1} \\ & & -8\alpha^4 k(k-1)(k-2)\delta_{k+l,2} \\ c\left(V_{2k+1}, V_{2l+1}\right) & = & -8\alpha^2(k+1)k(k-1)\delta_{k+l,0} - 4k(k+1)(2k+1)\delta_{k+l,-1} \\ c\left(V_{2k}, V_{2l+1}\right) & = & 0, \end{array} \tag{5}$$

for all $k, l \in \mathbb{Z}$.

Proof. Let us give the details for one calculation, the others can be done in the same way.

$$\begin{array}{rcl} c\left(V_{2k+1}, V_{2l+1}\right) & = & \displaystyle\frac{-1}{2i\pi}\frac{1}{2}\int_{\mathcal{C}_\alpha \cup \mathcal{C}_{-\alpha}} \{(z^2-\alpha^2)^{k+1}\}'''(z^2-\alpha^2)^{l+1} \\ & & \qquad -\{(z^2-\alpha^2)^{l+1}\}'''(z^2-\alpha^2)^{k+1}dz \\ & = & \displaystyle\frac{1}{2i\pi}\int_{\mathcal{C}_\infty} 6z\{(k+1)k-(l+1)l\}(z^2-\alpha^2)^{k+l} \\ & & \qquad +4z^3\{(k+1)k(k-1)-(l+1)l(l-1)\}(z^2-\alpha^2)^{k+l-1}dz \\ & = & \displaystyle -6\{(k+1)k-(l+1)l\}Res_0\left(\frac{(1-z^2\alpha^2)^{k+l}}{z^{2k+2l+3}}\right) \\ & & \displaystyle -4\{(k+1)k(k-1)-(l+1)l(l-1)\}Res_0\left(\frac{(1-z^2\alpha^2)^{k+l-1}}{z^{2k+2l+3}}\right). \end{array}$$

Consider the residues, if $k + l \leq -2$ the functions are holomorphics near 0 and the residues vanish and if $k + l \geq 1$ they also vanish taking into account the Taylor development. So, focus when

$$k+l=0, \quad \text{then} \quad Res_0\left(\frac{1}{z^3}\right)=0 \quad \text{and} \quad Res_0\left(\frac{(1-z^2\alpha^2)^{-1}}{z^3}\right)=\alpha^2$$

$$k+l=-1, \quad \text{then} \quad Res_0\left(\frac{(1-z^2\alpha^2)^{-1}}{z}\right)=1 \quad \text{and} \quad Res_0\left(\frac{(1-z^2\alpha^2)^{-2}}{z}\right)=1.$$

Finally, we obtain

$$\begin{aligned} c\,(V_{2k+1},V_{2l+1}) &= -6\{(k+1)k-(l+1)l\}\delta_{k+l,-1} \\ &\quad -4\{(k+1)k(k-1)-(l+1)l(l-1)\}(\alpha^2\delta_{k+l,0}+\delta_{k+l,-1}) \\ &= -8\alpha^2(k+1)k(k-1)\delta_{k+l,0}-4k(k+1)(2k+1)\delta_{k+l,-1}. \end{aligned}$$

Hence the result. □

4 Jordan superlagebras of Krichever-Novikov type

In this section, we consider a special type of Jordan superalgebras, introduced by V. Ovsienko in [21] under the name of "Lie antialgebras". Lie antialgebras were studied in [19], [21] and [16] and the cohomology theory of theses algebras is developed in [14]. Lie antialgebras of Krichever-Novikov type, $\mathcal{J}_{g,N}$, were introduced in [15]. We calculate a non-trivial 1-cocycle on $\mathcal{J}_{g,N}$ with values in the dual space $\mathcal{J}^*_{g,N}$. The construction is very similar to that of the 2-cocycle (4) and extends the cocycle found in [14].

4.1 Definition and examples of Lie antialgebras

Definition 4.1. A *Lie antialgebra* on $\mathbb{C}$ is a $\mathbb{Z}_2$-graded *supercommutative* algebra $\mathcal{A}=\mathcal{A}_0\oplus\mathcal{A}_1$ with a product:

$$x\bullet y=(-1)^{\bar{x}\bar{y}}y\bullet x,$$

for all homogeneous elements $x,y\in\mathcal{A}$, satisfying the following conditions.

(i) The subalgebra $\mathcal{A}_0$ is *associative.*

(ii) For every $a\in\mathcal{A}_1$, the operator of right multiplication by a is an (odd) derivation of $\mathcal{A}$, i.e.,

$$(x\bullet y)\bullet a=(x\bullet a)\bullet y+(-1)^{\bar{x}}\,x\bullet(y\bullet a)\,, \tag{6}$$

for all homogeneous $x,y\in\mathcal{A}$.

Note that, in the case where $\mathcal{A}$ is generated by its odd part $\mathcal{A}_1$, the first axiom of associativity is a consequence of (6), cf. [21].

Example 4.2. The first example of finite-dimensional Lie antialgebra is the famous tiny Kaplansky superalgebra, denoted by $\mathcal{K}_3$. It was first studied by K. McCrimmon in [18] and after by Morier-Genoud in [19] under the name of asl_2. The basis is $\{\varepsilon;a,b\}$ where ε is even and a,b are odds. It is defined by the following relations:

$$\begin{cases} \varepsilon\bullet\varepsilon=\varepsilon \\ \varepsilon\bullet a=\frac{1}{2}a \quad \varepsilon\bullet b=\frac{1}{2}b \\ a\bullet b=\frac{1}{2}\varepsilon. \end{cases}$$

The algebra $\mathcal{K}_3$ is an example of exceptional simple Jordan superalgebra.

Example 4.3. The second important example is an infinite-dimensional algebra, denoted by $\mathcal{AK}(1)$. Its geometric origins are related to the contact structure on the supercircle $S^{1|1}$. The basis of $\mathcal{AK}(1)$ is $\{\varepsilon_n : n \in \mathbb{Z}\} \oplus \{a_i : i \in \mathbb{Z} + \frac{1}{2}\}$ and the relations are

$$\begin{cases} \varepsilon_n \centerdot \varepsilon_m = \varepsilon_{n+m} \\ \varepsilon_n \centerdot a_i = \frac{1}{2} a_{i+n} \\ a_i \centerdot a_j = \frac{1}{2}(j-i)\varepsilon_{i+j}. \end{cases}$$

Note that we can see $\langle \varepsilon_0, a_{-1/2}, a_{1/2} \rangle$ as a subalgebra of $\mathcal{AK}(1)$ isomorphic to $\mathcal{K}_3$.

4.2 Relations to Lie superalgebras

A natural way to link Lie antialgebras and Lie superalgebras is to consider the Lie superalgebra of derivations $Der(\mathcal{A})$. In particular, one has : $Der(\mathcal{K}_3) \cong \mathrm{osp}(1|2)$ and $Der(\mathcal{AK}(1)) \cong \mathcal{K}(1)$, cf. [21].

An other way to associate a Lie superalgebra $\mathcal{G}_\mathcal{A}$ to an arbitrary Lie antialgebra $\mathcal{A}$, called the *adjoint Lie superalgebra*, was elaborated in [21]. Consider the $\mathbb{Z}_2$-graded space $\mathcal{G}_\mathcal{A} = \mathcal{G}_0 \oplus \mathcal{G}_1$ where, $\mathcal{G}_1 = \mathcal{A}_1$ and $\mathcal{G}_0 := (\mathcal{A}_1 \otimes \mathcal{A}_1)/S$ and where S is the ideal generated by

$$\{\, a \otimes b - b \otimes a,\; a \centerdot \alpha \otimes b - a \otimes b \centerdot \alpha \;\mid\; a, b \in \mathcal{A}_1, \alpha \in \mathcal{A}_0 \,\}.$$

If we denote by $a \odot b$ the image of $a \otimes b$ in $\mathcal{G}_0$, one can write the Lie (super) bracket :

$$\begin{aligned} [a,b] &= a \odot b \\ [a \odot b, c] &= a \centerdot (b \centerdot c) + b \centerdot (a \centerdot c) \\ [a \odot b, c \odot d] &= 2a \centerdot (b \centerdot c) \odot d + 2b \centerdot (a \centerdot d) \odot c \end{aligned}$$

There is a natural action of $\mathcal{G}_\mathcal{A}$ on the corresponding Lie antialgebra $\mathcal{A}$, so that there is a Lie algebra homomorphism

$$\mathcal{G}_\mathcal{A} \to Der(\mathcal{A}).$$

Indeed, the action of the odd part $\mathcal{G}_1$ is given by the right multiplication and this generates the action of $\mathcal{G}_0$, cf [21]. Note that, in the above examples, one has : $\mathcal{G}_{\mathcal{K}_3} \cong \mathrm{osp}(1|2)$ and $\mathcal{G}_{\mathcal{AK}(1)} \cong \mathcal{K}(1)$.

In general, the adjoint Lie superalgebra is not isomorphic to the Lie superalgebra of derivations.

4.3 Definition of $\mathcal{J}_{g,N}$

A new series of Lie antialgebras extended $\mathcal{AK}(1)$ was found by Leidwanger and Morier-Genoud, see [16]. These algebras are related to Riemann surfaces with marked points and are called the *Jordan superalgebras of Krichever-Novikov type*, $\mathcal{J}_{g,N}$. The even part of $\mathcal{J}_{g,N}$ is the space of meromorphic functions, $\mathfrak{a}_{g,N} \cong \mathcal{F}_0$, while the odd part is the space of $-1/2$-densities.

Definition 4.4. The Lie antialgebra $\mathcal{J}_{g,N}$ is the vector superspace $\mathfrak{a}_{g,N} \oplus \mathcal{F}_{-1/2}$ equipped with the product

$$\begin{aligned} e(z) \centerdot f(z) &= e(z) \bullet f(z) \\ e(z) \centerdot \psi(z)(dz)^{-1/2} &= \frac{1}{2} e(z) \bullet \psi(z)(dz)^{-1/2} \\ \varphi(z)(dz)^{-1/2} \centerdot \psi(z)(dz)^{-1/2} &= \{\varphi(z)(dz)^{-1/2}, \psi(z)(dz)^{-1/2}\}. \end{aligned}$$

More precisely, we can write:

$$\begin{cases} e \centerdot f = ef \\ e \centerdot \psi\,(dz)^{-1/2} = \frac{1}{2} e\,\psi\,(dz)^{-1/2} \\ \varphi\,(dz)^{-1/2} \centerdot \psi\,(dz)^{-1/2} = -\frac{1}{2}\varphi'\,\psi + \frac{1}{2}\varphi\,\psi' \end{cases}$$

It is shown in [16], that the adjoint Lie superalgebra of $\mathcal{J}_{g,N}$ coincides with $\mathcal{L}_{g,N}$.

Example 4.5. a) In the case of two marked points $A = \{0\} \cup \{\infty\}$ on the Riemann sphere, the algebra $\mathcal{J}_{0,2}$ can be identified with $\mathcal{AK}(1)$.

b) A beautiful example in the case of three punctures on the Riemann sphere is considered in [16]. One can fix $A = \{-\alpha, \alpha\} \cup \{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$. The Jordan superalgebra $\mathcal{J}_{0,3}$ has the basis

$$G_{2k}(z) = (z-\alpha)^k (z+\alpha)^k, \qquad G_{2k+1}(z) = z(z-\alpha)^k(z+\alpha)^k,$$
$$\varphi_{2k+\frac{1}{2}}(z) = \sqrt{2} z(z-\alpha)^k (z+\alpha)^k \, dz^{-1/2}, \quad \varphi_{2k-\frac{1}{2}}(z) = \sqrt{2}(z-\alpha)^k(z+\alpha)^k \, dz^{-1/2},$$

where $k \in \mathbb{Z}$. Remark that the generators of the odd parts of $\mathcal{L}_{g,N}$ and $\mathcal{J}_{g,N}$ are the same. The sub-superalgebra $\mathcal{J}_{0,3}^{-} = \left\langle G_n : n \leq 0\,, \varphi_i : i \leq \frac{1}{2} \right\rangle$ is isomorphic to $\mathcal{AK}(1)$. More precisely, the embedding $\iota : \mathcal{AK}(1) \hookrightarrow \mathcal{J}_{0,3}$ is defined on the generators as follows:

$$\iota(\varepsilon_{-1}) = G_0 + 2\alpha G_{-1} + 2\alpha^2 G_{-2}, \qquad \iota(\varepsilon_1) = G_0 - 2\alpha G_{-1} + 2\alpha^2 G_{-2}, \qquad \iota(\varepsilon_0) = G_0$$
$$\iota(a_{-\frac{1}{2}}) = \frac{1}{2\sqrt{\alpha}}(\varphi_{1/2} + \alpha\varphi_{-1/2}), \qquad \iota(a_{\frac{1}{2}}) = \frac{1}{2\sqrt{\alpha}}(\varphi_{1/2} - \alpha\varphi_{-1/2}),$$

see [16] for the details.

5 1-cocycles with values in the dual space

In this section, we construct 1-cocycles on $\mathcal{L}_{g,N}$ and $\mathcal{J}_{g,N}$ with values in the dual space. In the Lie case, existence of such a 1-cocycle is almost equivalent to the existence of a 2-cocycle with trivial coefficients (4). In the Jordan case, the situation is different. It was proved in [21] and [14] that the Jordan superalgebra $\mathcal{J}_{g,N}$ has no non-trivial central extensions. Therefore, there is no 2-cocycle on $\mathcal{J}_{g,N}$ analogous to (4). However, there exists a nice construction of 1-cocycle that has very similar properties.

5.1 1-cocycle on the K-N Lie superalgebras

Given a 2-cocycle on a Lie (super)algebra $c : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$, one can define a 1-cocycle, C, on $\mathcal{L}$ with values in the dual space $\mathcal{L}^*$. The definition is as follow

$$\langle C(x), y \rangle := c(x, y), \tag{7}$$

for all $x, y \in \mathcal{L}$. The 1-cocycle condition:

$$C\left([x,y]\right) = ad_x^*(C(y)) - (-1)^{\bar{x}\bar{y}} ad_y^*(C(x))$$

follows from the 2-cocycle condition for c. Note that the converse construction does not work since c is not necessarily skewsymmetric.

The 2-cocycle (4) defines, therefore, a 1-cocycle on every Lie superalgebra $\mathcal{L}_{g,N}$. Note that the dual space $\mathcal{L}_{g,N}^*$ has a nice geometric interpretation. Its even part coincides with the space $\mathcal{F}_2$ of meromorphic quadratic differentials, while the odd part consists in meromorphic 3/2-densities, $\mathcal{F}_{3/2}$. The coadjoint action of $\mathcal{L}_{g,N}$ reads:

$$\begin{aligned} ad^*_{\varphi(z)(dz)^{-1/2}}\left(u(z)(dz)^2 \oplus w(z)(dz)^{3/2}\right) &= \left(\frac{3}{2}\varphi' w + \frac{1}{2}\varphi w'\right)(dz)^2 \oplus -\frac{1}{2}\varphi u\,(dz)^{3/2} \\ ad^*_{e(z)(dz)^{-1}}\left(u(z)(dz)^2 \oplus w(z)(dz)^{3/2}\right) &= \left(2e'u + eu'\right)(dz)^2 \oplus \left(\frac{3}{2}e'w + ew'\right)(dz)^{3/2} \end{aligned}$$

where u, w, e and φ are some meromorphic functions on the surface.

Proposition 5.1. *On $\mathcal{L}_{g,N}$, a 1-cocycle is given by*

$$\begin{aligned} C\left(e(z)\tfrac{d}{dz}\right) &= -\left(e''' - 2Re' - R'e\right)dz^2, \\ C\left(\varphi(z)dz^{-1/2}\right) &= \left(\varphi'' - \tfrac{1}{2}R\varphi\right)dz^{3/2} \end{aligned} \tag{8}$$

Proof. Straightforward from (4). □

Example 5.2. In the case of Riemann sphere ($g = 0$), the 1-cocycle (7) related to (4) with $R \equiv 0$, reads simply:

$$C\left(e(z)\frac{d}{dz}\right) = -e'''(z)\,dz^2, \qquad C\left(\psi(z)\frac{d}{dz^{1/2}}\right) = \psi''(z)\,dz^{3/2}, \tag{9}$$

where z is the standard coordinate.

5.2 1-cocycle on $\mathcal{L}_{0,3}$

In the case of the Lie superalgebra $\mathcal{L}_{0,3}$ (and further with the Jordan superalgebra $\mathcal{J}_{0,3}$) the constructed 1-cocycle can be calculated explicitly.

The space $\mathcal{L}_{0,3}^*$ has the following basis:

$$\begin{aligned} \varphi^*_{2k-1/2} &= \frac{1}{\sqrt{2}}z(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, & V^*_{2k} &= (z^2-\alpha^2)^{-k-1}(dz)^2, \\ \varphi^*_{2k+1/2} &= \frac{1}{\sqrt{2}}(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, & V^*_{2k+1} &= z(z^2-\alpha^2)^{-k-2}(dz)^2, \end{aligned}$$

dual to (1) and (3).

Proposition 5.3. *The 1-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ related to (9) that vanishes on $osp(1|2)$ is given by:*

$$\begin{aligned} C(V_n) &= -n(n-1)(n+1)V^*_{-n} - 2\alpha^2 n(n-2)(n-1)V^*_{-n+2} - \alpha^4 n(n-2)(n-4)V^*_{-n+4} \\ C(V_m) &= -(m+1)m(m-1)V^*_{-m} - \alpha^2(m+1)(m-1)(m-3)V^*_{-m+2}, \\ C(\varphi_i) &= 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2}, \\ C(\varphi_j) &= 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2}, \end{aligned}$$

where $i - \frac{1}{2}$, n are even and $j - \frac{1}{2}$, m are odd.

Proof. This is a simple application of the general formulas (8) with $R \equiv 0$. □

5.3 Modules and cocycles on Lie antialgebras

Cohomology of Lie antialgebras was studied in [14]. In particular, a (unique) 1-cocycle $C : \mathcal{AK}(1) \longrightarrow \mathcal{AK}(1)^*$ vanishing on $\mathcal{K}_3$ was constructed.

Let us recall several basic notions from [14]. Let $\mathcal{B}$ be a $\mathbb{Z}_2$-graded vector space and $\rho : \mathcal{A} \to End(\mathcal{B})$ an even linear function. If $\mathcal{A} \oplus \mathcal{B}$ equipped with the product

$$(a, b) \centerdot (a', b') = (a \centerdot a', \rho_a(b') + (-1)^{\bar{a'}\bar{b}} \rho_{a'}(b))$$

for all homogeneous elements $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, is a Lie antialgebra then $(\mathcal{B}, \rho)$ is called an $\mathcal{A}$-*module*. This structure is called a *semi-direct sum* and denoted by $\mathcal{A} \ltimes \mathcal{B}$. Given an $\mathcal{A}$-module $\mathcal{B}$, the dual space $\mathcal{B}^*$ is also an $\mathcal{A}$-module, the $\mathcal{A}$-action being given by

$$\langle \rho_a^* u, b \rangle := (-1)^{\bar{a}\bar{u}} \langle u, \rho_a b \rangle \,, \tag{10}$$

for all homogeneous elements $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $u \in \mathcal{B}^*$.

A 1-*cocycle* on a Lie antialgebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module $\mathcal{B}$, is an even linear map $\mathcal{C} : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\mathcal{C}\,(u \centerdot v) = \rho_u(\mathcal{C}(v)) + (-1)^{\bar{u}\bar{v}} \rho_v(\mathcal{C}(u)). \tag{11}$$

A Lie antialgebra is tautologically a module over itself, the adjoint action $ad : \mathcal{A} \longrightarrow End(\mathcal{A})$ defined such that $ad_a(a') = a \centerdot a'$ for all $a, a' \in \mathcal{A}$. So that, the dual space, $\mathcal{A}^*$, is an $\mathcal{A}$-module as well.

5.4 1-cocycles on $\mathcal{J}_{g,N}$

It was already proved in [21] that a Lie antialgebra has no non-trivial central extensions, provided the even part contains a unit element. It follows that the algebras $\mathcal{J}_{g,N}$ have no non-trivial 2-cocycles. However, one has the following :

Theorem 2. *(i) The expression*

$$\mathcal{C}\,(\varepsilon(z)) = -\varepsilon'(z)dz, \qquad \mathcal{C}\left(\psi(z)dz^{-1/2}\right) = \left(\psi''(z) - \frac{1}{2} R\,\psi(z)\right) dz^{3/2} \tag{12}$$

defines a 1-cocycle on $\mathcal{J}_{g,N}$ with coefficients in $\mathcal{J}^*_{g,N}$.

(ii) The cocycle (12) with $R \equiv 0$ is the unique (up to a multiplicative constant) 1-cocycle $C : \mathcal{J}_{0,N} \longrightarrow \mathcal{J}^*_{0,N}$ vanishing on the subalgebra $\mathcal{K}_3$.

Proof. Part (i). Similarly to formula (4), the expression in the right-hand-side of (12) is independent of the choice of the coordinate z. One now easily checks that this expression indeed satisfies the condition (11) of 1-cocycle. This follows from the relations :

$$\begin{array}{rcl} ad^*_{\varphi(z)(dz)^{-1/2}}\left(u(z)dz \,\oplus\, w(z)(dz)^{3/2}\right) & = & -\frac{1}{2}\varphi w\; dz \oplus -\left(\frac{1}{2}\varphi u' + \varphi' u\right)\; (dz)^{3/2} \\ ad^*_{\varepsilon(z)}\left(u(z)dz \oplus w(z)(dz)^{3/2}\right) & = & \varepsilon u\; dz \,\oplus\, \frac{1}{2}\varepsilon w\; (dz)^{3/2} \end{array}$$

where u, w, ε and φ are some meromorphic functions on the surface.

Part (ii). Let us first consider the case $N = 2$ and show that the cocycle (12) with $R \equiv 0$ from $\mathcal{AK}(1)$ to $\mathcal{AK}(1)^*$ is the unique (up to a multiplicative constant) 1-cocycle that vanishes on $\mathcal{K}_3$. The explicit formula in the basis is given in [14], for all $n \in \mathbb{Z}$ and all $i \in \mathbb{Z} + \frac{1}{2}$:

$$\mathcal{C}(\varepsilon_n) = -n\varepsilon^*_{-n}, \quad \mathcal{C}(a_i) = \left(i - \tfrac{1}{2}\right)\left(i + \tfrac{1}{2}\right) a^*_{-i}. \tag{13}$$

Assume that $\mathcal{C} : \mathcal{AK}(1) \longrightarrow \mathcal{AK}(1)^*$ is a 1-cocycle. Since $\mathcal{C}$ is even, it is of the form

$$\mathcal{C}(\varepsilon_n \oplus a_i) = \mathcal{C}(\varepsilon_n) \oplus \mathcal{C}(a_i) = \sum_{r\in\mathbb{Z}} \lambda_n^r \varepsilon_r^* \oplus \sum_{k\in\mathbb{Z}+\frac{1}{2}} \mu_i^k a_k^*.$$

The condition of 1-cocycle (11) gives :

$$\begin{aligned}
\mathcal{C}(\varepsilon_n \centerdot \varepsilon_m) &= ad^*_{\varepsilon_n}\mathcal{C}(\varepsilon_m) + ad^*_{\varepsilon_m}\mathcal{C}(\varepsilon_n) &\Leftrightarrow&\quad \lambda^r_{n+m} = \lambda_m^{r+n} + \lambda_n^{r+m}\\
\mathcal{C}(\varepsilon_n \centerdot a_i) &= ad^*_{\varepsilon_n}\mathcal{C}(a_i) + ad^*_{a_i}\mathcal{C}(\varepsilon_n) &\Leftrightarrow&\quad \mu^k_{i+n} = \mu_i^{k+n} + (k-i)\lambda_n^{i+k}\\
\mathcal{C}(a_i \centerdot a_j) &= ad^*_{a_i}\mathcal{C}(a_j) - ad^*_{a_j}\mathcal{C}(a_i) &\Leftrightarrow&\quad (j-i)\lambda^r_{i+j} = -\mu_j^{r+i} + \mu_i^{r+j},
\end{aligned}$$

for all $n, m, r \in \mathbb{Z}$ and all $i, j, k \in \mathbb{Z} + \frac{1}{2}$. Since this cocycle vanishes on the Lie antialgebra $\mathcal{K}_3$ generated by $\langle \varepsilon_0, a_{-1/2}, a_{1/2}\rangle$, by induction one has the following (unique) solution:

$$\lambda_n^r = -n\delta_{r,-n} \quad \text{and} \quad \mu_i^k = (k^2 - \tfrac{1}{4})\delta_{k,-i} \qquad \forall n, r \in \mathbb{Z};\ \forall i, k \in \mathbb{Z} + \tfrac{1}{2},$$

and thus obtains the cocycle (13).

Now, let us show the uniqueness for $N = 3$. As proved in [16], the subalgebra $\mathcal{J}^-_{0,3} = \langle G_n : n \leq 0;\ \varphi_i : i \leq 1/2\rangle$ is isomorphic to $\mathcal{AK}(1)$, cf. Example 4.5 b). Suppose that we have a 1-cocycle $\mathcal{C} : \mathcal{J}_{0,3} \longrightarrow \mathcal{J}^*_{0,3}$ and writing it with the elements of the basis as the same way than in the first part of the proof (*ii*). Using the 1-cocycle condition (11), we can can show that if we know the 1-cocycle $\mathcal{C}$ on $\mathcal{J}^-_{0,3}$ (i.e. on $\mathcal{AK}(1)$), then the cocycle is uniquely determined on $\mathcal{J}_{0,3}$ entirely. Hence the result on $\mathcal{J}_{0,3}$ since the 1-cocycle on $\mathcal{AK}(1)$ is unique when it vanishes on $\mathcal{K}_3$.

We conclude by induction using the fact that $\mathcal{J}_{0,N}$ is generated by N copies (well chosen) of $\mathcal{J}_{0,N-1}$, since we have the embedding $\mathcal{J}_{0,N-1} \subset \mathcal{J}_{0,N}$. □

Remark 5.4. The 1-cocycle (12) has a very simple and, geometrically, very natural form : this is the De Rham differential of a function combined with the Sturm-Liouville equation associated to a projective connection, applied to a $-1/2$-density.

5.5 An explicit formula of the 1-cocycle on $\mathcal{J}_{0,3}$

We finish the paper with an explicit formula of the 1-cocycle (12) in the case of 3 marked points.

Proposition 5.5. *The 1-cocycle (12) on the algebra $\mathcal{J}_{0,3}$ is given by*

$$\begin{aligned}
C(G_n) &= -nG^*_{-n},\\
C(G_m) &= -mG^*_{-m} - \alpha^2(m-1)G^*_{-m+2},\\
C(\varphi_i) &= 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2},\\
C(\varphi_j) &= 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2},
\end{aligned}$$

where $i - \frac{1}{2}$ and n are even and $j - \frac{1}{2}$, m are odd.

Proof. The basis of the dual space $\mathcal{J}_{0,3}^{*}$ is as follows:

$$\varphi_{2k-1/2}^{*} = \frac{1}{\sqrt{2}} z(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, \qquad G_{2k}^{*} = z(z^2-\alpha^2)^{-k-1}dz,$$
$$\varphi_{2k+1/2}^{*} = \frac{1}{\sqrt{2}} (z^2-\alpha^2)^{-k-1}(dz)^{3/2}, \qquad G_{2k+1}^{*} = (z^2-\alpha^2)^{-k-1}dz.$$

The computations are then straightforward. □

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