Abelian symmetries in N-Higgs-doublet models

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Multi-Higgs-Doublet Models

• We introduce N complex Higgs doublets with electroweak isospin $Y = 1/2$:

$$\phi_a = \begin{pmatrix} \phi_a^+ \\ \phi_a^0 \end{pmatrix}, \quad a = 1, \dots, N$$

• The generic Higgs potential can be written in a tensorial form:

$$V = Y_{\bar{a}b}(\phi_a^\dagger \phi_b) + Z_{\bar{a}b\bar{c}d}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)$$

where all indices run from 1 to N.

• There are N^2 independent components in Y and $N^2(N^2+1)/2$ independent components in Z.

• The explicit analysis of the most general case is impossible.

Symmetries

Several questions concerning symmetry properties of the scalar sector of NHDM arise;

- What groups G are realizable as symmetry groups of some potential V?

- How to write examples of the Higgs potential whose symmetry group is equal to a given realizable group G?

G is a realizable symmetry group if there exists a G-symmetric potential and there's no larger group which includes G and keeps this potential invariant.

For $N = 2$ the model has been studied extensively, but for $N > 2$, these questions have not been answered yet.

Here we introduce a strategy to find all Abelian subgroups in NHDM.

Reparametrization trasformations

• Reparametrization transformations: non-degenerate linear transformations which mix different doublets ϕ_a without changing the intradoublet structure and which conserve the norm $\phi_a^\dagger \phi_a$.

• All such transformations must be unitary (Higgs family transformation) or antiunitary (generaliazed CP transformation):

$$U: \quad \phi_a \to U_{ab}\phi_b$$
$$U_{CP} = U \cdot J: \quad \phi_a \to U_{ab}\phi_b^\dagger$$

with unitary matrix U_{ab}.

Unitary transformations

• Such transformations form the group $U(N)$. The overall phase factor multiplication is taken into account by the $U(1)_Y$.

• This leaves us with $SU(N)$, which has a non-trivial center $Z(SU(N)) = Z_N$ generated by the diagonal matrix $\exp(2\pi i/N) \cdot 1_N$.

• Therefore, the group of physically distinct unitary reparametrization transformations is

$$G_u = PSU(N) \simeq SU(N)/Z_N$$

anti-unitary transformations

Consider $U_{CP} = U \cdot J$, $U \in U(N)$
Define an action of J on the group $U(N)$ by

$$J \cdot U \cdot J = U^*$$

This action leaves invariant both the overall phase subgroup $U(1)_Y$ and the center of the $SU(N)$.

We again can consider only such $U_{CP} = U \cdot J$ that $U \in PSU(N)$.
Once this action is defined, we can represent all distinct reparametrization transformations as a semidirect product

$$G = PSU(N) \rtimes Z_2$$

Strategy

- Write maximal Abelian subgroups of $PSU(N)$.
- Find all the subgroups of each maximal Abelian subgroup.
- Check the potential is not symmetric under a larger group.

Complicated mathematical theorems state that there are two sorts of maximal Abelian subgroups inside $PSU(N)$:
- The maximal tori, which will be constructed here.

- Certain finite Abelian groups, which are not subgroups of maximal tori and must be treated separately

Constructing maximal torus in $PSU(N)$

• Starting from $SU(N)$; all maximal Abelian subgroups are maximal tori:

$$[U(1)]^{N-1} = U(1) \times U(1) \times \cdots \times U(1)$$

and all such maximal tori are conjugate inside $SU(N)$.

• Therefore without loss of generality one could pick up a specific maximal torus, for example, the one that is represented by phase rotations of individual doublets

$$\mathrm{diag}[e^{i\alpha_1},\, e^{i\alpha_2},\, \ldots,\, e^{i\alpha_{N-1}},\, e^{-i\sum \alpha_i}]$$

and study its subgroups.

• The diagonal transformation matrix could conveniently be written as a vector of phases:

$$\left(\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_{N-1}, -\sum \alpha_i\right)$$

- We construct a maximal torus in $PSU(N)$ which has the form

$$T = U(1)_1 \times U(1)_2 \times \cdots \times \overline{U(1)}_{N-1}$$

where

$$\begin{aligned}
U(1)_1 &= \alpha_1(-1, 1, 0, 0, \ldots, 0)\,, \\
U(1)_2 &= \alpha_2(-2, 1, 1, 0, \ldots, 0)\,, \\
U(1)_3 &= \alpha_3(-3, 1, 1, 1, \ldots, 0)\,, \\
&\vdots \\
\overline{U(1)}_{N-1} &= \alpha_{N-1}\left(-\frac{N-1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right)
\end{aligned}$$

with all $\alpha_i \in [0, 2\pi)$.

Identifying the symmetry groups

• Now, using the strategy, we check which subgroups of maximal torus T are realizable in the scalar sector of NHDM:

• We start from the most general T-symmetric potential:

$$V(T) = -\sum_a m_a^2(\phi_a^\dagger\phi_a) + \sum_{a,b}\lambda_{ab}(\phi_a^\dagger\phi_a)(\phi_b^\dagger\phi_b) + \sum_{a\neq b}\lambda'_{ab}(\phi_a^\dagger\phi_b)(\phi_b^\dagger\phi_a)$$

• A sufficiently general potential of this form has no other unitary symmetry

- We start from the T-symmetric potential and add more terms.
- Our task now is to find which realizable subgroups of T can be obtained in this way.

- Each monomial gets a phase change under T:

$$\phi_a^\dagger \phi_b \to \exp[i(p_{ab}\alpha_1 + q_{ab}\alpha_2 + \cdots + t_{ab}\alpha_{N-1})] \cdot \phi_a^\dagger \phi_b$$

- The coefficients $p, q, \ldots, t$ of such terms can be easily calculated for every monomial.

Identifying the symmetry groups

• Consider a Higgs potential V which is a sum of k terms, with coefficients $p_1, q_1, \dots t_1$ to $p_k, q_k, \dots t_k$. This potential defines the following $(N-1) \times k$ matrix of coefficients:

$$X(V) = \begin{pmatrix} p_1 & q_1 & \cdots & t_1 \\ p_2 & q_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ p_k & q_k & \cdots & t_k \end{pmatrix}$$

• The symmetry group of this potential can be constructed from the set of solutions for α_i of the following equations:

$$X(V) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} 2\pi n_1 \\ \vdots \\ 2\pi n_{N-1} \end{pmatrix}$$

- There are two major possibilities depending on the rank of matrix X:

- If rank of this matrix is less than $N-1$, there exists a hyperplane in the space of angles α_i, which solves this equation for $n_i = 0$. The potential is symmetric under $[U(1)]^D$, where $D = N - 1 - rank(X)$.

- If $\text{rank}X(V) = N - 1$, there is no continuous symmetry. Instead, there exists a unique solution for any n_i.
All such solutions form the finite group of phase rotations of the given potential.

finite groups

• with exactly $N-1$ monomials, the matrix $X(V)$ becomes a square matrix (with integer entries), which one could easily diagonalize.

• Diagonalizing the $X(V)$ matrix results in:

$$X(V) = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{N-1} \end{pmatrix}$$

Then we will have the finite symmetry group:

$$Z_{d_1} \times Z_{d_2} \times \cdots \times Z_{d_{N-1}}$$

Summary of the strategy to identify all finite subgroups of torus:

• Write down all possible monomials with N-doublets
• Consider all possible subsets with exactly $N-1$ distinct monomials
• Construct the matrix X for this subset and find its symmetry group following the above scheme.

Although this strategy is far from being optimal, its algorithmic nature allows it to be easily implemented in a machine code.

Antiunitary transformation

We extend the realizable subgroups of torus T to larger abelian groups that would include not only unitary but also antiunitary transformations

• Define the action of the CP-transformation J on $PSU(N)$ given by: $J \cdot U \cdot J = U^*$.

• To embed A in a larger abelian group, we must find such an antiunitary transformation $J' = b \cdot J$ that commutes with any element $a \in A$:

$$(J')^{-1} a J' = a \quad \Leftrightarrow \quad aba = b \quad \Leftrightarrow \quad ba^{-1}b^{-1} = a$$

• If at least one matrix b satisfying this equation is found, all the other matrices can be constructed with the help of the last form of this equation.

Example

Let us apply this strategy to the full maximal torus T.

A generic $a \in T$ acting on a doublet ϕ_i generates a non-trivial phase rotation $\psi_i(\alpha_1, \ldots, \alpha_{N-1})$.
Then we have

$$aba = b \quad \Rightarrow \quad e^{i(\psi_i + \psi_j)} b_{ij} = b_{ij} \quad \Rightarrow \quad b_{ij} = 0$$

This means that there is no J' that would commute with every element of the torus T.
T cannot be embedded into a larger abelian group T_{CP} that would include antiunitary transformations.

Now we're done with the strategy of the work
It's time for some examples in 3HDM and 4HDM

The 3HDM example

• In the 3HDM the representative maximal torus $T \subset PSU(3)$ is parametrized as

$$T = U(1)_1 \times U(1)_2\,, \quad U(1)_1 = \alpha(-1,\,1,\,0)\,, \quad U(1)_2 = \beta\left(-\frac{2}{3},\,\frac{1}{3},\,\frac{1}{3}\right)$$

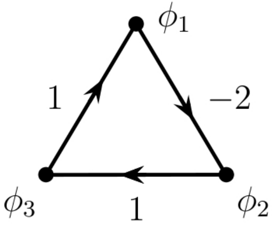
• There are six bilinear combinations of doublets transforming non-trivialy under T

$$(\phi_a^\dagger \phi_b) \to \exp[i(p\alpha + q\beta)](\phi_a^\dagger \phi_b)$$

	p	q
$(\phi_2^\dagger \phi_1)$	-2	-1
$(\phi_3^\dagger \phi_2)$	1	0
$(\phi_1^\dagger \phi_3)$	1	1

and their conjugates with opposite coefficients p and q.

• These coefficients are shown graphically as labels of the edges of two oriented graphs shown here:



- Any monomial term is symmetric under a $U(1)$, because $rank X(V) = 1$.
- In order to have a finite group we need at least 2 terms.

- To find all realizable groups, one has to write the full list of possible terms and then calculate the symmetry group of all distinct pairs of terms. For example, if the two monomials are $v_1 = (\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3)$ and $v_2 = (\phi_2^\dagger \phi_1)(\phi_2^\dagger \phi_3)$, then the matrix $X(v_1 + v_2)$ has the form

$$X(v_1+v_2) = \begin{pmatrix} 3 & 2 \\ -3 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

and it produces the symmetry group Z_3. The solution of the equation

$$X(v_1+v_2)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2\pi n_1 \\ 2\pi n_2 \end{pmatrix}$$

yields $\alpha = 2\pi/3 \cdot k$, $\beta = 0$.

Full list of subgroups of 3HDM

- Checking all possible combination of monomials, we arrive at the full list of unitary Abelian subgroups of the maximal torus:

$$Z_2, \quad Z_3, \quad Z_4, \quad Z_2 \times Z_2,$$
$$U(1), \quad U(1) \times Z_2, \quad U(1) \times U(1)$$

$Z_3 \times Z_3$

The only finite abelian group that is not contained in any maximal torus in $PSU(3)$ is $_3 \times _3$. Although there are many such groups inside $PSU(3)$, all of them are conjugate. Thus, only one representative case can be considered.

But it turns out that $Z_3 \times Z_3$ is not realizable for 3HDM.

Antiunitary symmetries in 3HDM

Applying the strategy described before, we obtain the following additional realizable Abelian groups:

$$Z_2^* \,, \quad Z_2 \times Z_2^* \,, \quad Z_2 \times Z_2 \times Z_2^* \,, \quad Z_4^*$$

where the asterisk indicates that the generator of the corresponding cyclic group is an anti-unitary transformation.

The 4HDM example

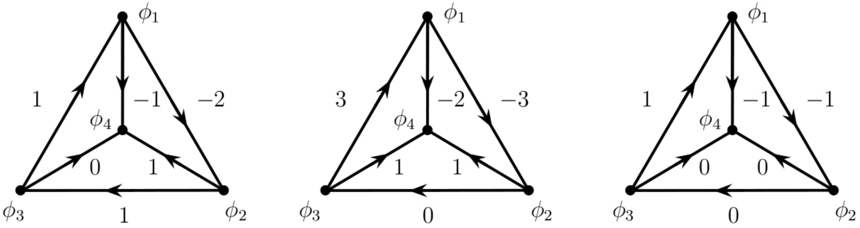
• In the case of 4 Higgs doublets, the representative maximal torus in $PSU(4)$ is $T = U(1)_1 \times U(1)_2 \times U(1)_3$, where

$$U(1)_1 = \alpha(-1, 1, 0, 0)\,, \quad U(1)_2 = \beta(-2, 1, 1, 0)\,, \quad U(1)_3 = \gamma\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

• The phase rotations of a generic bilinear of doublets under T is characterized by three integers p, q, r,

$$(\phi_a^\dagger \phi_b) \to \exp[i(p\alpha + q\beta + r\gamma)](\phi_a^\dagger \phi_b)$$

These coefficients can again be represented graphically as label of edges of three simplices



- Using the strategy we have found all finite unitary Abelian groups

Z_k with $k = 2, \ldots, 8;$ $\quad Z_2 \times Z_k$ with $k = 2, 3, 4;$ $\quad Z_2 \times Z_2 \times Z_2$

which interestingly all have order ≤ 8

- And all realizable continuous groups:

$$U(1) \times U(1) \times U(1)\,, \quad U(1) \times U(1) \times Z_2\,, \quad U(1) \times Z_k\,, \quad k = 2, 3, 4, 5, 6$$

General NHDM

• The algorithm described above can be used to find all Abelian groups realizable as the symmetry groups of the Higgs potential for any N.

Several statements:

• Upper bound on the order of finite Abelian groups:
For any given N there exists an upper bound on the order of finite Abelian groups realizable as symmetry groups of the NHDM potentials: The order of any such group must be $\leq 2^{\frac{3}{2}(N-1)}$. We suspect that this bound could be improved to 2^{N-1}.

• Cyclic groups:
The cyclic group Z_p is realizable for any positive integer $p \leq 2^{N-1}$.

• Polycyclic groups:
Let $(N-1) = \sum_{i=1}^{k} n_i$ be a partitioning of $(N-1)$ into a sum of non-negative integers n_i. Then, the finite group

$$G = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_k}$$

is realizable for any $0 < p_i \leq 2^{n_i}$.

Conclusion

To summarize:

• NHDM are interesting because one can introduce many non-trivial symmetries. Finding such symmetries is one of the hot topics.

• In this work we have focused on Abelian symmetries and developed a strategy to find all Abelian groups realizable for any NHDM. Specific examples of 3HDM and 4HDM have been shown.

• We have derived some general conclusion for NHDM.