A DIDACTIC SURVEY OF THE MAIN CHARACTERISTICS OF LAGRANGE'S THEOREM IN MATHEMATICS AND IN ECONOMICS

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Abstract: Because of its many uses, the constrained optimization problem is presented in most undergraduate mathematics courses dealing with calculus for both mathematicians and economists. Our research focuses on the teaching of Lagrange's Theorem in both branches of study, mathematics and economics. This paper addresses two objectives. First, we describe the methodology of our research project concerning the didactic transposition of Lagrange's Theorem in university courses. Secondly, we compare two mathematics courses dealing with calculus given at the universities of Namur and Louvain by means of the "Anthropological Theory of Didactics" of Yves Chevallard and emphasize its explanatory power to describe mathematical activity.

Keywords: Lagrange's Theorem, optimization, didactic transposition, Anthropological Theory of Didactics

INTRODUCTION

Constrained optimization plays a central role in optimization theory but also in economics. In fact, constrained optimization can be seen as one of the fundamental techniques that economists use to solve economic problems. Since the Theorem of Lagrange and the consequential method of Lagrange multipliers [named after Joseph-Louis Lagrange (1736-1813)] provide an appealing strategy for finding the maxima and minima of a function subject to equality constraints, we are interested in studying the teaching of this theorem in both branches of study, mathematics and economics.

Based on the author's own teaching experiences at the University of Namur (Belgium), it is apparent that a considerable number of first year students struggle with calculus courses and, in particular, with Lagrange's Theorem. Furthermore, the mathematical exercises in these classes involve students using a large number of standardized procedures for obtaining answers to clearly delimited types of exercise questions. Dreyfus mentions in this context that

> they end up with a considerable amount of mathematical knowledge but without the working methodology of the mathematician, that is they lack the know-how that allows them to use their knowledge in a flexible manner to solve problems of a type unknown to them. (Dreyfus, 1991, p.28)

Hence, we question in our research project whether the choice of the didactic transposition of Lagrange's Theorem may influence the students' perception and understanding. Therefore, we question whether our findings help to enlighten mathematics professors concerned with increasing students' comprehension of this theorem. In fact, we also would like to show how teachers' practices are influenced inside the didactic transposition by a combination of didactic reasons and mathematical reasons.

Next, we describe the theoretical framework used to guide our research. In order to investigate the constraints under which a professor should operate when conceiving and carrying out the teaching of Lagrange's Theorem, we analyzed existing didactic transpositions by means of the “Anthropological Theory of Didactics” (ATD) of Chevallard (1992, 1999). This model describes mathematical activity in terms of *mathematical (*or *didactic) organisations* or *praxeologies.* The third section provides a description of our methodology, which used ideas from the ATD, a useful tool for the analysis of mathematical and teaching activities. In the fourth section, we briefly describe the *epistemological reference model* (ERM), which constitutes our basic theoretical model used to describe the didactic transposition. Related, mathematical praxeologies are then used to describe and compare the knowledge to be taught around the Lagrange's multiplier rule as it is proposed at the universities of Namur and of Louvain. Finally, we provide conclusions and a brief survey of perspectives of our research work.

THE ANTHROPOLOGICAL APPROACH

As we utilize an institutional perspective[1] in our research, the choice of the “Anthropological Theory of Didactics” (ATD) proposed by Chevallard (1992, 1999) appears pertinent to investigate characteristics of teachers' practices. This model of mathematical knowledge envisions mathematics as a human involving the study of *types of problems*. Below, we provide a brief summary of the principal content of this theory based on ~~work by~~ a paper written by Barbé, Bosch, Espinoza and Gascón (2005).

In the anthropological approach, an object exists from the moment one person or an institution individually recognizes this object as existing, and more precisely, if there exists a relation to it. These relationships can be established through activities making use of the object. We identify two inseparable aspects of mathematical activity. First, the *pratico-technical block* (or *know-how*) is formed by *types of problems* or *problematic tasks,* T, and by the *techniques,* τ, used to resolve them. Studying problems of a given type (with an aim of solving them) is considered to be “*doing mathematics*”. Furthermore, procedural methods, resolutions of problems or accomplishments of tasks suppose the existence of a technique in the anthropological approach. This holds true even if the given technique can scarcely be explained or shown to others or even to ourselves.

Secondly, we assume in the anthropological approach that one can rarely find human practices without a copious environment of discourse. The objective of this “spoken surround” is to describe, explain and justify what is done. Therefore, there is the

1 Institutional perspective means that we observe practices relative to a (mathematical) object, these practices are expected to differ from one institution to another.

knowledge block of mathematical activity that offers the mathematical conversation necessary to justify and interpret the practical block. The knowledge block is divided into two parts: the *technology*, θ, refers to the technique used and the *theory*, Θ, establishes profound justifications of the technology. In ATD, this second block is called the *technological-theoretical* block.

Types of problems, techniques, technologies and theories can be seen as the fundamental elements of the anthropological model of mathematical activity. We also employ them to describe mathematical knowledge, which can be considered both a means and a product of this activity. When we examine various types of problems, techniques, technologies and theories together, we entitle them *mathematical praxeological organisations* or, in short, *mathematical organisations* or *mathematical praxeologies*. An examination of the etymology of the word "praxeology" shows how practice (praxis) and the discourse about practice (logos) are closely connected.

The "Anthropological Theory of Didactics" posits that we can analyse more than only mathematical activities. Any form of human activity can be interpreted in terms of praxeological organisation. Therefore, we also introduce the concept of *didactic praxeologies* when speaking about the process of study of mathematical constructions.

Given the increasing interest and demand to investigate on teachers' practices and their role in the didactic relationship, an investigation of these didactic praxeologies appears warranted.

In order to achieve the aforementioned objectives, our research was guided by the following research questions amongst others:

- What are the essential characteristics of the teaching of Lagrange's Theorem in mathematics and economics?
- Are there similarities and differences between the teaching of Lagrange's Theorem in mathematics and in economics with respect to the mathematical praxeologies?
- May it be possible to improve students' understanding/interpretation of Lagrange's Theorem by exchanging ideas between the two disciplines?

FOUR KINDS OF KNOWLEDGE

Teaching and learning are not isolated, but take place in the complex process of *didactic transposition* (Chevallard, 1991). With regard to this transposition, we need to distinguish among four kinds of knowledge: "scholarly" mathematical knowledge, mathematical knowledge "to be taught" and mathematical knowledge "as it is actually taught" by professors to students. When including students' comprehension and learning into the process, we have to add the fourth kind of knowledge – mathematical knowledge "learnt", which is generally difficult to access. A basic theoretical model, the *epistemological reference model* (ERM) (Bosch & Gascón, 2005), established from information about the scholarly knowledge and the knowledges to be taught and taught completes this presentation.

We used the following framework procedural methodology in order to investigate our research questions and related questions:

“Scholarly” mathematical knowledge

To understand “scholarly” mathematical knowledge, our first step consisted of an epistemological analysis of Lagrange's Theorem and of associated mathematical literature in mathematics and economics.

The mathematical knowledge “to be taught”

To gain deeper insight in the mathematical knowledge to be taught, we analyzed textbooks and course notes about Lagrange's Theorem from different mathematics courses, both for mathematics and economics students, using the “Anthropological Theory of Didactics”. In doing so, we exercised caution because only the “knowledge to be taught” can be reproduced from these textbook elements. The “knowledge actually taught” unfortunately appears only in the students' notes and in the specific teaching practices carried out in the day-to-day teaching praxeologies in classrooms.

Mathematical knowledge “taught” and “learnt”

Next, we explored teachers' and students' conceptions about Lagrange's Theorem. We contacted professors who are (or were) responsible for teaching Lagrange's Theorem in either an economics or a mathematics course (or both) dealing with calculus, or more particularly with optimization. This was done at Belgian universities in the French-speaking part of the country. By means of a questionnaire composed of 27 multiple-choice and open-ended questions, we attempted to identify, as precisely as possible, the environment and conditions of the teaching of Lagrange's Theorem. A second questionnaire built upon the first one was then designed in order to obtain information about students' conceptions and ideas. This questionnaire was composed of 14 multiple-choice and open-ended questions. Courses observations and students' presentations and evaluations at exams (involving three different tasks to solve) completed our data collection process.

Note that our research is still in progress. “Scholarly” mathematical knowledge and mathematical knowledge “to be taught” are already analysed, mathematical knowledges “taught” and “learnt” are the topics we are momentarily concerned with.

EPISTEMOLOGICAL REFERENCE MODEL OF LAGRANGE'S THEOREM

The reference mathematical model we are considering about Lagrange's Theorem includes five mathematical organisations MO_1, MO_2, MO_3 , MO_4 and MO_5 that respectively address the following type of tasks:

- T_1 : Find candidates to be optimal solutions for a constrained optimization problem subject to equality constraints.

- T_2 : Solve a constrained optimization problem subject to equality constraints.
- T_3 : Develop the theory concerning Lagrange's Theorem.
- T_4 : Use an interpretation of Lagrange's multiplier.
- T_5 : Develop the theory concerning Lagrange's multipliers.

Many relationships among these mathematical organisations can be described but we limit ourselves in this paper to the following brief remarks: MO_1 originates from the the original works of Lagrange, whereas MO_2 is concerned with the solving of particular constrained optimization problems. In fact, Lagrange's Theorem may or may not intervene in the technique of MO_2. Accomplishing T_1 can therefore be one step in the process of accomplishing T_2. MO_3 can be regarded as part of the theory of MO_1 (and also MO_2), but it is also the self-contained praxeology that deals, amongst other tasks, with the proving of Lagrange's Theorem. MO_4 uses Lagrange's multipliers as a mathematical tool (Douady, 1986), whereas MO_5 constitutes an additional mathematical organisation concerned with Lagrange's multiplier being seen as a mathematical object in the sense of Douady (1986).

COMPARISON BETWEEN TWO TEXTBOOKS

To describe emerging mathematical and didactic praxeologies we are going to analyse existing textbooks with regard to our ERM. However, due to space limitations of this paper, the presented analysis only considers two mathematics textbooks, one from each discipline – economics and mathematics. Nevertheless, we will be able to show how the anthropological approach renders a comparison possible.

The components of Lagrange's Theorem we are considering are integrated in the course “Mathematics for Economic Analysis I” (Thiry, 2006) for first-year students in economics and business management at the University of Namur and in the course “Mathematical Analysis II” (Ponce & Van Schaftingen, 2010) for first-year students in mathematics at the University of Louvain (Belgium).

Mathematics for Economic Analysis I (Thiry, 2006)

Twelve pages in the textbook deal with Lagrange's Theorem (in Chapter 4, “Multivariable Optimization”). We analysed these pages in terms of praxeologies. Looking at this knowledge to be taught, we can name it “Analytic solving of equality constrained optimization problems”. In fact, the textbook starts with one well-known economics-based optimization problem: the utility maximization problem[2] the consumer faces, and tries to solve it:

- first, by the substitution method,
- then, by noticing that substitution is not always possible. Therefore, the use of the method of Lagrange multipliers is announced.

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2 This problem can be resumed by "How should I spend my money in order to maximize my utility?"

After this short introduction, the textbook mathematically defines the problem it is going to solve. The textbook poses the following type of problems:

T_I : Find all the candidates to be optimal solutions for the following constrained optimization problems

$$(P_1)\left\{\begin{array}{ll} \max & f(x,y) \\ \text{sc} & g(x,y)=k \end{array}\right. , \qquad \text{and} \qquad (P_2)\left\{\begin{array}{ll} \min & f(x,y) \\ \text{sc} & g(x,y)=k \end{array}\right. .$$

T_{II} : Solve (P_1) (or (P_2) respectively).

T_{III} : Approximately determine the maximum of (P_1) (or the minimum of (P_2)) when k is increased (or decreased) by ϵ.

The section about Lagrange's Theorem is followed by exercises where we can find one additional type of problem: exercises that are mainly like T_{II} but require *mathematical modelling.* We do not consider these problems in the paper due to space limitations.

Let us start with T_I. Before obtaining an appropriated technique to solve this type of problem, we switch to the technological-theoretical block and read the technology θ_I used to justify the appropriated resolution process. In terms of a geometric interpretation and by the use of the Implicit Function Theorem, we get a characterization of the solution: a point (x^*, y^*) solution of the equality constrained optimization problem has to verify

- $g(x^*, y^*) = k$ and
- There exists $\lambda \in \mathbb{R}$ such that $\nabla f(x^*, y^*) = \lambda \nabla g(x^*, y^*)$.

This necessary optimality condition is then formally stated in the third section: "Lagrange's Theorem".

> […] the observation of the contour lines shows us that a point (x^*, y^*), extremum of f under the constraint $g(x, y) = k$ necessarily verifies the aforementioned equations. We now formulate these conclusions in the form of a theorem (that will not be proved here). (Thiry, 2006, p. 164)

The author does not give a rigorous proof, reason why we say that the theory which is closest to the given statement and which justifies the technology presented before is nearly absent from the notes. This section continues by defining Lagrange's function, and reformulates the theorem by means of this function. The whole section can be considered as the technological discourse justifying the technique of "finding candidates to be optimal solutions". Section 4 finally marks down algorithmically the technique τ_I to follow to find candidates for problems T_I. Combining type of problem T_I, technique τ_I and technology θ_I permits us to obtain a first praxeology $[T_I, \tau_I, \theta_I, /]$ where "/" symbolizes the "non-explicitly-phrased" theory.

Finding all the candidates does not cope with the intended solution of the type of problem T_{II}, even if a solution to T_I is necessary to solve T_{II}. We therefore need a second step in the resolution process, which is presented in the textbook in Section 5.

No technological-theoretical block is presented for this second step in the resolution process, but only a proposition is provided that details the strategy τ_{II} used to identify whether a candidate is effectively an optimum of the equality constrained optimization problem.

> The following proposition (that will not be proved) furnishes a test based on second-order derivatives to decide whether a stationary point is effectively a~~n~~ maximum or a minimum. (Thiry, 2006, p. 169)

This technique is illustrated by means of an example. We say that we obtain the following praxeology $[T_{II}, \tau_I \cup \tau_{II}, \theta_I \cup /, /]$. Again, the “/” means that these praxeological elements are not rendered explicitly.

As it is the case of T_I, the technology θ_{III} concerned with the solving of type of problems T_{III} is presented before the presentation of the technique τ_{III} and is resumed in the proposition 4.12 (Thiry, 2006, p. 171). The associated technique then is exposed in the proposition 4.13 (Thiry, 2006, p. 172) and is completed with one exercise. We get a third praxeology $[T_{III}, \tau_{III}, \theta_{III}, /]$ where the slash indicates again that the theory is practically absent in the sense that it doesn't explicitly appear in the textbook.

Combining the three aforementioned praxeologies, we find that the considered knowledge to be taught is principally composed of the traces left by MO_1, MO_2 and MO_4. In fact, the first type of tasks is a particular case of tasks of MO_1. Furthermore, with regard to our ERM, the first praxeology constitutes a particular reconstruction of MO_1. The theoretical element absent for this mathematical organisation attests to the predominant missing MO_3 in the textbook. The second praxeology arises from MO_2 and covers an activity that can't be solved only by means of the technique of Lagrange's multiplier rule. Hence, students have to seek techniques by means of “external arguments” in the sense of arguments that are out of range of Lagrange's Theorem. We make the assumption that students who stop the resolution of a constrained optimization problem subject to equality constraints after having applied the multiplier rule do not distinguish between the solving of tasks T_I and T_{II}. This may be founded in the presentation of Lagrange's Theorem as the main technique to solve equality constrained optimization problems without insisting that this theorem constitutes only necessary optimality conditions. The third praxeology then clearly comes from MO_4 as regarded in an economical context. Finally, only a few remarks and definitions can be considered as traces left by MO_3 and MO_5.

Mathematical Analysis II (Ponce & Van Schaftingen, 2010)

Chapter 5 in this textbook covers optimization problems. The first section deals with unconstrained optimization problems, whereas the second deals with equality constraints and the third deals with inequality constraints. A last section offers multiple exercises. We are interested in the six pages treating optimization problems with equality constraints at section 5.2. As for the first analysis, we call this knowledge to be taught “Analytic solving of equality constrained optimization problems by penalty method”.

Before looking at this particular section, let us mention that each chapter of this textbook starts with a list of questions students are going to be confronted with at the final exams. We therefore can say that the student is informed about the specific type of problems treated in this course. The final section of each chapter provides more exercises. For the constrained optimization problem and Lagrange's Theorem in particular, we therefore find two relevant types of problems:

- T_{IV} : Give a geometric interpretation of Lagrange's multipliers.
- T_V : Given function , $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \rightsquigarrow f(x, y)$ determine the minima and maxima of f constrained to $g(x, y) = 0$.

Most of problems of type T_V in the textbook treat geometric problems and use therefore concepts from geometry (e.g., distances, planes, surfaces). Furthermore, the type of problems T_V resembles T_{II}, affirming that mathematics and economics students are confronted with the same type of problems.

The section about equality constrained optimization problems opens directly with a theoretical discourse and gives the mathematical formulation of Lagrange's Theorem. Its proof needs lemma 5.5 and both propositions are rigorously proved by a so-called *penalty method*[3]. One remark is then given which concerns the values the multipliers can take. No further explanations are provided and the section ends with two example tasks and their resolution. We add these problems to the list of types of problems and therefore define:

- T_{VI} : Minimize function $f : \mathbb{R}^n \to \mathbb{R} : x \rightsquigarrow f(x)$ under the equality constraint $g(x) = 0$, where $g : \mathbb{R}^n \to \mathbb{R} : x \rightsquigarrow g(x)$.
- T_{VII} : Prove the “inequality of arithmetic and geometric means”.

In summary, we conclude that, for the types of problems T_{IV} and T_V, only the most relevant theory Θ is stated and proven even if, from a mathematical point of view, technique τ_{VI} could be used to solve T_V. The formulation of Lagrange's Theorem is then used as technology (without giving further argumentation) to solve T_{VI} and T_{VII}. However, neither the complete pratico-technical block nor the complete technological-theoretical block is explicitly presented in the textbook to solve the introductory tasks. We obtain the following praxeology $[T_{IV} \cup T_V, /, /, \Theta]$, where “/” symbolizes that these praxeological elements are not stated. As far as the two problems solved as examples are concerned, the associated technique is furnished, so that we get $[T_{VI}, \tau_{VI}, \theta, \Theta]$ and $[T_{VII}, \tau_{VII}, \theta, \Theta]$, where θ is the formulation of Lagrange's Theorem. The technology θ is a minimal discourse in the sense of furnishing only the theorem that justifies the technique. We presume that the understanding of Lagrange's Theorem may be hindered if no supplementary

3 This method consists of neglecting the constraints while adding a penalty term to the function to be optimized if the constraints are violated. A consequence of this method is the avoidance of the Implicit Function Theorem.

information is given (for example, during lectures) and that students will encounter some problems in solving type of problems T_{IV} and T_V. However, this analysis only concerns the textbook and the mathematical knowledge "to be taught" and does not represent the mathematical knowledge "as it is actually taught".

With regard to our ERM, the textbook does a nearly complete presentation of MO_3 (with one possible proof amongst others). As this mathematical organisation is presented before solving particular tasks arising from MO_2, we see that the technological discourse concerned with these tasks of MO_2 is replaced by adding MO_3 to the theoretical discourse. Furthermore, the students have to reason the associated technique to solve tasks of type T_V by themselves. We do not find traces of MO_1 regarded as self-contained praxeology in the textbook. Finally, as far as MO_4 is concerned, the type of tasks T_{IV} arises from this mathematical organisation. However, MO_5 is completely absent.

Comparison

In order to compare the knowledge to be taught as it is presented in textbooks provided to students, we have to take into account that they target different audiences. We therefore obtain a first discrepancy between the teaching of Lagrange's Theorem in mathematics and economics. Students in economics are confronted with a detailed pratico-technical bloc of MO_1 and often obtain profound technological arguments to justify the technique of Lagrange's multiplier before completely solving the equality constrained optimization problem. Traces of MO_1 and MO_2 can be found. Conversely, mathematics students are directly confronted with the more "general" task of finding solutions and tasks of MO_1, which are incorporated in the solving process of tasks of MO_2. Students in mathematics directly have access to a praxeology of type MO_3. This affirms that proving is one of the dominant activities in mathematical studies. The technological discourse of MO_2 is then reduced to the formulation of the theorem in question, and students are left to find the associated technique by themselves (by learning or by assisting the theoretical course or exercise sessions).

CONCLUDING REMARKS

Data collected from our experiences still need to be analysed in more detail in order to answer our research questions. The methodology presented will be pursued to obtain more insight into the didactic transposition of Lagrange's Theorem. This paper highlighted some discrepancies between the mathematical knowledge to be taught in mathematics and in economics due to the fact that, first and foremost, the role of mathematics in each discipline is different. Even if this is not a surprising result, we give a foretaste of the descriptive power of the ATD of Chevallard as a tool for our ongoing analyses. In fact, the second objective of the paper was to show how ATD can render an analysis of the complex process of didactic transposition possible. It provides a classification of the didactic material presented in the textbooks and, with regard to our epistemological reference model, makes a comparison possible and significant between these textbooks. As already mentioned, we need to be cognizant that textbooks do not represent the mathematical knowledge “as it is actually taught”. We therefore have to refine our epistemological reference model and to carry out further analyses concentrating on the latter to get deeper access in teachers' practices and students' perceptions.

The intended outcome of this research project may probably not achieve the aim of a didactic engineering in the sense of Artigue (1989), but we expect to understand better the essential qualities of Lagrange's Theorem and how teacher can effectively intervene in its teaching to improve their practices.

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