Extended Formulations for Gomory Corner Polyhedra

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1 Introduction

More than 30 years ago, Ralph Gomory [3] introduced an approach to solve integer linear programs based on group relaxations. His approach has later been refined by Wolsey [?] and by Gomory and Johnson [4, 5] leading to a theory of superadditive functions [6]. The central starting point of this line of research is the Gomory corner polyhedron that may be derived from an integer programming problem as follows. Starting with a linear integer program

$$\begin{array}{ll} \max & c^T x \\ \text{s. t.} & Ax = b \\ & x \in \mathbb{Z}^n_+, \end{array} \tag{1}$$

one may select a subset of linearly independent columns A_B and relax the integer program by

$$S = \{x \in \mathbb{Z}^n : A_B x_B + A_{\bar{B}} x_{\bar{B}} = b,\ x_{\bar{B}} \geq 0\},$$

i.e., one ignores the nonnegativity requirement for the variables in the index set B but maintains integrality of all the variables and the nonnegativity conditions for the variables in the complementary set of B. Of course, the set S is in bijection with its projection to the space of complementary variables

$$S' = \{x \in \mathbb{Z}^{|\bar{B}|} : A_{\bar{B}} x_{\bar{B}} \equiv b \ (\mathrm{mod}\ L(A_B)),\ x_{\bar{B}} \geq 0\},$$

where $L(A_B) = \{A_B z :\ z \in \mathbb{Z}^{|B|}\}$ denotes the lattice generated by the column vectors of A_B.

The convex hull of all points in S' is denoted as the Gomory corner polyhedron. Gomory and coauthors [7, 8, 2] claim that although the corner polyhedron seems to be a quite general mathematical object, its geometry might be substantially easier to describe than the one of the original integer programming polyhedron. The reason is that a Gomory corner polyhedron is based on a cyclic group. In particular, authomorphisms from the cyclic group into itself can be used to characterize facets of the corner polyhedron. In a similiar vein composition rules can be used to "lift" facets of corner polyhedra associated with small groups to corner polyhedra associated with bigger groups. All these results certainly provide theoretical evidence that Gomory corner polyhedra ought to be investigated into more depth. This is a starting point of this paper that is particlarly motivated by the following comment in Evans,Gomory, Johnson [8], page 338, section 3:

"If we were able to come close to solving the Corner Polyhedron - say by having an adequate supply of cutting planes or perhaps, in other ways, such as finding solutions to the group problems, we could come close to a different kind of algorithm - one based on solving a sequence of Corner Polyhedron problems..."

The central theme of this paper is to "come close to such a different kind of integer programming algorithm" that is based on feasible solutions in corner polyhedra. We regard the following results as the major contribution of this paper.

- It is shown that taking any group problem and providing a reformulation with the nondecomposable solutions of the corresponding system provides the convex hull of all its solutions. This demonstrates that reformulating a corner polyhedron problem by means of computing irreducible solutions is the strongest possible operation so as to move from the linear programming relaxation to the integer optimum. In other words, the reformulation is as strong as deriving a convex hull representation by adding all the facet-defining inequalities to the system.
- If there are many irreducible solutions in a Gomory corner polyhedron, then one may resort to linear transformations and composition methods in order to derive a compact representation for the irreducible solutions. This increases drastically our ability to determine practically the irreducible solutions to a group problem.
- Evidence is given that Gomory's group approach can not only theoretically but also practically be applied to solve integer programs. The key is however not to make use of the inequality description for small corner polyhedra, but to systematically study the non-decomposable integer solutions of the corner polyhedron. This opens up the possibility of solving an integer program iteratively through a sequence of corner prolyhedron problems as Ralph Gomory suggested.

The algorithm that we propose works in an iterative fashion. Each iteration requires the analysis of a group relaxation of the original integer program. The key step is to define an extended formulation for such a group relaxation whose variables are in one-to-one correspondence with the nondecomposable solutions of the group relaxation.

More precisely, the general model that we study in this paper is

$$Y^G = \{x \in \mathbb{Z}^n_+ : Bx \equiv f \pmod{\Delta}\},$$

where Δ is a regular diagonal positive integer matrix, columns, and $B \in \mathbb{Z}^{r \times n}$. As before, the convex hull of points in Y^G defines a Gomory corner polyhedron. An extended formulation of Y^G is a representation of the form

$$Y^G = \{x : x = C\mu,\ D\mu = d,\ \mu \in \mathbb{Z}_+^p\}.$$

Of particular relevance are reformulations that produce the convex hull, i.e. formulations for which

$$\text{conv}(Y^G) = \{x : x = C\mu, D\mu = d,\ \mu \in \mathbb{R}_+^p\}.$$

In the following sections, we examine such relaxations. In Section 2 we present four possible extended formulations for the group relaxation. The first one uses all the irreducible solutions of the group problem. We call it the disaggregated formulation. The second one is a first generalization that tries to reduce the number of new variables by aggregating variables with identical residuum class. (The aggregated formulation). The third reformulation is based on an advanced aggregation technique that reduces even further the number of new variables. Finally we present a reformulation based on the representation of groups by paths in a digraph.
Section 3 analyzes the different formulations. We show that the extended, the aggregated and the path reformulation fulfill the convex hull property. We also characterize conditions under which the advanced aggregation formulation produces the convex hull.
Section 4 discusses the algorithmic possibilities to compute the extended formulations introduced before. Section 5 contains the outline of the "different kind of algorithm based on solving a sequence of corner polyhedron problems" and reports on computational experiments with the method.

2 Extended Formulations for Group Relaxations

We consider a set of the form

$$S(d) = \{x \in \mathbb{Z}_+^n : Bx \equiv d \ (\text{mod } \Delta)\}, \tag{2}$$

where Δ is a regular diagonal positive integer matrix, and $B \in \mathbb{Z}^{r \times n}$, and $d \in \mathbb{Z}_+^r$. Associated with $S(d)$ is a group

$$G = \{x \in \mathbb{Z}_+^r : x_i \in \{0, \cdots, \Delta_i - 1\}, i = 1, \cdots, r\},$$

consisting of all potential residue classes modulo the vector diag(Δ). For a set $S(d)$, we now examine several possible extended formulations. To do this we need to introduce the notion of *irreducible* solutions.

Definition 1 *A vector x is an* irreducible solution *of $S(d)$ if $x \in S(d)$, and there is no other distinct nonzero $\tilde{x} \in S(d)$ with $\tilde{x} \leq x$. Every irreducible vector x in $S(0)$ is called* homogeneous*. An irreducible vector x in $S(d)$ is called* inhomogeneous *whenever $d \neq 0$.*

An important property is that every integer point in $S(d)$ can be represented as the sum of exactly one inhomogeneous irreducible solution of $S(d)$ and an integer combination of the homogeneous irreducible solutions of $S(0)$. In order to come up with a first reformulation, we investigate a group relaxation

$$Y(f) = \{x \in \mathbb{Z}_+^n : Bx \equiv f \ (\mathrm{mod}\ \Delta)\} \tag{3}$$

of the integer program

$$\begin{array}{ll} \max & c^T x \\ \text{s. t.} & Ax = b \\ & x \in \mathbb{Z}_+^n. \end{array}$$

We determine a matrix C whose column vectors correspond to all inhomogeneous solutions of $Y(f)$. Accordingly we introduce a matrix D whose column vectors are all the homogeneous solutions of $Y(0)$. We associate integer λ and μ variables with the columns of C and D, respectively, to define the first formulation.

Proposition 1

$$Y(f) = \{x \in \mathbb{R}_+^n : x = C\lambda + D\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}_+^s, \mu \in \mathbb{Z}_+^t\}.$$

Therefore $\{x \in \mathbb{R}_+^n : x = C\lambda + D\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}_+^s, \mu \in \mathbb{Z}_+^t\}$ *is a valid extended formulation for* $Y(f)$ *referred to as the* disaggregated formulation.

Example: Consider the set $X = \{x_1, x_2, x_3 \in \mathbb{Z}_+ : 3x_1 + 7x_2 + 9x_3 = 22\}$. By taking the equation (mod 4, we obtain the valid group relaxation

$$Y(2) = \{x_1, x_2, x_3 \in \mathbb{Z}_+ : 3x_1 + 3x_2 + x_3 \equiv 2 \ (\mathrm{mod}\ 4)\}. \tag{4}$$

The inhomogeneous irreducible solutions of $Y(2)$ are represented in the matrix

$$C = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The homogeneous irreducible solutions of $Y(0)$ are represented in the matrix

$$D = \begin{pmatrix} 1 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

We refer to Section 4 for more details on how to compute these matrices of irreducibles. A valid reformulation for $Y(2)$ is thus to associate a new variable to each irreducible solution and hence write

$$\begin{aligned} Y(2) = \{x \in \mathbb{Z}_+^3 :& \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = &\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_8 \end{pmatrix}, \\ &\lambda_1 + \cdots + \lambda_4 = 1 \\ &\lambda_1, \cdots, \lambda_4 \in \{0,1\},\ \mu_1, \cdots, \mu_8 \in \mathbb{Z}_+ \quad \}. \end{aligned}$$

By looking in more details the irreducibles given in the example, it appears that we can classify some of the irreducible solutions according to the value given by the sum of the first and second component. For instance, all the vectors $\begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ have the property that the sum of the two first components is 4. This property can be highlighted for some other sets of vectors of C and D and is due to the fact that x_1 and x_2 have the same coefficient in the group equation of (4). This suggests to aggregate these two variables in one variable w so as to reduce the size of the reformulation. We now let $N = \{1, \cdots, n\}$ and $N_\alpha = \{j \in N : B_{:j} = \alpha\}$, where α is any element of the group G (mod Δ) and where $B_{:j}$ denotes the j^{th} column of B. We aggregate the variables with the same coefficient into $w_\alpha = \sum_{j \in N(\alpha)} x_j$ and consider the set

$$W(f) = \{w \in \mathbb{Z}_+^{|\tilde{G}|} : \sum_{\alpha \in \tilde{G}} \alpha w_\alpha = f \text{ (mod } \Delta)\},$$

where $\tilde{G} = \{\alpha \in G : \text{there exists } j \text{ with } B_{:j} = \alpha\}$. By denoting $\tilde{C} \in \mathbb{Z}^{n \times \tilde{s}}$ and $\tilde{D}i \in \mathbb{Z}^{n \times \tilde{t}}$ the matrices whose columns are the inhomogeneous and homogeneous irreducibles of $W(f)$ and $W(0)$ respectively, we are now able to write a second formulation for $Y(f)$.

Proposition 2

$$Y(f) = \{x \in \mathbb{Z}_+^n : w = \tilde{C}\lambda + \tilde{D}\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}_+^{\tilde{s}},\ \mu \in \mathbb{Z}_+^{\tilde{t}}, \\ w_\alpha = \textstyle\sum_{j \in N(\alpha)} x_j,\ w \in \mathbb{Z}_+^{|\tilde{G}|}\}.$$

Therefore $\{x \in \mathbb{Z}_+^n : w = \tilde{C}\lambda + \tilde{D}\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}_+^{\tilde{s}},\ \mu \in \mathbb{Z}_+^{\tilde{t}}, w_\alpha = \sum_{j \in N(\alpha)} x_j,\ w \in \mathbb{Z}_+^{|\tilde{G}|}\}$ *is a valid extended formulation for* $Y(f)$ *referred to as the* aggregated formulation.

Example: Consider again the group relaxation (4),

$$Y(2) = \{x \in \mathbb{Z}_+^3 : 3x_1 + 3x_2 + x_3 \equiv 2 \text{ (mod 4)}\}.$$

We aggregate the first two variables in $w = x_1 + x_2$ and now consider an aggregated version of $Y(2)$,

$$W(2) = \{w, x_3 \in \mathbb{Z}_+ : w + 3x_3 \equiv 2; \text{(mod 4)}\}.$$

The correponding matrices of irreducibles of $W(2)$ and $W(0)$ are

$$\tilde{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{D} = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}.$$

The aggregated reformulation of $Y(2)$ is

$$\begin{aligned} Y(2) = \{x \in \mathbb{Z}^3_+ : x_1 + x_2 = w_1 & \\ & \begin{pmatrix} w_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \\ & \lambda_1 + \lambda_2 = 1,\ \lambda \in \mathbb{Z}^2_+,\ \mu \in \mathbb{Z}^3_+,\ w \in \mathbb{Z}_+\}. \end{aligned}$$

Compared to the disaggregated formulation that we gave earlier, the aggregated formulation is much more compact.

One way of further reducing the size of the reformulation of $Y(f)$ consists in generalizing our aggregation technique to variables with different coefficients. We study here the case of aggregation of variables whose residue classes are integer multiples of each other. As before, we let $N(\alpha) = \{j \in N : B_{:j} = \alpha\}$. For some $\alpha \in G$, we define $h_\alpha \in \mathbb{Z}_+$ and

$$z_{h_\alpha \alpha} = \sum_{j \in N(\alpha)} x_j + \sum_{k \in N(h_\alpha \alpha)} h_\alpha x_k. \tag{5}$$

The set of all $\alpha \in G$ for which h_α is defined and therefore define some aggregation of type (5) is denoted by L. Let us now consider the set

$$Z(f) = \{z \in \mathbb{Z}^{|\hat{G}|}_+ : \sum_{\alpha \in \hat{G}} \alpha z_\alpha \equiv f \pmod{\Delta}\},$$

where $\hat{G}$ is the set of all elements of the group G that remain after having removed $h_\alpha \alpha$ for every $\alpha \in L$. By denoting $\hat{C} \in \mathbb{Z}^{n \times \hat{s}}$ and $\hat{D} \in \mathbb{Z}^{n \times \hat{t}}$ the matrices whose columns are the inhomogeneous and homogeneous irreducibles of $Z(f)$ and $Z(0)$ respectively, we are now able to present a third formulation for $Y(f)$.

Proposition 3

$$\begin{aligned} Y(f) = \{x \in \mathbb{Z}^n_+ : z = \hat{C}\lambda + \hat{D}\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}^{\hat{s}}_+,\ \mu \in \mathbb{Z}^{\hat{t}}_+, & \\ & z_{h_\alpha \alpha} = \textstyle\sum_{j \in N(\alpha)} x_j + \sum_{k \in N(h_\alpha \alpha)} h_\alpha x_k, \quad \textit{for } \alpha \in L \\ & z \in \mathbb{Z}^{|\hat{G}|}_+\}. \end{aligned} \tag{6}$$

Therefore the right hand side of (6) is a valid extended formulation of $Y(f)$. We refer to it as the advanced aggregation reformulation*.*

Example: We study the group problem

$$Y(3) = \{x \in \mathbb{Z}^3_+ : x_1 + 2x_2 + 4x_3 \equiv 3 \pmod{5}\}. \tag{7}$$

We decide to aggregate x_2 and x_3 and to write $z = x_2 + 2x_3$. The advanced aggregation version of $Y(3)$ is

$$Z(3) = \{z, x_1 \in \mathbb{Z}_+ : x_1 + 2z \equiv 3 \pmod{5}\}.$$

The inhomogeneous irreducible solutions of $Z(3)$ are $\hat{C} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \end{pmatrix}$ and the homogeneous irreducible solutions of $Z(0)$ are $\hat{D} = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 1 & 5 & 0 \end{pmatrix}$. This allows us to write an advanced aggregation reformulation of $Y(3)$,

$$
\begin{aligned}
Y(3) = \{x \in \mathbb{Z}^3_+ : x_2 + 2x_3 = z, \\
\begin{pmatrix} x_1 \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_4 \end{pmatrix} \\
\lambda_1 + \lambda_2 + \lambda_3 = 1,\ \lambda \in \mathbb{Z}^3_+,\ \mu \in \mathbb{Z}^4_+,\ z \in \mathbb{Z}_+\}.
\end{aligned}
$$

Finally we present a fourth reformulation that is based on the "path" structure of the group equation. Let (V, A) be a digraph with $|G|$ nodes corresponding to each element of the group G, and arcs $(\alpha, \alpha + B_{:j} \pmod{\Delta})$ for all $\alpha \in G$ and $j \in N$. Figure 1 shows such a graph related to the $x_1 + 2x_2 \pmod 4$ group problem. The arcs above the nodes correspond to x_1 while the arcs below the nodes correspond to $2x_2$. Now if we go back to the general case, any walk from

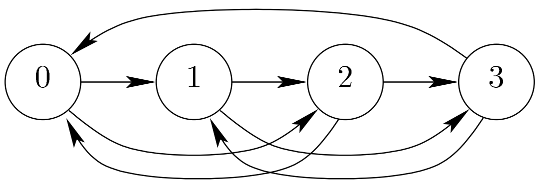

Figure 1: The path representation of $x_1 + 2x_2 \pmod 4$, $x \in \mathbb{Z}^2_+$.

0 to f corresponds to a point in $Y(f)$. Specifically let $w(B_{:j})$ be the number of times the arc $(\alpha, \alpha + B_{:j})$ occurs in the walk. Then

$$\sum_{\alpha \in G} \sum_{j \in N(\alpha)} B_{:j} w_\alpha(B_{:j}) \equiv f \pmod{\Delta}.$$

We can now formulate the group problem by flow constraints on each node of the graph, with a flow of 1 coming into node 0 and a flow of 1 going out of f, the new variables being the flow variables w.

Proposition 4

$$
\begin{aligned}
Y(f) = \{x: \quad & x_j = \sum_{\alpha \in G} w_\alpha(B_{:j}) \\
& \sum_{j\in N} w_0(B_{:j}) - \sum_{j\in N} w_{D-B_{:j}}(B_{:j}) = 1 \\
& \sum_{j\in N} w_\beta(B_{:j}) - \sum_{j\in N} w_{\beta-B_{:j}}(B_{:j}) = 0 \qquad \textit{for } \beta \neq 0, f \\
& \sum_{j\in N} w_f(B_{:j}) - \sum_{j\in N} w_{f-B_{:j}}(B_{:j}) = -1 \\
& w \in \mathbb{Z}_+^{n|G|} \qquad\qquad \},
\end{aligned}
$$

The right hand side of the equation above is a valid extended formulation of $Y(f)$ *referred as to the* path reformulation *of* $Y(f)$.

Example: Let us consider the following knapsack

$$x_1 + 5x_2 - 3x_3 = 3$$

and a corresponding group relaxation

$$Y(3) = \{x \in \mathbb{Z}_+^3 : x_1 + x_2 + 2x_3 = 3 \ (\mathrm{mod}\ 4)\}.$$

The path approach provides the following extended formulation

$$
\begin{aligned}
Y(3) = \{\, & x \in \mathbb{Z}_+^3 : \\
& x_1 = z(0,1) + z(1,1) + z(2,1) + z(3,1) \\
& x_2 = z(0,2) + z(1,2) + z(2,2) + z(3,2) \\
& x_3 = z(0,3) + z(1,3) + z(2,3) + z(3,3) \\
& z(0,1) + z(0,2) + z(0,3) - z(3,1) - z(3,2) - z(2,3) = 1 \\
& z(1,1) + z(1,2) + z(1,3) - z(0,1) - z(0,2) - z(3,3) = 0 \\
& z(2,1) + z(2,2) + z(2,3) - z(1,1) - z(1,2) - z(0,3) = 0 \\
& z(3,1) + z(3,2) + z(3,3) - z(2,1) - z(2,2) - z(1,3) = -1 \\
& z \in \mathbb{Z}_+^{12}. \qquad \}
\end{aligned}
$$

A solution $z(0,2) = z(1,3) = 1$ corresponds to $x_2 = x_3 = 1$.

The following example illustrates the sizes of the different formulations for the single row group problem

$$3x_1 + 3x_2 + 3x_3 + 6x_4 + 5x_5 + 10x_6 + 7x_7 \equiv 1 \ (\mathrm{mod}\ 11).$$

	Homogeneous Irreducibles	Inhomogeneous Irreducibles	Total Number of Variables
Disaggregated form.	378	76	454
Aggregated form.	54	26	80
Advanced agg. form.	13	8	21
Path formulation			77

3 Analysis of the reformulations

In this section we analyze the four formulations presented in the previous section. We focus on showing under which conditions the convex hull property is satisfied for each formulation. We denote by P_Y^1 the corresponding polyhedron when the integrality requirement of λ and μ is dropped in the disaggregated reformulation. Similarly we denote by P_Y^2, P_Y^3, P_Y^4 the polyhedra corresponding to the aggregated, the advanced aggregation and the path reformulation respectively, when all the integrality constraints are dropped.

An important result is that the convex hull property presented in Section 1 holds for three among the four reformulations, namely for the disaggregated, the aggregated and the path reformulation. To prove this result, two intermediate propositions are needed.

Proposition 5 *For $f \neq 0$, every extreme point of conv$(Y(f))$ is an irreducible inhomogeneous solution.*

Proof: Let y be a vertex of conv$(Y(f))$. It is obvious that y is an inhomogeneous solution of (2). Let us suppose now that y is reducible. Therefore there exists $y_1 \leq y, y_1 \neq y$, inhomogeneous solution of (2). We also have that $z = y - y_1$ is nonnegative and is an homogeneous solution of the group problem. Hence $y_2 = y + z$ is also a solution of (2) different from y. Furthermore

$$y = \frac{1}{2}y_1 + \frac{1}{2}y_2,$$

contradicting the fact that y is a vertex. □

It can also be noticed that every unit vector multiplied by the product of all Δ_i, for $i = 1, \cdots, r$ is definitely an homogeneous solution of the problem. Therefore there always exist a parallel vector which is irreducible.

Proposition 6 *For all $\alpha \in \tilde{G}$, $\prod_{i=1}^{r} \Delta_i e_\alpha \in W(0)$. Therefore conv$(W(0)) = \mathbb{R}_+^{|\tilde{G}|}$. Similarly, conv$(Y(0)) = \mathbb{R}_+^{|N|}$.*

Based on Proposition 5 and 6 it can be shown that the disaggregated, the aggregated and the path formulation do not only model a group problem, but define "ideal formulations" in the sense that they produce the Gomory corner polyhedron, i.e. the convex hull of all the integer points of a group problem.

Theorem 1 $P_Y^1 = P_Y^2 = P_Y^4 =$ *conv*$(Y(f))$.

Proof: For P_Y^1, the proof follows immediately from Proposition 5 and 6 as the extreme points are columns of C, and the extreme rays $k_j e_j$ are columns of D.

For P_Y^2, we note that if $x \in P_Y^2$, there exist w, λ, μ such that

$$w = \tilde{C}\lambda + \tilde{D}\mu, \; 1\lambda = 1, \; w_\alpha = \sum_{j \in N(\alpha)} x_j, \; x, w, \lambda, \mu \geq 0.$$

For α with $w_\alpha > 0$, we have that

$$w_\alpha = \sum_s \tilde{C}_{\alpha s}\lambda_s + \sum_t \tilde{D}_{\alpha t}\mu_t,$$

or

$$\begin{aligned}
\begin{pmatrix} x_{j1} \\ \vdots \\ x_{j|N(\alpha)|} \end{pmatrix} &= \begin{pmatrix} w_\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \lambda_s \frac{x_{j1}}{w_\alpha} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ w_\alpha \end{pmatrix} \lambda_s \frac{x_{j|N(\alpha)|}}{w_\alpha} \\
&= \sum_{j \in N(\alpha)} w_\alpha e_j \lambda_s \frac{x_j}{w_\alpha}.
\end{aligned}$$

Therefore we can write

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_s \sum_\alpha \sum_{ji \in N(\alpha)} \begin{pmatrix} w^s_{\alpha_0} e_{j_1} \\ \vdots \\ w^s_{\alpha_{D-1}} e_{j_D} \end{pmatrix} \lambda_s \frac{\prod x_{ji}}{\prod w_\alpha}.$$

The results follows since each of the vector in the sum lies in C.

For P^4_Y, the result follows from the fact that the matrix of flow constraints on the arcs w is totally unimodular. Hence every extreme point has integer values for w. Therefore x is integer as well. $\square$

The structure of the path reformulation is interesting because it leads to a totally unimodular matrix. This fact can be further used. We can also produce the convex hull for a bounded corner polyhedron as we outline below. Consider a bounded group relaxation

$$Y_B(f) = \{x \in \mathbb{Z}^n_+ : Bx \equiv f \text{ (mod } \Delta),\ x \leq u\}, \tag{8}$$

where $u \in \mathbb{Z}^n_+$ represent the bounds on the variables. We construct a digraph (V,A) with $n+1$ levels of nodes, corresponding to each variable and a source level. Specifically the digraph has $(n+1)|G|$ nodes denoted by $V_{0\alpha}$ for the source level and by $V_{i\alpha}$ for $i = 1, \cdots, n$, $\alpha \in G$, corresponding to a group element at the i^{th} level. For each variable i, and each group element α, the arcs

$$(V_{(i-1)\alpha}, V_{i\alpha}), \cdots, (V_{(i-1)\alpha}, V_{i((\alpha + u_i B_{:i}) \text{ (mod } \Delta))})$$

belong to the graph. A solution to (8) is now any walk from the source node V_{00} to the "target node" V_{nf}. If we denote by $w(V_{(i-1)\alpha}, V_{i\beta})$ the flow going into the arc $(V_{(i-1)\alpha}, V_{i\beta})$, for any $i \in N, \alpha, \beta \in G$, we have

$$\sum_{\alpha \in G} \sum_{i=1}^{n} \sum_{k=0}^{u_i} kB_{:i} w\left(V_{(i-1)\alpha}, V_{i((\alpha + kB_{:i}) \text{ mod } \Delta))}\right) \equiv f \text{ (mod } \Delta).$$

Hence in any solution, the value of the variables can be expressed by

$$x_i = \sum_{k=0}^{u_i} \sum_{\alpha \in G} kw \left(V_{(i-1)\alpha}, V_{i\left((\alpha + kB_{:i}) \bmod \Delta\right)} \right),$$

for all $i = 1, \cdots, n$. Figure 2 shows such a graph for the $x_1 + 2x_2 \pmod 4$ group problem with $x_1 \in \{0, 1, 2\}$ and $x_2 \in \{0, 1\}$. A walk from V_{00} to V_{22} represent a solution with 2 as right hand side.

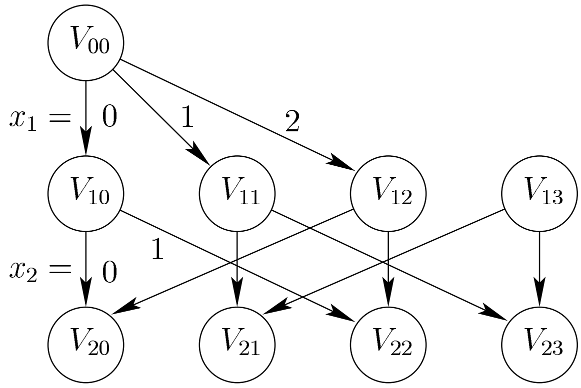

Figure 2: The graph related to the group problems $x_1 + 2x_2 \pmod 4$, $x_1 \in \{0, 1, 2\}, x_2 \in \{0, 1\}$.

Proposition 7

$$\begin{aligned}
&Y_B(f) = \{x \in \mathbb{Z}^n_+ : \\
&\quad x_i = \sum_{k=0}^{u_i} \sum_{\alpha \in G} kw \left(V_{(i-1)\alpha}, V_{i\left((\alpha + kB_{:i})\ mod\ \Delta\right)} \right), \quad \textit{for } i = 1, \cdots, n \\
&\quad \sum_{k=0}^{u_1} w(V_{00}, V_{1(kB(:1))}) = 1 \\
&\quad \sum_{k=0}^{u_i} w(V_{(i-1)(\alpha - kB_{:(i-1)})}, V_{i\alpha}) - \sum_{k=0}^{u_{i+1}} w(V_{i\alpha}, V_{(i+1)(\alpha + kB_{:i})}) = 0 \\
&\qquad\qquad\qquad\qquad\qquad \textit{for all } i = 1, \cdots, n, \textit{ and } \alpha \in G,\ (i, \alpha) \neq (n, f) \\
&\quad \sum_{k=0}^{u_n} w(V_{(n-1)(f - kB_{:i})}, V_{nf}) = 1 \\
&\quad w \in \mathbb{Z}^M_+ \qquad\qquad \},
\end{aligned}$$

where $M = |G| \sum_{i=1}^{n} (u_i + 1)$. Therefore, the expression in brackets is a valid extended formulation for $Y_B(f)$, referred as to the bounded path reformulation*.*

The key argument to prove that the bounded path reformulation provides the convex hull of $Y_B(f)$ is again that the flow conservation constraints form a totally unimodular matrix. If we denote by P_Y^5 the polyhedron obtained from the bounded path reformulation by relaxing the integrality constraints on the variables, we have the following result.

Proposition 8

$$P_Y^5 = conv(Y_B(f)).$$

We have seen that the convex hull property holds for the disaggregated, the aggregated and the path formulations. Unfortunately the advanced aggregation formulation is not as strong from a linear programming point of view. Indeed, in this case some fractional extreme points appear from the fact that

$$P = \{x_1, x_2, \lambda_i, \mu_j \in \mathbb{R}_+ : x_1 + hx_2 = \sum_{i=1}^{s} C_{li}\lambda_i + \sum_{j=1}^{t} D_{lj}\mu_j,\ \sum_{i=1}^{s} \lambda_i = 1\}, \quad (9)$$

for some $l = 1, \cdots, |\hat{G}|$, is not an integral polyhedron. We can, however, strengthen the formulation by adding cutting planes that are valid for

$$\bar{P} = \{x_1, x_2, \lambda_i, \mu_j \in \mathbb{Z}_+ : x_1 + hx_2 = \sum_{i=1}^{s} C_{li}\lambda_i + \sum_{j=1}^{t} D_{lj}\mu_j,\ \sum_{i=1}^{s} \lambda_i = 1\}. \quad (10)$$

Furthermore when $h = 2$ or 3, one can completely recover integrality by adding known valid inequalities.

Proposition 9 *The inequality*

$$x_2 \leq \sum_{i=1}^{s} \lfloor \frac{C_{li}}{h} \rfloor \lambda_i + \sum_{j=1}^{t} \lceil \frac{D_{lj}}{h} \rceil \mu_j \quad (11)$$

is valid for the set $\bar{P}$.

Proof: We know that, for every integer feasible solution, exactly one of the λ_i is equal to one. Let us fix $1 \leq k \leq s$, and consider all the integer points satisfying (10) such that $\lambda_k = 1$. For those points, we certainly have that

$$x_1 + hx_2 - \sum_{j=1}^{t} D_{lj}\mu_j = C_{lk}.$$

By dividing by h and rounding down, we can generate the Gomory cut

$$x_2 - \sum_{j=1}^{t} \lceil \frac{D_{lj}}{h} \rceil \mu_j \leq \lfloor \frac{C_{lk}}{h} \rfloor,$$

which is valid for all the points in $\bar{P}$ for which $\lambda_k = 1$. This can also be written as

$$x_2 - \sum_{j=1}^{t} \lceil \frac{D_{lj}}{h} \rceil \mu_j \leq \lfloor \frac{C_{lk}}{h} \rfloor \lambda_k. \quad (12)$$

We can write similar constraints for all k, $k = 1, \cdots, s$. Furthermore it is clear that every equation (12) is still valid by adding integer combinations of the λ_i, $i \neq k$, since $\lambda_i = 0$, for $i \neq k$. Therefore it follows that (11) is globally valid for every integer point in $\bar{P}$. $\square$

It is also clear that one can multiply the equation (9) by any integer κ before generating the Gomory cut. Therefore the following proposition is also true.

Proposition 10 *For any $\kappa \in \mathbb{Z}_+$, the inequality*

$$\kappa x_2 \leq \sum_{i=1}^{s} \lfloor \frac{\kappa C_{li}}{h} \rfloor \lambda_i + \sum_{j=1}^{t} \lceil \frac{\kappa D_{lj}}{h} \rceil \mu_j \tag{13}$$

is valid for the set $\bar{P}$.

In the case when $h = 2$ or 3, it can be shown that one or two inequalities of type (13) are required to guarantee the integrality of the polyhedron P.

Theorem 2 *The polyhedron*

$$\begin{aligned} \hat{P} \quad = \quad & \{x_1, x_2, \lambda_i, \mu_j \geq 0, i = 1, \cdots, s,\ j = 1, \cdots, t : \\ & x_1 + 2x_2 - \sum_{j=1}^{t} \hat{D}_{lj} \mu_j = \sum_{i=1}^{s} \hat{C}_{li} \lambda_i && (14) \\ & x_2 - \sum_{j=1}^{t} \lceil \frac{\hat{D}_{lj}}{2} \rceil \mu_j \leq \sum_{i=1}^{s} \lfloor \frac{\hat{C}_{li}}{2} \rfloor \lambda_i && (15) \\ & \sum_{i=1}^{s} \lambda_i = 1 \qquad \} \end{aligned}$$

is integral.

Proof: We show that every extreme point of $\hat{P}$ is integer. First we compute the extreme points for which exactly one λ_i is nonzero. Let us fix $1 \leq k \leq s$ such that $\lambda_k = 1$. Let us consider an arbitrary linear objective function that we maximize.

First if $\hat{C}_{lk} = 0$, the unique extreme point is $x_1 = x_2 = 0$ and $\mu_j = 0$ for all $j = 1, \cdots, q$, which is integer. Henceforth we assume $\hat{C}_{li} > 0$.

We then consider the case where the optimal solution has only one non zero variable. To satisfy the equation (14), this variable cannot be any μ_j, because the coefficients of all μ_j are non-positive. It remains two cases to analyze. If the non zero variable is x_1, then $x_1 = \hat{C}_{lk}$ which is integer and satisfies the second inequality. If the non zero variable is x_2, then $x_2 = \frac{\hat{C}_{lk}}{2}$ which satisfies the second constraint only if $\hat{C}_{lk}$ is even. In that case, the vertex is integer. In the case where $\hat{C}_{lk}$ is odd, two variables have to be non zero. Therefore, all the extreme points having one non zero variable are integer.

Let us now consider the case where the optimal solution has two non zero variables, and therefore the two constraints (14) and (15) are tight. These two variables cannot be of the form μ_{j_1} and μ_{j_2}, for $j_1, j_2 = 1, \cdots, q$, $j_1 \neq j_2$, since it would not satisfy the equation (14). We can also note that these two variables cannot be x_1 together with μ_j, for $j = 1, \cdots, q$, because the constraint (15) could not be tight, and it would certainly not be optimal. Two possible cases remain. Let us first suppose that both x_1 and x_2 are non zero. In this case, the optimal solution satisfies

$$\begin{cases} x_1 + 2x_2 = \hat{C}_{lk} \\ x_2 = \lfloor \frac{\hat{C}_{lk}}{2} \rfloor \end{cases}$$

If $\hat{C}_{lk}$ is even, this leads to an already known solution, i.e. with $x_1 = 0$. If $\hat{C}_{lk}$ is odd, the corresponding extreme point is $x_1 = 1$ and $x_2 = \lfloor \frac{\hat{C}_{lk}}{2} \rfloor$, which is an integer point. The second and last case is when x_2 and μ_j are non zero for some $j = 1, \cdots, q$. We know that the optimal solution satisfies

$$\begin{cases} 2x_2 - \hat{C}^j_{lk} = \hat{C}_{lk} \\ x_2 - \lceil \frac{\hat{D}_{lj}}{2} \rceil \mu_j = \lfloor \frac{\hat{C}_{lk}}{2} \rfloor . \end{cases} \tag{16}$$

Now we consider (16a)-2(16b), namely

$$(2\lceil \frac{\hat{D}_{lj}}{2} \rceil - \hat{D}_{lj})\mu_j = \hat{C}_{lk} - 2\lfloor \frac{\hat{C}_{lk}}{2} \rfloor . \tag{17}$$

If $\hat{C}_{lk}$ is odd and $\hat{D}_{lj}$ is even, the system has no solution. If both are odd, this leads to the extreme point $x_2 = \frac{\hat{C}_{lk}+\hat{D}_{lj}}{2}$ and $\mu_j = 1$, which is integer. If both are even, $\mu_j = 0$ and this leads to a known integer extreme point, with only x_2 non zero. Finally, if $\hat{C}_{lk}$ is even and $\hat{D}_{lj}$ is odd, μ_j can take any non negative value. But, related to the objective function, either it is not lucrative to increase both x_2 and μ_j, in which case the optimal solution will be the known integer extreme point with only x_2 non zero, or it is lucrative, and the problem is unbounded and this objective function does not lead to an extreme point of the polyhedron.

Finally we show that no extreme point x^* can be such that $\lambda^*_{i_1}, \lambda^*_{i_2} \neq 0$, for $i_1 \neq i_2$. If this were the case, then exactly one other variable would be non zero making the three constraints tight. It clearly cannot be one of the other μ variables beacause it would not fulfill (14). Suppose $x^*_1 \neq 0$. From (14), we have $x^*_1 = \hat{C}_{li_1}\lambda^*_1 + \hat{C}_{li_2}\lambda^*_2$. This is the convex combination of the two feasible points $(x^1_1 = \hat{C}_{li_1}, \lambda^1_{i_1} = 1, \lambda^1_{i_2} = 0)$ and $(x^2_1 = \hat{C}_{li_2}, \lambda^2_1 = 0, \lambda^2_2 = 1)$. Therefore, x^* cannot be extreme. Suppose now that $x^*_2, \lambda^*_{i_1}, \lambda^*_{i_2} \neq 0$. As (14) and (15) are tight, it means that $\hat{C}_{li_1}$ and $\hat{C}_{li_2}$ are even because otherwise (14) cannot be tight. Then x^* is a convex combination of two points with $\lambda_{i_1} = 1$ and $\lambda_{i_2} = 1$ respectively like in the case where x_1 is nonzero. Therefore no point with more than one λ_i nonzero can be extreme.

Now, all the possible cases have been explored, and they all lead to either an integer extreme point or to an unbounded solution. Therefore, all the extreme points of the polyhedron are integer. □

Theorem 2 can be extended to the case where $h = 3$. The idea of the proof is similar to the proof of Theorem 2 but requires the analysis of many subcases that we refrain from explaining here in detail.

Theorem 3 *The polyhedron*

$$
\begin{aligned}
\hat{P} \quad = \quad & \{x_1, x_2, \lambda_i, \mu^j \geq 0,\ i = 1, \cdots, s,\ j = 1, \cdots, t : \\
& x_1 + 3x_2 - \sum_{j=1}^{t} \hat{D}_{lj}\mu_j = \sum_{i=1}^{s} \hat{C}_{li}\lambda_i \\
& x_2 - \sum_{j=1}^{t} \lceil \frac{\hat{D}_{lj}}{3} \rceil \mu_j \leq \sum_{i=1}^{s} \lfloor \frac{\hat{C}_{li}}{3} \rfloor \lambda_i \\
& 2x_2 - \sum_{j=1}^{t} \lceil \frac{2\hat{D}_{lj}}{3} \rceil \mu_j \leq \sum_{i=1}^{s} \lfloor \frac{2\hat{C}_{li}}{3} \rfloor \lambda_i \\
& \sum_{i=1}^{s} \lambda_i = 1 \qquad \}
\end{aligned}
$$

is integral.

A straightforward extension of Theorem 3 where $h \geq 4$ is not true as the following example shows.

Example: Consider the polyhedron

$$
\begin{aligned}
Q = \{ & x_1, x_2, \lambda_1, \cdots, \lambda_3, \mu_1, \cdots, \mu_4 \in \mathbb{R}_+ : \\
& x_1 + 4x_2 = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \mu_1 + 2\mu_2 + 3\mu_3 + 4\mu_4 \\
& \lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad \}.
\end{aligned}
$$

As Proposition 10 states, we can add some valid inequalities like

$$
\begin{aligned}
x_2 &\leq \mu_1 + \mu_2 + \mu_3 + \mu_4 \\
2x_2 &\leq \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + 2\mu_3 + 2\mu_4 \\
3x_2 &\leq \lambda_2 + 2\lambda_3 + \mu_1 + 2\mu_2 + 3\mu_3 + 3\mu 4.
\end{aligned}
$$

If we add these three inequalities to Q, there remain two fractional extreme points

$$
\begin{array}{l}
(\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad 1 \quad 0 \quad \frac{1}{2} \quad 0 \quad 0 \quad 0) \text{ and} \\
(0 \quad 1 \quad 0 \quad 1 \quad 0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 0),
\end{array}
$$

that satisfy the three valid inequalities presented above. In this case, the inequality

$$2x_2 \leq \lambda_3 + \mu_1 + 2\mu_2 + 2\mu_3 + 2\mu_4,$$

which is not of the type of Proposition 10, is missing to make the polyhedron integral.

4 Computation of irreducible group solutions

A way of computing the irreducibles of a group problem is by using a Buchberger-type algorithm (see [1]) or via a recursive computation of "Primitive Partition Identities (PPI)" [9, 10]. These algorithms have the tendency of being slow and are therefore not suited to be used within an iterative integer programming algorithm. This fact suggests the idea to build up a database for certain group master problems and reuse the precomputed database in the iterative integer programming algorithm. Suppose, for example, that we construct a table $\mathcal{T}_D(D_0)$ of the irreducibles of the master group problem

$$x_1 + 2x_2 + 3x_3 + \cdots + (D-1)x_{D-1} \equiv D_0 \pmod{D},$$

for some $D \in \mathbb{Z}_+$ and $0 \leq D_0 \leq D-1$. Once this table is precomputed, we are able to read off the irreducibles of every (mod D) group problem from the irreducibles given in the table. Table 1 shows the number of irreducible solutions of single and two rows master group problems.

Modulus	Number of Irreducibles
2	2
3	7
4	15
5	38
6	56
7	143
8	209
9	402
10	598
11	1267
12	1445
13	3238

Moduli	Number of Irreducibles
2,2	9
3,3	246
3,4	1436
3,5	6363
3,6	12939
3,7	63454
3,8	147210
4,4	5215
4,5	40014
5,5	169870

Table 1: The number of irreducibles for single and two rows master group problems

The operation of reading the irreducibles in the table is fast. We explain it below. Suppose that we want to find the irreducibles for the group problem

$$\sum_{i=1}^{D-1} \sum_{k=1}^{\rho_i} i x_{ik} \equiv D_0 \pmod{D}, \tag{18}$$

where ρ_i denotes the number of times the coefficient i appears in (18) for $i = 1, \cdots, D-1$. The following algorithm gives the set $\mathcal{I}$ of irreducibles for (18).

Reconstruction Algorithm

Set $\mathcal{I} := \emptyset$
For each vector $v \in \mathcal{T}_D(D_0)$ do
  If for all $i \in \{0, \cdots, D-1\}$, $\rho_i = 0$ implies $v_i = 0$ **then**
    Create as many irreducibles w^j as there are integer solutions of
      $\sum_{k=1}^{\rho_i} w_{ik} = v_i$ for all i such that $\rho_i > 0$.
    Set $\mathcal{I} := \mathcal{I} \cup \left(\cup_j \{w^j\}\right)$.
Return $\mathcal{I}$.

Proposition 11 *(i) The Reconstruction Algorithm provides the set of all irreducibles of (18)*

(ii) The number of irreducibles generated from a vector v for which $\rho_i = 0$ implies $v_i = 0$ for all $i \in \{0, \cdots, D-1\}$ is $\prod_{i:v_i>1} \binom{\rho_i+v_i-1}{\rho_i}$.

The proof of (i) can be found in [9]. An easy analysis of the Reconstruction Algorithm provides the proof of (ii). From Proposition 11 we can thus deduce that the number of irreducible solutions of a group problem grows exponentially with the size of ρ_i. This affects particularly the disaggregated formulation. Indeed, the other formulations based on irreducibles gather the variables having the same coefficients which implies that ρ_i are small for all i.

It is computationally intractable to build tables for single-row master group problems if the modulus D is too big. However, there is an alternative for composite groups with D non prime, in the case of a single-row group problem. It is possible to build up the set of irreducibles from the irreducibles of smaller groups. Suppose specifically that $D = pq$ with $p, q > 1$ integer. Starting from $W(a_0) = \{w \in \mathbb{Z}_+^D : \sum_{j=1}^{D-1} a_j w_j = a_0 \pmod{D}\}$, we consider the relaxation

$$\tilde{W}(g_0) = \{w \in \mathbb{Z}_+^D : \sum_{j=1}^{D-1} g_j w_j = g_0 \pmod{p}\}$$

where $a_j = f_j p + g_j$ for all $0 \leq j \leq D-1$. Given the matrices of irreducibles $\tilde{C}$ and $\tilde{D}$ for $\tilde{W}(g_0)$, we have that

$$\tilde{W}(g_0) = \{w : w = \tilde{C}\lambda + \tilde{D}\mu,\ 1\lambda = 1, \lambda_i \in \mathbb{Z}_+, \mu_j \in \mathbb{Z}_+\}.$$

Substituting for w in $W(a_0)$, we obtain that

$$a^T\tilde{C}\lambda + a^T\tilde{D}\mu = a_0 \pmod{D},$$

where $a^t = pf^t + g$, $g\tilde{C} = g(1 \cdots 1), g\tilde{D} = 0$. In other words

$$\begin{array}{l} pf^T\tilde{C}\lambda + g^T\tilde{C}\lambda + pf^T\tilde{D}\mu + g\tilde{D}\mu = f_0 p + g_0 \pmod{pq} \\ 1\lambda = 1 \end{array}$$

or

$$\begin{aligned}&f^T\tilde{C}\lambda + f^T\tilde{D}\mu = f_0 \pmod q\\&1\lambda = 1\end{aligned}$$

Now let $\{A^s\}_{s=1}^S$ be the matrix of inhomogeneous irreducibles for $\{\mu : f^T\tilde{D}\mu = f_0 - f^T\tilde{C}^s \pmod q\}$ and B the matrix of homogeneous irreducibles. We now have

$$\begin{aligned}\mu &= A^s\alpha^s + B\beta\\ \lambda^s &= 1\end{aligned}$$

Finally we obtain

$$\mu = \sum_s A^s\alpha^s + B\beta$$

$$\sum_s \alpha^s = 1$$

This opens up the possibility of computing the set of irreducibles of bigger groups from the database that we precomputed.

Example: Let us consider the problem of finding irreducibles for

$$\begin{aligned}&w_1 + 2w_2 = 3 \pmod 4\\&w \in \mathbb{Z}_+^2.\end{aligned}$$

By first considering the equation (mod 2), it yields

$$\begin{aligned}&w_1 + 0w_2 = 1 \pmod 2\\&w \in \mathbb{Z}_+^2,\end{aligned}$$

which means that

$$\begin{aligned}&\begin{pmatrix} w_1\\ w_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\lambda_1 + \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\mu\\&\lambda_1 = 1.\end{aligned} \tag{19}$$

Substituting gives

$$\begin{aligned}&\lambda_1 + 2\mu_1 + 2\mu_2 = 3 \pmod 4\\&\lambda_1 = 1.\end{aligned}$$

By replacing λ_1 by its value and by dividing by 2, we now have

$$\mu_1 + \mu_2 = 1 \pmod 2.$$

By using the representation by irreducibles, we can write

$$\begin{aligned}&\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\alpha + \begin{pmatrix}2 & 1 & 0\\ 0 & 1 & 2\end{pmatrix}\beta\\&\alpha_1 + \alpha_2 = 1.\end{aligned} \tag{20}$$

By replacing (20) in (19), we finally obtain

$$\begin{aligned}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \alpha + \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \beta \\ &= \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \alpha + \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \beta \end{aligned} \tag{21}$$

with $\alpha_1 + \alpha_2 = 1$ and $\alpha, \beta \geq 0$ and integer.

5 Computational experiments

We now discuss an algorithm that incorporates any of the previous reformulations presented above. Our algorithm is primal-dual. The primal part comes from the fact that we always keep a primal feasible solution and an integer simplex tableau in which we perform the reformulations. These reformulations also provide the possibility of finding new augmenting vectors in the tableau. For choosing the right group relaxations, we "ask" the dual. Indeed, we select the group relaxations from rows derived from a fractional simplex tableau, for example at the optimal solution of the linear relaxation.

For the ease of exposition, we assume that we have a feasible integer starting point. If it is not the case, then one starts the algorithm with an infeasible integer point and apply a phase 1-method that uses the sum of the violated constraints as an objective function. Suppose that we are trying to solve

$$\begin{array}{ll} \max & c^T x \\ \text{s. t.} & Ax = b \\ & x \in \mathbb{Z}^n_+. \end{array} \tag{22}$$

As said before, we assume that we know $x_0 \in \mathbb{Z}^n_+$ feasible for (22). The Algorithm can be outlined as follows.

1. **Initialization**
   Compute an integer tableau representing x_0.

2. **Geometric Search for Fractionality**
   Compute a fractional point x_f of the polyhedron, for instance the LP optimum.
   If no such point exists, return the LP optimum as optimal solution of (22).

3. **Group Relaxation**
   Generate a group relaxation $Y^G = \{x \in \mathbb{Z}^n_+ : Bx \equiv f \pmod{\Delta}$ from a subset of tight constraints defining x_f.

4. **Reformulation**
   Compute an extended reformulation for Y^G.

5. **Make Compact**
   Use bounds on variables, as well as other problem constraints or knowledge of the problem to eliminate as many of the new variables, possibly adding also GUB or SOS constraints to the description of Y^G. The goal here is to prevent an excessive increase in the number of variables. If the number is still too large, the set Y^G must perhaps be relaxed further.

6. **Update Simplex Tableau**
   Update the integer tableau introducing the new columns and rows. Select the right variables to enter the basis in order to recover a tableau.

7. **Augmentation**
   Check for augmentation, i.e. check if there exists a new column v of positive reduced cost such that $x_i + v$ is feasible. In that case, pivot in v in an integer fashion and obtain an integer tableau representing the new feasible solution $x_{i+1} = x_i + v$.

8. Return to Step 2.

The Integral Basis Method introduced by Haus, Köppe and Weismantel [9] provides the algorithmic frame for testing our algorithm. In particular the following four phases have been described in [9]: Initialization, Make Compact, Update Simplex Tableau, Augmentation. In the following, we address some questions related to the phases Geometric Search for Fractionality, Group Relaxation and Reformulation.

Geometric Search for Fractionality

We start from an integer point x. Associated with this integer point is an algebraic tableau representation that encodes the geometry of the underlying linear programming relaxation. Investigating a small subset of the rows of this tableau means geometrically that we only consider the vicinity of x. However the "interesting" part of the linear programming polyhedron is hidden. These heuristic arguments motivate to inspect the linear programming optimum and use its tableau representation to select few rows from which a group relaxation can be built that cuts off the fractional point.

Group Relaxation

We compute the linear programming optimum of our current tableau using floating point arithmetic. We extract one or two fractional basic variables whose fractional part is closest to $1/2$. For those variables, we reconstruct in exact arithmetic the corresponding rows. It turns out that on our entire test set, the least common multiple of all the denominators occuring in a fractional row is gigantic, namely up to hundreds of digits. Clearly one cannot work with a group of this size. Our strategy is to define a tractable group problem by selecting a modulus (or two moduli) between 2 and 20 (between 2 and 8) for which the nondecomposable solutions have been tabulated in our database. Within this range, we choose a modulus that

(i) is not a divisor of the right hand side, in order to cut off the optimum point with our group relaxation,

(ii) produces the maximum number of 0-residue coefficients in order to reduce the size of the reformulation.

Example: Suppose that an interesting row provided by the fractional tableau at the optimum LP point is

$$x_0 + \frac{930}{1000}x_1 + \frac{1724}{1000}x_2 - \frac{937}{1000}x_3 - \frac{620}{1000}x_4 + \frac{30}{1000}x_5 = \frac{127}{1000},$$

which can also be written in integer form

$$1000x_0 + 930x_1 + 1724x_2 - 937x_3 - 620x_4 + 30x_5 = 127.$$

As we pointed before, it is computationnaly intractable to work with a (mod 1000) group. We decide therefore to chose some modulus between 2 and 20 in order to be able to use our precomputed table of irreducible solutions. If we choose 5 or 10 as a modulus, four variables will disappear from the group relaxation, which can be interesting in the point of view of having a reformulation that is not too large. Furthermore, as 127 is not divisible by 5, the group relaxation will cut off the fractional point. That is why we choose to compute the group relaxation

$$Y(2) = \{x_2, x_3 \in \mathbb{Z}_+ : 4x_2 + 3x_3 \equiv 2 \pmod 5\}.$$

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